Lesson 34 – Coordinate Ring of an Affine Variety

In mathematics we often understand an object by studying the functions on that object. In order to understand groups, for instance, we study homomorphisms; to understand topological spaces, we study continuous functions; to understand manifolds in differential geometry, we study smooth functions. In algebraic geometry, our objects are varieties, and because varieties are defined by polynomials, it will be appropriate to consider polynomial functions on them.

I. The Coordinate Ring of an Affine Variety Recall that any polynomial in n variables, say $f \in k[x_1, x_2, \dots, x_n]$, is a function mapping k^n to k; i.e., $f: k^n \to k$. Given an affine algebraic variety $V \subset k^n$, the restriction of f to V defines a map from V to k; that is, $f|_V: V \longrightarrow k$. Under the usual pointwise operations of addition and multiplication, these functions form a k-algebra

$$k[x_1, x_2, \dots, x_n]|_V$$

which we call the *coordinate ring of* V and denote by $k[V]$.

Definition. Given an affine variety $V \subseteq k^n$, the **coordinate ring of $\boldsymbol{V}$** is donoted by $k[V]$ and defined to be the set of polynomials $f|_V: V \longrightarrow k$, where $f \in k[x_1, x_2, \dots, x_n]$. The ring $k[V]$ is often described as "the ring of polynomial functions on V".

Remarks

1. If $V = k^n$, then clearly $k[x_1, x_2, \dots, x_n]|_V = k[x_1, x_2, \dots, x_n]$.

2. The elements in $k[V] = k[x_1, x_2, \dots, x_n]|_V$ are *restrictions* of polynomials on k^n, but we typically denote them by their original polynomials. This is perhaps a source of confusion because two different polynomials in $k[x_1, x_2, \dots, x_n]$ could be the same polynomial when considered as a function on V.

Exercise 1 Consider the variety $V = \mathbf{V}(x^2 + y^2 - 1) \subseteq k^2$. List three distinct polynomials in $k[x, y]$ that are the same function when considered as elements in $k[V]$.

Theorem 1 Given an affine algebraic variety $V \subset k^n$,

$$k[V] \cong k[x_1, x_2, \dots, x_n]/\mathbf{I}(V)$$

More Remarks

3. It is *very* common to write $k[V] = k[x_1, x_2, \ldots, x_n]/\mathbf{I}(V)$ even though $k[V] \cong k[x_1, x_2, \ldots, x_n]/\mathbf{I}(V)$.

4. An equivalence class in $k[x_1, x_2, \ldots, x_n]/\mathbf{I}(V)$ is represented by a single polynomial, and typically denoted by $[f]$ or $f + \mathbf{I}(V)$, though the former is more prevalent since it is easier to write. In actuality, people often write f when they mean $[f]$, under the assumption that the context makes the meaning clear.

5. Two two polynomials f and g define the same equivalence class in $k[x_1, x_2, \ldots, x_n]/\mathbf{I}(V)$ iff $f - g \in \mathbf{I}(V)$. Hence two restrictions $f|_V,\ g|_V$ are equal in $k[V]$ iff $f - g \in \mathbf{I}(V)$.

Exercise 2. Find the coordinate ring of the cone $V = \mathbf{V}(x^2 + y^2 - z^2)$ in $\mathbb{C}^3$.

Exercise 3 Show that the two polynomials $f, g \in \mathbb{C}[x, y, z]$ defined by $f(x, y, z) = x^3 + 2xy^2 - 2xz^2 + x$ and $g(x, y, z) = x - x^3$ are the same when considered as elements of $\mathbb{C}[V]$, where V is the variety defined in the previous exercise. $f|_V = g|_V$.

III. Lots of Examples Now let us gain some intuition for $k[V]$ by examining some examples.

(1) **The Coordinate Ring of Lines:** Consider $\ell: f(x, y) = 0$ for $f(x, y) = y - mx - b$. Then $k[\ell] = k[x, y]/\langle f \rangle$.

Exercise 4 Show that $k[\ell] \cong k[x]$.

(2) **The Coordinate Ring of a Parabola**: The coordinate ring of the parabola $\mathcal{C}: y - x^2 = 0$ is given by $k[\mathcal{C}] = k[x, y]/\langle y - x^2 \rangle$.

Exercise 5 Show that $k[\mathcal{C}] \cong k[x]$.

(3) **The Coordinate Ring of the Unit Circle**

Exercise 6

a) If $\mathcal{C}: f(x, y) = x^2 + y^2 - 1$ and $\varphi: k[x, y] \to k[x, y]/\langle f \rangle = k[\mathcal{C}]$, what is the image of the polynomial $g(x, y) = x^4 + x^2y + xy^2$ in $k[\mathcal{C}]$?

b) Is the coordinate ring for the unit circle isomorphic to $k[x]$?

Algebraic Properties of the Affine Coordinate Ring

Proposition 1 The ring $\mathbb{Z}$ cannot be the coordinate ring of any curve $\mathcal{C}: f(x, y) = 0$, where $f \in k[x, y]$ and $\deg f \geq 1$.

Exercise 7 Prove Proposition 1.

Proposition 2 (Plane Curves) Let $f \in k[x, y]$ be a nonconstant polynomial with coefficients from some algebraically closed field, and $\mathcal{C}: f(x, y) = 0$ the corresponding affine curve. The following assertions are equivalent:
(1) f is irreducible in $k[x, y]$;
(2) $\langle f \rangle$ is a prime ideal in $k[x, y]$;
(3) the coordinate ring $k[\mathcal{C}]$ of $\mathcal{C}$ is an integral domain

Proof. This proposition is a "baby case" of the next proposition.

Proposition 3 (Affine Varieties) Let $V \subset k^n$ be an affine variety. The following assertions are equivalent:
(1) V is irreducible
(2) $\mathbf{I}(V)$ is a prime ideal
(3) $k[V]$ is an integral domain.

Proof. The equivalence (1) $\Leftrightarrow$ (2) is Proposition 3 of section 4.5 in your text. The equivalence (2) $\Leftrightarrow$ (3) follows from the fact that an ideal *P* is prime in some ring *R* iff *R/P* is an integral domain.

Exercise 8 We have seen that $k[V] \simeq k[x_1, x_2, \ldots, x_n]/\mathbf{I}(V)$. True or False: If J is some other ideal defining the variety V, then $k[V] \simeq k[x_1, x_2, \ldots, x_n]/J$ **.**

Exercise 9 True or False: If V is an affine variety in k^n, where k is algebraically closed, then $k[V] \simeq k[x_1, x_2, \ldots, x_n]/\mathbf{I}(V)$ contains no nonzero nilpotent elements.

Exercise 10 $\mathbb{R}[x, y]/\langle x^3, y\rangle \simeq \mathbb{R}[x, y]/\langle x^2, y^2\rangle$.