Review of Convex Polyhedra by A. D. Alexandrov

Robert Connelly [1]

March 4, 2006

[1]Research supported in part by NSF Grant No. DMS-0209595

This book was first published in Russion in 1950, then translated into German in 1958, and V. A. Zalgaller has updated and substantially increased this English edition with more recent related results and translations of hard-to-find related results. Nevertheless the earlier russian and german versions of this book have made a significant contribution to research concerning the geometry of polyhedra, especially convex polyhedra in three-space, despite their lack of accessibility. (It seems that there are only two copies of the original Russian edition available in the United States through the interlibrary loan system.) It is a mystery to me why such a translation into English has not been done long ago.

For me, the star result in this book has to do with the realizability of developments of convex polyhedra. Suppose you have a compact, convex (bounded) three-dimensional polyhedron P siting on your table made out of cardboard. Take a knife and slit it open in such a way that the resulting cardboard lies flat on the table as one connected piece possibly with overlaps. This is a developement. It can be thought of a flat polygonal disk represented in the plane, with identifications on its boundary to determine how it fits onto the surface of P. The developement is essentially just the intrinsic metric surface of P.

If you look at the developement in a neighborhood of what was a vertex v of P, the sum s of the internal angles at v of the faces of P will be strictly less than π, and $\pi - s$ is the intrinsic *curvature* of P at v. It is a basic result that the sum of the curvatures of all the vertices of P is 4π. The convexity of P implies that this intrisic curvature at each vertex is positive, and for other point in the developement, the curvature is 0.

Suppose you start with a developement, thought of as a metric disk with identifications, such that the intrinsic curvature at each point is non-negative, only a finite number are positive, and the total curvature is 4π. So topologically you get a surface homeomorphic to a two-dimensional sphere. One of the most important results explained in the book is that for each such intrinsically convex developement, there is a unique convex polyhedron P having that developement, with the possible degenerate case when P becomes a doubly covered convex planar polygon. This is Alexandrov's existence theorem. It is explained carefully in this book, and is a substantial accomplishment.

After Alexandrov's first proof of this result appeared in the 1940's [A**], A. V. Pogorelov [P**] generalized it considerably to the case where the surface was any sort of metric space homeomorphic to a two-dimensional sphere but still intrinsically convex, while at the same time extending the definition of intrinsic convexity to allow for the boundary of any convex set in three-space. Pogorelov's result was also described in a book by him that was translated into English [P**] in the 1950's. But Pogorelov's proof was quite complicated and lengthy. In the present translation of A. D. Alexandrov's book, in addition to several footnotes concering more recent results, Zalgaller has included supplements which are translations of papers of Yu. A. Volkov that explain a somewhat simpler and shorter (but still quite non-trivial) proof of Pogorelov's result above. This gives the book a completeness and accesability that has so far been sadly lacking, at least for English speakers in the west.

Existence and uniqueness results, especially for polyhedra, have a long and distinquished history going back at least to Cauchy in 1813 and even, in spirit, to Euclid's Elements. Cauchy showed that if two convex polyhedra P and Q in three-space are such that there is a continuous correspondence from the surface of P to the surface of Q, which restricted to each face of P and Q is a rigid congruence, then P and Q are congruent (allowing for the possibility of reflected images). This is one of the first uniqueness (or rigidity) theorems.

Cauchy's proof had some hiccups that were not noticed for some time [XX], but the basic ideas were very insightful and still form a foundation for the theory of rigid polyhedra.

There are at least two points of view concerning such existence and rigidity results for polyhedra. One is to emphasize the surface of the polyhedron as a metric manifold as if it were a piece of paper. The other is to regard the vertices as universal joints and edges as fixed-length bars connecting those joints, thinking of the polyhedron as a framework, giving the subject a more discrete and combinatorial flavor. The Russian school with A. D. Alexandrov, V. A. Pogorelov, N. V. Efimov, and many others tended toward the first viewpoint, while some others such as W. Whiteley, H. Crapo, others, and myself, while coming much later to the subject, have tended to the more discrete viewpoint. But both viewpoints are quite compatable, where they overlap.

In 1916, M. Dehn [XX] proved an analogous result to Cauchy's for infinitesimal deformations convex polyhedra in three-space. An *infinitesimal deformation (or infinitesimal flex)* of a polyhedron is a vector field associated to the polyhedron that is trivial when restriced to each face. (A vector field being *trivial* means that it extends to the derivative of a differientiable family of congruences, starting at the identity, of all of Euclidean space at time 0.) Dehn's result says that any infinitesimal flex of a convex polyhedron in three-space that is trivial when restricted to each face, is globally trivial. In a formal sense, this proof is somewhat simpler than Cauchy's and is nicely described in Alexandrov's book. Indeed, in [XX] H. Gluck describes Alexandrov's proof of Dehn's theorem, while he points out that from the point of view of frameworks, almost all (in the measure theoretic sense, for example) triangulated spheres in three-space are rigid. Gluck also repeated the rigidity conjecture that says that any embedded triangulated surface in three-space is rigid, using this result as "evidence" for it. This paper was the starting point for my interest in this subject which led to my discovery of a flexible embedded triangulated sphere, a counterexample to the rigidity conjecture [XX].

Alexandrov's idea is to use the Cauchy-Dehn uniqueness result about the realization of polyhedral metrics to show the existence of polyhedra with the given metric. There is a natural map from the space of convex polyhedral metrics to the space of convex polyhedra that is essentially locally one-to-one by the uniqueness property. Then he applies the topological invariance of domain to show that the map is onto.

But Alexandrov's book has several interesting discussions of other closely related subjects in addition to the existence and uniqueness of polyhedra having given metrics. For example, a theorem of Minkowski says that if normal vectors in three-space and corresponding positive scalars are given, necessary and sufficient conditions are given for the existence of convex polytope in three-space whose face normals are the given vectors and whose corresponding areas are the given scalars. Furthermore, it is shown how the polytope that realizes these normal vectors and face areas is unique. All this is done using very similar argument as with the realization problem of polyhedral metrics. In all these problems there is also a lot of discussion about the situation for unbounded polyhedra and polyhedra with boundary, situations that are complicated and need to handled with care. There is also a very interesting discussion connecting the Cauchy-type problem with the Minkowski-type problem formally and showing how the Brunn-Minkowski inequalities can be brought to bear.

I would definitately recommend this book to a student who would like to get acquainted with some of the ideas in this sort discrete geometry. There are little goodies throughout that are very enlightening, and the discussion is very conversational. But this book would be very difficult to use as a text without guidance. There are few exercises, and complicated extensions are treated along with the basic more interesting ones without pause. The book has a lot of rich ideas to be mined.

Figure 1: The compete graph K_4

The author thanks

Bibliography

[Asimow-Roth] Asimow, L.; Roth, B., The rigidity of graphs. Trans. Amer. Math. Soc. 245 (1978), 279–289. MR 80i:57004a

[Asimow-Roth II] Asimow, L.; Roth, B., The rigidity of graphs. II. J. Math. Anal. Appl. 68 (1979), no. 1, 171–190. MR 80i:57004b

[Benedetti-Risler] Benedetti, Riccardo; Risler, Jean-Jacques, Real algebraic and semi-algebraic sets. Actualits Mathmatiques. [Current Mathematical Topics] Hermann, Paris, 1990. 340 pp. ISBN: 2-7056-6144-1 MR 99a:57012

[Berg-Jordán] Berg, Alex R.; Jordán, Tibor, A proof of Connelly's conjecture on 3-connected circuits of the rigidity matroid, Journal of Combinatorial Theory B, 88, (2003) 77-97.

[Connelly1] Connelly, Robert, Rigidity and energy. Invent. Math. 66 (1982), no. 1, 11–33. MR 83m:52012

[Connelly2] Connelly, Robert, Generic global rigidity. Applied geometry and discrete mathematics, 147–155, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 4, Amer. Math. Soc., Providence, RI, 1991. MR 83m:52012

[Connelly3] Connelly, Robert, Rigidity. Handbook of convex geometry, Vol. A, B, 223–271, North-Holland, Amsterdam, 1993. MR 94j:52041

[Connelly-Whiteley] Connelly, Robert; Whiteley, Walter, Second-order rigidity and prestress stability for tensegrity frameworks. (English. English summary) SIAM J. Discrete Math. 9 (1996), no. 3, 453–491. MR 97e:52037

[Gluck] Gluck, Herman, Almost all simply connected closed surfaces are rigid. Geometric topology (Proc. Conf., Park City, Utah, 1974), pp. 225–239. Lecture Notes in Math., Vol. 438, Springer, Berlin, 1975. MR 53 #4074

[Graver-Servatius-Servatius] Graver, Jack; Servatius, Brigitte; Servatius, Herman, Combinatorial rigidity. (English. English summary) Graduate Studies in Mathematics, 2. American Mathematical Society, Providence, RI, 1993. MR 95b:52034

[Guillemin-Pollack] Guillemin, Victor; Pollack, Alan, Differential topology. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1974. MR 50 #1276

[Hendrickson1] Hendrickson, Bruce, The Molecule Problem: Determining Conformation from Pairwise Distances, Ph.D dissertation, Cornell University (1991).

[Hendrickson2] Hendrickson, Bruce, Conditions for unique graph realizations. SIAM J. Comput. 21 (1992), no. 1, 65–84. MR 92m:05182

[Hendrickson3] Hendrickson, Bruce The molecule problem: exploiting structure in global optimization. SIAM J. Optim. 5 (1995), no. 4, 835–857. MR 96g:90093

[Jackson-Jordán] Jackson, Bill; Jordán, Tibor, Connected Rigidity Matroids and Unique Realizations of Graphs, (preprint) (2003).

[Marker] Marker, David, Model theory. (English. English summary) An introduction. Graduate Texts in Mathematics, 217. Springer-Verlag, New York, 2002.

[Milnor] Milnor, John W., Topology from the differentiable viewpoint. Based on notes by David W. Weaver. Revised reprint of the 1965 original. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. MR 98h:57051

[Saxe] Saxe, James B., Embeddability of weighted graphs in k-space is strongly NP-hard. Technical report, Computer Science Department, Carnegie Mellon University, 1979.

[Seidenberg] Seidenberg, A., A new decision method for elementary algebra. Ann. of Math. (2) 60, (1954). 365–374. MR 16,209a

[Tay-Whiteley] Tay, Tiong-Seng(SGP-SING); Whiteley, Walter Generating isostatic frameworks. Dual French-English text. Structural Topology No. 11, (1985), 21–69. MR 87e:05139

Department of Mathematics, Malott Hall, Cornell University, Ithaca, NY 14853
connelly@math.cornell.edu