Math 762 Homework Assignment, Due Thursday, March 8

1. With this exercise you will prove that any distance-preserving transformation of $\mathbb{E}^d$ is given by an orthogonal transformation followed by a translation. We call such a transformation a *congruence* of $\mathbb{E}^d$. Let $f : \mathbb{E}^d \to \mathbb{E}^d$ be any function such that for all p_1, p_2 in $\mathbb{E}^d$, $|f(p_1) - f(p_2)|^2 = |p_1 - p_2|^2$.

a. Assume $f(0) = 0$. Show that for any p_1, p_2 in $\mathbb{E}^d$, $f(p_1) \cdot f(p_2) = p_1 \cdot p_2$. (Hint: Use the vector 'polarization' identity $p_1 \cdot p_2 = (p_1^2 + p_1^2 - |p_1 - p_2|^2)/2$.)

b. Assume $f(0) = 0$. Use part a.) to show that f is linear. Show that for any p_1, p_2 in $\mathbb{E}^d$, $|f(\alpha p_1 + p_2) - \alpha f(p_1) - f(p_2)|^2 = 0$, where α is an arbitrary scalar.

c. Use part a.) and part b.) to show that $f(p) = Ap + b$, where A is a d-by-d orthogonal matrix, and b is a vector in $\mathbb{E}^d$.

2. Let $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ be two configurations of points in $\mathbb{E}^d$, such that for all $1 \leq i < j \leq n$, $|p_i - p_j| = |q_i - q_j|$. Show that there is a congruence $f : \mathbb{E}^d \to \mathbb{E}^d$ such that for $1 \leq i \leq n$, $f(p_i) = q_i$. Furthermore, if the affine span of p is d-dimensional, the congruence f is unique.

3. Suppose that p_1, p_2, p_3, p_4, p_5 are five points in Euclidean 3-space such that $|p_i - p_{i+1}|$ is constant for $i = 1, \ldots, 5$ (indices $\mod 5$), and the angles from p_{i-1} to p_i to p_{i+1} are equal to θ, constant for all i. Let $f : p \to p$ be the function defined by $f(p_i) = p_{i+1}$, indices $\mod 5$.

a. Use problem 2 to show that f extends to a congruence of $\mathbb{E}^3$, and that this extension can be taken to be of order 5. In other words f composed with itself 5 times is the identity.

b. Show that the congruence f of part a.) fixes the centroid $(p_1+p_2+p_3+p_4+p_5)/5$.

c. Assume that the centroid of p of part b.) above is the origin, so the congruence of part a.) is a linear transformation given by an orthogonal matrix A. Since $A^5 = I$ by part a.), conclude that the determinant of A is 1. Thus A is a rotation about some line through the origin.

d. Conclude that the affine span of p must be 2-dimensional.