Torus Maps from Weak Coupling of Strong Resonances

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Abstract

This paper investigates a family of diffeomorphisms of the two dimensional torus derived from a model of an array of driven, coupled lasers (Braiman et al. 1995). It gives a negative answer to a question raised by Baesens et al. (1991a) about two parameter families of diffeomorphisms and flows of the two dimensional torus by exhibiting resonance regions in which the two parameter family has no Hopf or Taken-Bogdanov points. The example can be interpreted as a general model for single-mode coupling of two strongly nonlinear oscillators driven near resonance. We give a complete analysis of the steady state bifurcations in this model that depends upon four parameters and a description of the observed global bifurcations in a two parameter subfamily.

1 Introduction

The phase dynamics of a system of n oscillators take place on the n-dimensional torus. Coupled oscillators exhibit *mode locking* and *resonance* in which trajectories approach a limit that lies on a torus whose dimension is smaller than n. For a pair of oscillators ($n = 2$), the geometry associated with mode-locking is quite well understood. Flows on the two dimensional torus with global cross-sections have return maps that are diffeomorphisms of a circle. The theory of rotation numbers characterizes the flows that exhibit mode-locking as those whose return maps have rational rotation numbers. Rotation numbers are defined as elements of R/Z, or equivalently can be regarded as real numbers whose integer part is determined by the lift of a diffeomorphism of the circle to a diffeomorphism of the line (Herman 1979). The work of Arnold (1965), Herman (1979) and Yoccoz (1984) gives a very thorough analysis of the dynamics of families of diffeomorphisms of the circle. We shall say that a pair of oscillators has a strong resonance if the corresponding circle diffeomorphism has a fixed point, or equivalently the oscillators are mode-locked with a rotation number

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that is an integer. There is a close correspondence between the analysis of diffeomorphisms of the circle with fixed points and the theory of flows of the circle. For diffeomorphisms that are close to rotations, this correspondence has a simple expression in terms of the Euler integration algorithm: the time ϵ map of the equation $\dot{x} = f(x)$ is approximated by the diffeomorphism $F(x) = x + \epsilon f(x)$. The equilibrium points of f and the fixed points of F are identical. Using a similar correspondence, we can learn much about mode-locking of several oscillators by approximating their return maps by flows.

The mathematical theory of families of torus flows and diffeomorphisms in higher dimensions is less complete. Baesens et al. (1991b) defined resonance regions of a family of torus diffeomorphisms to be the set of parameters for which the system has periodic orbits. They studied the bifurcations within and on the boundary of resonance regions of two parameter families of two dimensional torus maps. Their starting point was the family of *twist maps*, a model for uncoupled oscillators in which the relative frequencies of the oscillators are parameters. They explored nonlinear perturbations of twist map families as models of the typical behavior of weakly coupled oscillators. They gave extensive descriptions of the topology of resonance regions in the parameter spaces and of the global bifurcations that interacted with the resonance regions. The resulting picture displays the structure of a higher dimensional analog of a resonance tongue for a family of circle diffeomorphisms. The dynamics of diffeomorphisms of the two dimensional torus is much less constrained than that of the circle, so there is much more variability in the bifurcation diagrams of the resonance regions. Figure 1 shows the bifurcation diagram of a two parameter family of flows on the two torus that corresponds to a *simple resonance region.* A simple resonance region of a family of diffeomorphisms is one in which no parameter value yields more than two periodic orbits of the resonance type of the region. Figure 1 displays several global bifurcation curves at which the number of periodic orbits of the flows change. The diagram is *generic* in the sense that all of the singular points of the bifurcation diagram are codimension two bifurcations of flows whose qualitative structure is unchanged by perturbations of the family of flows. (Perturbations of the time t map of these flows as a family of diffeomorphisms introduce much additional complexity, including the presence of chaotic invariant sets.) In all of the cases they examined, the resonance regions of torus maps contained codimension one Hopf bifurcations and codimension two *Takens-Bogdanov* bifurcations of periodic orbits. The Takens-Bogdanov bifurcations of diffeomorphisms are characterized as periodic orbits for which the linearization has one as an eigenvalue of algebraic multiplicity two. Baesens et al. (1991a) proved that Takens-Bogdanov points must occur in the boundaries of simple resonance regions and asked the question as to whether all resonance regions have Takens-Bogdanov points.

This paper analyzes a family of flows of the two dimensional torus without Hopf bifurcations or Taken-Bogdanov bifurcations. While our results are about flows, they translate directly into comparable results about diffeomorphisms through the correspondence between flows and the Euler time ϵ step of the flows. Thus, we give a negative answer to the question asked above. More specifically, if $\dot{x} = f(x)$ defines a flow on the torus T^2, then the equilibrium points of f are the fixed points of the diffeomorphisms $F(x) = x + \epsilon f(x)$ for all $0 < \epsilon < \sup(|Df(x)|^{-1})$. Moreover, the stability of the equilibrium points of f and the fixed points of F agree. The examples we analyze arose in the study of solid-state laser arrays where entrainment is due to external periodic forcing (Braiman et al. 1995, Khibnik et al. 1998). They are defined by the equations

$$\begin{aligned} \dot{x_1} &= a + b - c\sin(x_1) - d\sin(x_2 - x_1) \\ \dot{x_2} &= a - b - c\sin(x_2) - d\sin(x_1 - x_2) \end{aligned} \tag{1}$$

where we fix c and d and regard a and b as the active parameters of the family. In the laser models, a and b represent the frequencies of the two lasers, c is the amplitude of external forcing and d is the coupling strength. These equations are derived by applying singular perturbation theory to a more complex model and then restricting the reduced equations to an invariant torus. The single equation $\dot{x} = a + c\sin(x)$ is the result of applying this procedure to a single resonantly forced oscillator. This is the vector field analog of a family of diffeomorphisms displaying strong resonance. When $|a| \le |c|$, the system has equilibrium point(s) corresponding to strongly resonant oscillations. When $|a| > |c|$, there are no equilibria and the system is no longer resonant. The system we study represents two such oscillators with slightly different resonant frequencies. The frequency difference is represented by the parameter b. The two oscillators are coupled by a term that depends symmetrically upon the phase difference of the two oscillators. We have not tried to pinpoint other physical problems that lead to this system, but we think that its behavior is typical of symmetry breaking perturbations of weakly coupled, resonantly forced oscillations of two identical oscillators.

Rescaling time in the the vector fields (1) multiplies each of the parameters by a scalar. Thus the bifurcation set of the family has a conical structure, and it suffices to study a three parameter family to determine all of the dynamics of the system. For example, setting $c^2 + d^2 = 1$ is a normalization that fixes the size of the resonance region in the (a, b) plane and allows us to easily explore both the limits of strong coupling of weakly forced oscillators ($c = 0$) and weak coupling of strongly forced oscillators ($d = 0$). Here we focus upon global bifurcations that occur for weak coupling of strongly forced oscillators ($|d/c| \ll 1$.)

2 Local Bifurcations

This section discusses the local bifurcations of the two parameter family of vector fields (1) obtained by varying a and b. For each value of c and d, the local bifurcations form curves in the (a, b) parameter space that we describe. There are three types of local codimension one bifurcations of equilibria: saddle-nodes, Hopf bifurcations and resonant saddles. Resonant saddles are not always regarded as bifurcations since they do not affect the local topological properties of a phase portrait near an equilibrium point, but they play an important role in some kinds of global bifurcations. Since they are significant for the global bifurcations of the family (1), we include their analysis here. The different local bifurcations occur when the Jacobian matrix J of Equations (1)

$$\begin{bmatrix} -c\cos(x_1) + d\cos(x_1 - x_2) & -d\cos(x_1 - x_2) \\ -d\cos(x_1 - x_2) & -c\cos(x_2) + d\cos(x_1 - x_2) \end{bmatrix}$$

has a zero eigenvalue, pure imaginary eigenvalues or a pair of real eigenvalues whose sum is zero, respectively. Zero eigenvalues occur when

$$\det(J) = c^2\cos(x_1)\cos(x_2) - cd\cos(x_1 - x_2)(\cos(x_1) + \cos(x_2)) = 0$$

and pure imaginary or resonant eigenvalues imply that

$$\mathrm{trace}(J) = 2d\cos(x_1 - x_2) - c(\cos(x_1) + \cos(x_2)) = 0$$

Pure imaginary and real, resonant eigenvalues are distinguished by the sign of $\det(J)$: imaginary eigenvalues occur when $\det(J) > 0$ and real, resonant eigenvalues occur when $\det(J) < 0$. When $\det(J)$ and $\mathrm{trace}(J)$ are both zero, we say that Takens-Bogdanov bifurcation occurs. This is a codimension two bifurcation. Our first result has a simple proof, but it answers negatively a question posed by Baesens et al. (1991a).

Theorem 1 *Assume that $c \neq 0$ and $d \neq 0$. The vector field defined by*

$$\begin{aligned} \dot{x_1} &= a + b - c\sin(x_1) - d\sin(x_2 - x_1) \\ \dot{x_2} &= a - b - c\sin(x_2) - d\sin(x_1 - x_2) \end{aligned}$$

has no Hopf or Takens-Bogdanov bifurcations as a and b vary.

Proof: Assume that c and d are both non-zero. The curve defined by $\mathrm{trace}(J) = 0$ consists of points with either pure imaginary eigenvalues (Hopf points) or points at which there are real eigenvalues of opposite sign and equal magnitude (resonant saddles). When $\mathrm{trace}(J) = 0$, J is readily seen to have negative determinant since $\cos(x_1 - x_2) = c(\cos(x_1) + \cos(x_2))/(2d)$ implies $\det(J) = c^2(\cos(x_1)\cos(x_2) - (\cos(x_1) + \cos(x_2))^2/2) = -c^2((\cos(x_1))^2 + (\cos(x_2))^2)/2 < 0$ unless $\cos(x_1) = \cos(x_2) = 0$. However, $\cos(x_1) = \cos(x_2) = 0$ implies $|\cos(x_1 - x_2)| = 1$, contradicting the equation $\mathrm{trace}(J) = 0$. Thus $\mathrm{trace}(J) = 0$ implies that $\det(J) < 0$ and that the only points where $\mathrm{trace}(J) = 0$ are resonant saddles. This proves the theorem.

We next examine the geometry of the bifurcation curves in more detail, beginning with the resonant saddles. The gradient of $\mathrm{trace}(J)$ is $(c\sin(x_1) - 2d\sin(x_1 - x_2), c\sin(x_2) + 2d\sin(x_1 - x_2))$. Points where $c \neq 0$ and the gradient vanishes satisfy $\sin(x_1) + \sin(x_2) = 0$. This implies that $x_2 = x_1 + \pi$ or $x_2 = -x_1$. In the first case, we have $\cos(x_1) + \cos(x_2) = 0$ and $\mathrm{trace}(J) = 0$ implies that $d = 0$. In the second case, $\cos(x_2) = \cos(x_1)$, $\cos(x_1 - x_2) = 2\cos^2(x_1) - 1$ and $\sin(x_1 - x_2) = 2\cos(x_1)\sin(x_1)$. Substitution of these expression into the equations trace$(J) = 2d\cos(x_1 - x_2) - c(\cos(x_1) + \cos(x_2)) = 0$ and $c\sin(x_1) - 2d\sin(x_1 - x_2) = 0$ yields $-2c\cos(x_1) + 4d(\cos(x_1))^2 - 2d = 0$ and $\sin(x_1)(c - 4d\cos(x_1)) = 0$. If $cd\sin(x_1) \neq 0$, this pair of equations has no solution. We conclude that the singularities of the curves defined by trace$(J) = 0$ occur when $x_1 = x_2 = 0$ and $c = d$ or when $x_1 = x_2 = \pi$ and $c = -d$. A simple computation of the Hessian of trace(J) shows that the singular points are index 1 Morse critical points of the function trace(J). For $|c| > |d|$, we assert that the set of resonant saddles is a connected curve of trivial homotopy type on the torus of the (x_1, x_2) variables. To see this, note that, when $d = 0$, the solutions of trace$(J) =$ are given by $\cos(x_1) + \cos(x_2) = 0$ or $x_1 = \pi \pm x_2$. Singularities of this curve occur at $(\pi, 0)$ and $(0, \pi)$. When $d > 0$ is small, the deformations of the solutions near the singularities are readily computed by expanding the cosine functions as Taylor series to obtain the approximate equations $4d = x_1^2 - (x_2 - \pi)^2$ near the point $(0, \pi)$ and $4d = -(x_1 - \pi)^2 + x_2^2$ near the point $(\pi, 0)$. If $|c| < |d|$, the set of resonant saddles has two components, each of which is a closed curve of homotopy type $(1, 1)$ on the (x_1, x_2) torus. This is readily seen by setting $c = 0$ in the equation trace$(J) = 0$ to obtain the curves given by $x_1 - x_2 = \pm\pi$. The following theorem summarizes this discussion.

Theorem 2 *Assume $|c| > |d|$. As (a, b) vary, the curve of resonant saddle points of vector fields (1) is connected and nonsingular. The curve of resonant saddle points has two nonsingular components if $|c| < |d|$. If $|c| = |d|$, the curve of resonant saddle points has a double point.*

The geometry of the saddle-node bifurcation curves is somewhat more complex than that of the resonant saddle curves. We prove the following result.

Theorem 3 *For $|c/d| > 2$, the saddle-node curve of the family (1) has two components. For $0 < |c/d| < 2$, the saddle-node curve has three components. All the components are homotopically trivial and have nonsingular projections onto the (x_1, x_2) torus.*

Proof: The defining equation for saddle-node bifurcations is $\det(J) = c^2\cos(x_1)\cos(x_2) - cd\cos(x_1 - x_2)(\cos(x_1) + \cos(x_2)) = 0$. The gradient of $\det(J)$ is

$$\begin{gathered}(-c^2\sin(x_1)\cos(x_2) + cd(\sin(x_1 - x_2)(\cos(x_1) + \cos(x_2)) + \cos(x_1 - x_2)\sin(x_1)),\\ -c^2\sin(x_2)\cos(x_1) - cd(\sin(x_1 - x_2)(\cos(x_1) + \cos(x_2)) + \cos(x_1 - x_2)\sin(x_2)))\end{gathered}$$

Thus, the singularities of of the saddle-node bifurcation curves in their domains occur when the 3×2 matrix

$$\begin{pmatrix} \cos(x_1)\cos(x_2) & \cos(x_1 - x_2)(\cos(x_1) + \cos(x_2)) \\ -\sin(x_1)\cos(x_2) & \sin(x_1 - x_2)(\cos(x_1) + \cos(x_2)) + \cos(x_1 - x_2)\sin(x_1) \\ -\sin(x_2)\cos(x_1) & \sin(x_2 - x_1)(\cos(x_1) + \cos(x_2)) - \cos(x_1 - x_2)\sin(x_2) \end{pmatrix}$$

has rank one. Computing the minors of this matrix and using $\sin^2(x_i) = 1 - \cos^2(x_i)$ yields the following pair of equations:

$$\begin{array}{rcl} \cos^2(x_1)\cos(x_2)\sin(x_1) - \cos^3(x_1)\sin(x_2) - \cos(x_2)\sin(x_2) &=& 0 \\ \cos^2(x_2)\cos(x_1)\sin(x_2) - \cos^3(x_2)\sin(x_1) - \cos(x_1)\sin(x_1) &=& 0 \end{array} \tag{2}$$

If $\sin(x_1)$ and $\sin(x_2)$ are not both zero, then eliminating $\sin(x_1)$ and $\sin(x_2)$ from the equations (2) yields

$$\cos^4(x_1) + \cos(x_1)\cos(x_2) + \cos^4(x_2) = 0 \tag{3}$$

Using $\sin^2(x_i) = 1 - \cos^2(x_i)$ once again to eliminate $\sin(x_i)$ from the first equation of the system (2), we obtain

$$\cos^4(x_1)\cos^2(x_2) + 2\cos^3(x_1)\cos^3(x_2) + \cos^4(x_1) = \cos^6(x_1) + \cos^2(x_1) + 2\cos^3(x_1)\cos(x_2) \tag{4}$$

Using the computer program Maple to simultaneously solve equation (3) and equation (4) produces three solutions: $\cos(x_1) = \cos(x_2) = 0$, $\cos(x_1) = -\cos(x_2) = \pm\sqrt{(1/2)}$ and solutions with $\cos(x_1)$ a root of the polynomial $p(z) = z^8 - 3z^6 + 4z^4 - 2z^2 + 1$. All roots of $p(z)$ are complex, so there are no roots in the third family. If $\cos(x_1) = \cos(x_2) = 0$, then $\cos(x_1 - x_2) = \pm 1$ and hence $d = 0$. If $\cos(x_1) = -\cos(x_2) = \pm\sqrt{(1/2)}$, then $\cos(x_1 - x_2) = 0$ and hence $c = 0$. We conclude that the only singularities of the saddle-node curves occur when $\sin(x_1)$ and $\sin(x_2)$ are both zero. In this case $\det(J) = 0$ implies that $c/d = \pm 2$. Calculating the Hessian of $\det(J)$, we find that this is a Morse singularity of index one for the zero level set of the function $\det(J)$.

We finish the proof of the theorem by perturbation calculations from the families with $c = 0$ and $d = 0$ that determine the number of components of the saddle node curves in the (x_1, x_2) torus. For $d = 0$ and $c > 0$, the 0 level curve of $\det(J) = 0$ consists of the four circles $x_i = \pm\pi/2$. An easy calculation shows that the function $\det(J)$ is a Morse function in the neighborhood of each of the intersection points $(x_1, x_2) = (\pm\pi/2, \pm\pi/2)$. We proved that the saddle-node curves for $0 < d \ll c$ are non-singular, so they will be approximated by hyperbolas near $(x_1, x_2) = (\pm\pi/2, \pm\pi/2)$ when $d > 0$ is small. To determine the topology of the saddle node curves, we only need to determine which direction the crossing curves for $d = 0$ split as we perturb to $c > 0$. Expanding the Taylor series of $\det(J)$ near the points $(x_1, x_2) = (\pm\pi/2, \pm\pi/2)$, we find that the saddle-node curve becomes two homotopically trivial closed curves, one surrounding the origin and one surrounding the point $(x_1, x_2) = (\pi, \pi)$. The perturbation arguments for the case $0 < c \ll d$ are similar. When $c = 0$, the 0 level curve of $\det(J) = 0$ is given by four circles $x_1 \pm x_2 = \pi$ and $x_1 - x_2 = \pm\pi/2$. There are six points of intersection of these circles at $(x_1, x_2) = (\pi/4, 3\pi/4), (3\pi/4, \pi/4), (-\pi/4, -3\pi/4), (-3\pi/4, -\pi/4), (\pi, 0)$ and $(0, \pi)$. The function $\det(J)$ is a Morse function at each of these points. As c is perturbed to be positive, the saddle-node curve becomes three homotopically trivial closed curves which contain the points $(11\pi/8, 3\pi/8), (3\pi/8, 11\pi/8)$ and $(0, 0)$ in their interiors.

Since the saddle-node curves lie at singularities of the map from the (x_1, x_2) torus to the (a, b) parameter plane, additional singularities appear when the curves are projected into the (a, b) parameter plane. The singularity in the (x_1, x_2) torus that occurs when $c/d = \pm 2$ is mapped to a curve that has the local structure of the level set of $u^2 = v^4$ in suitable coordinates. Furthermore, cusps and swallowtails appear along the saddle-node curves in the (a, b) plane. These do not seem to be simple to compute exactly, so we report observations from numerical calculations of the saddle-node curves. The outer component of the saddle-node curve in the (a, b) plane appears to be convex and nonsingular for all values of $c \neq 0$ and $d \neq 0$. For $|c/d| > 3$, the inner component of the saddle-node curve has four cusps. These are located on the a and b coordinate axes. When $|c/d| = 2\sqrt{2}$, there are a pair of swallowtail singularities that occur on the a axis. For $|c/d| < 2\sqrt{2}$, the inner component of the saddle node curve has eight cusps, of which two lie on the b axis and two lie on the a axis. As $|c/d|$ decreases through 2, the two cusps on the a axis merge with one another and branches of the two cusps reconnect smoothly with one another. For $0 < |c/d| < 2$, there are two inner components of the saddle-node curve that are reflections of one another in the a axis. Each of these inner components has three cusps. Additionally, the two components intersect. Denote $c_t = \sqrt{2}(\sqrt{5} - 2)\sqrt{\sqrt{5} + 1} \approx 0.60057$. There are six points of intersection between the inner two components if $c_t < |c/d| < 2$. If $0 < |c/d| < c_t$, there are only two points of intersection between these curves. This completes our description of the local bifurcations of the equilibria for this family of vector fields. Figure 2 is a collection of diagrams showing the saddle-node and resonant saddle curves in the (a, b) plane for different values of c/d.

3 Periodic Orbits and Global Bifurcations

We begin the discussion of periodic orbits of the family (1) by discussing what happens for large values of (a, b). The return maps of torus flows with global cross-sections (and hence no equilibrium points) have rotation numbers (Guckenheimer and Holmes 1983). The rotation number of a return map determines whether the flow has periodic orbits and characterizes the homotopy types of those orbits. More precisely, while traversing a large circle $a^2 + b^2 = r^2$ in the (a, b) plane, the flows are perturbations of rotations whose return maps assume all rotation numbers. Heuristically, we picture a family of "Arnold tongues," one for each rational number, that extend into the (a, b) plane from infinity. Formally, we define the (p, q) Arnold tongue to be the set of flows that have a periodic orbit of homotopy type (p, q). The tongues are disjoint from one another. The primary objective of this section is to describe the boundaries of the Arnold tongues. Our results are not rigorous, but rather an interpretation of numerical computations based upon the assumption that the two parameter family (1) with varying a and b has only generic bifurcations for most values of c and d. Here we focus upon values of $c/d > 3$ for which the saddle-node curve has only two components and four cusp points and report the results of these investigations. A more complete investigation is required to characterize other cases and the codimension three bifurcations that occur with smaller values of c/d.

We discuss briefly the case $d = 0$, viewing this as a singular limit as c/d increases. When $d = 0$, the system (1) separates into two uncoupled equations. The resonance region is the square $|a| + |b| = 1$. All points on the boundary of the resonance region except its corners are flows that have one saddle node equilibrium point. In the interior of the resonance region, all flows are "fully mode locked" with four equilibrium points and no periodic orbit. Outside the resonance region in the strip $|a + b| \leq 1$, the flow has periodic orbits of homotopy type $(0, \pm 1)$. Outside the resonance region in the strip $|a - b| \leq 1$, the flow has periodic orbits of homotopy type $(\pm 1, 0)$. When $|a + b| > 1$ and $|a - b| > 1$, the flow has a rotation number $\rho \neq 0$. The corners of the saddle-node curve are in the closure of all the Arnold tongues in the quadrant of $|a \pm b| > 1$ adjacent to the corner.

Now assume that $c/d > 3$. The local bifurcation diagram of the (a, b) plane is shown in the top row of Figure 2. The resonance region is bounded by a nonsingular curve of saddle node bifurcations S_o that lies close to the square $|a| + |b| = 1$. There is a second saddle-node curve S_i inside S_o with four cusps located on the coordinate axes. The curve R of resonant saddles lies between the two saddle-node curves near the cusp points, though much of R lies inside S_i if c/d is large enough. The Arnold tongues are perturbations of the Arnold tongues for $d = 0$. Apart from the tongues with homotopy types $(\pm 1, 0)$ and $(0, \pm 1)$, they all collect in the vicinity of the cusps. To make our discussion more concrete, we focus on the region near the cusp on the positive b axis in the (a, b) plane. The structure of the bifurcations near the other cusps is similar. The largest tongues near this cusp have homotopy types $(1, 0)$, $(0, -1)$ and $(1, -1)$. The tongues with homotopy types $(1, 0)$ and $(0, -1)$ are wide and separate the tongues with homotopy types $p > 0 > q$ from those that approach the other cusps. The $(1, -1)$ tongue is special because it is the only tongue that intersects the interior of S_i near the positive b axis.

The bifurcation diagram is symmetric with respect to the b axis, so we further restrict our attention to the left side of the b axis where the tongues have homotopy types with $p < -q$. Arnold tongues $A_{(p,q)}$ with homotopy type $(n, -(n+1))$ have a somewhat different structure from those with homotopy types (p, q) with $-q > p + 1$. Figure 3 portrays the structure of these tongues. Our first comments apply to all of the Arnold tongues $A_{(p,q)}$ without restrictions on (p, q). The boundary of $A_{(p,q)}$ is formed from curves of codimension one bifurcations with endpoints at codimension two bifurcations. There are two types of codimension one bifurcations that terminate families of periodic orbits: saddle-nodes of periodic orbits and homoclinic bifurcations. Four types of codimension two bifurcations appear on the boundary of these tongues: saddle-node loops, resonant homoclinic bifurcations, transversal intersections of saddle-node bifurcations of equilibria with saddle-node bifurcations of periodic orbits and saddle-node fan points. The intersection of each tongue $A_{(p,q)}$ with S_o is a connected arc $S_{o(p,q)}$ bounded by two codimension two points at which there are transverse intersections of S_o with curves of saddle-nodes of periodic orbits that bound $A_{(p,q)}$ in the exterior of S_o. In addition, there are a pair of saddle-node loop points s_h in $S_{o(p,q)}$ at which the saddle-node equilibrium has a homoclinic orbit that lies in the boundary of its two dimensional unstable manifold. Along S_o between the points s_h, the flows have a single periodic orbit which is stable. Emanating from the points s_h are curves of homoclinic bifurcations that extend into the interior of the resonance region. These homoclinic curves terminate on the saddle-node curve S_i at *saddle-node fan* points s_f (Baesens et al. 1991b, p.432). Each of these complex codimension two bifurcation points s_f lies at the end of homoclinic bifurcation curves of a countable number of Arnold tongues $A_{(p,q)}$. The saddle-node fans also lie at the endpoints of heteroclinic bifurcation curves that are described below. For tongues $A_{(p,q)}$ with $-q > p+1$, both homoclinic curves bounding the tongue terminate at the same saddle-node fan point. The homoclinic curves of the tongues $A_{(n,-(n+1))}$ end at adjacent saddle-node fan points. No periodic orbits of homotopy type (p,q), $|pq| > 1$, are found in the interior of S_i. The points where the homoclinic curves cross the resonant saddle curve R are resonant homoclinic bifurcation points. At each of these a bifurcation curve of saddle-nodes of periodic orbits meets the homoclinic curve and terminates. In the phase space, the periodic orbits emerging from the homoclinic orbits are unstable outside the resonant saddle curve R, while those emerging from the homoclinic curves inside R are stable. The Arnold tongues $A_{(p,q)}$ are separated by flows with quasiperiodic minimal sets. Outside the curve S_o, these flows have dense trajectories. In the annulus between S_o and S_i, the flows are Cherry flows (Baesens et al. 1991b) with minimal sets that are solenoids.

We next describe the additional bifurcations associated with the Arnold tongue $A_{(1,-1)}$ of homotopy type $(1,-1)$. This resonance region contains codimension one heteroclinic bifurcations and codimension two heteroclinic cycles that appear inside the region bounded by S_i. For parameters that lie on the b axis, the systems (1) have a reflection symmetry. Figure 4 shows a collection of four phase portraits for decreasing values of b along this axis. Outside S_o, there are a pair of periodic orbits, one of which is the invariant circle $x_1+x_2 = 2\pi$. The periodic orbits that emerge at S_o lie on this invariant circle and destroy the unstable periodic orbit. The intersection of S_i with the b axis is a cusp at which a pair of asymmetric saddle points p_1 and p_2 emerge from the invariant line, leaving behind a sink on the axis. At a parameter value $b = b_{\text{het}}$ below the cusp point, p_1 and p_2 have a heteroclinic cycle with vertices at p_1 and p_2. The periodic orbit that exists for $b > b_{\text{het}}$ disappears at $b = b_{\text{het}}$. For $b_{\text{het}} > b > 0$, there are no periodic orbits.

The parameter regions inside the cusp of S_i on the positive b axis contains sequences of heteroclinic bifurcations associated with $A_{(1,-1)}$. Refer to Figure 5 for a picture of the phase space for parameters in this region. We use the unit square as a fundamental domain for T^2. There are four equilibrium points, a sink, a source and two saddles. The two saddle points are labeled p_1 and p_2. Each saddle has one stable separatrix that goes to the sink without intersecting the boundary of the unit square and one unstable separatrix that comes from the source without intersecting the boundary of the unit square. The other separatrices of the saddles p_1 and p_2 do intersect the boundary of the unit square in points s_1, s_2, u_1, u_2. Points to the left of s_1 and above s_2 tend to the sink and those below u_1 and to the right of u_2 come from the source. We need to follow points to the right of s_1 and below s_2 to determine the remainder of the phase portrait of these fields.

Let $I_1 = (s_1, 1)$ be the interval on the top of the unit square to the right of s_1, and let $I_2 = (0, s_2)$ be the interval on the left edge of the square below s_2. We examine the return map of I_1 to the top of the unit square. Points that intersect the left side of the square above s_2 tend to the sink and do not return. Thus, if $u_1 > s_2$, there is no recurrence for the map and all points have α- and ω- limit sets that are equilibria. When $u_1 < s_2$ and $s_1 < u_2$, there is an interval $(s_1, x_2) = J \subset I_1$ that does return to the top of the square with x_2 denoting the first point on the backwards trajectory of s_2 to intersect the top of the square. Let θ be the first return map to the top of the square. The map θ is monotone increasing and has a continuous extension to the closed interval $[s_1, x_2]$. The derivative of θ appears to be smaller than one in our numerical studies, so we assume that $\theta' < 1$ in the following discussion. There are three cases to consider:

1. $\theta(J) \subset J$
2. $s_1 \in \theta(J)$
3. $\theta(J) \subset (0, s_1)$.

In the first of these cases, there is a fixed point that is the intersection of J with a periodic orbit of homotopy type $(1, -1)$. In the third case, all of the points of J tend to the sink after a single return to the top of the square. The second case is more complicated than the first and third. All points of J either tend to the sink or lie in the stable manifold of p_1, but they may make repeated returns to J. If $\theta^n(x_2) = s_1$, there is a heteroclinic orbit that intersects the top of the square in $n+1$ points. As we proceed along the interior of the inner saddle-node curve S_i towards the cusp with negative values of a increasing towards 0, we move from the third case described above to the second and then to the first. We encounter a sequence of heteroclinic bifurcations which accumulate at parameters for which $\theta(s_1) = s_1$. These parameter values are ones for which the point p_1 has a homoclinic orbit, and they bound the Arnold tongue $A_{(1,-1)}$. All of these curves of heteroclinic and homoclinic bifurcations meet at the codimension two point $(a, b) = (0, b_{\text{het}})$ of the heteroclinic cycle. The bifurcation curves to the right of the b axis are reflections of the ones we have described to the left of the b axis.

We want to connect our analysis of the global bifurcations inside the saddle-node curve S_i with that for the annulus bounded by the two saddle-node curves S_i and S_o. The end points of the heteroclinic bifurcations on S_i are the codimension two saddle-node fan points described above. At these parameters, the branch of the stable manifold of p_1 passing through s_1 coincides with the unstable manifold of the saddle-node point p_2. We can extend these connections to closed curves by adjoining either branch of the unstable manifold of p_1. The two closed curves formed by adjoining one of the branches of the unstable manifold of p_1 have homotopy types of the form $(n-1, -n)$ and $(n, -(n+1))$. This saddle-node fan point is the end point of homoclinic bifurcation curves bounding $A_{(p,q)}$ for $p/q \in (-1 + \frac{1}{n+1}, -1 + \frac{1}{n})$. Thus the codimension two points on S_i are the common endpoints of heteroclinic bifurcation curves inside S_i and a countable set of homoclinic bifurcation curves lying in the annulus between S_i and S_o. This completes our description of the bifurcation diagram inside S_i near the cusp along the positive b axis in the (a, b) plane. Figure 6 depicts the bifurcation diagram of this region of the (a, b) plane.

Our description of the bifurcations of two parameter families (1) with varying a and b and $|c/d|$ large contains only generic codimension one and two bifurcations. We conjecture that this description is not only consisent but also correct. In particular, we believe that it includes all of the bifurcations that occur in these families. The unfoldings of each of the codimension two bifurcations have been described previously in the literature. The only novelty is the appearance of the countable sequences of heteroclinic bifurcations that meet at the heteroclinic cycles. It is evident that there are significant changes in the global bifurcation diagrams as $|c/d|$ decreases, but we have not tried to classify these or to determine which codimension three global bifurcations are present within the family (1).

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Figure Captions:

Figure 1: This figure, reproduced from Baesens et al. (1991b), shows the geometry of a simple resonance region of a two parameter family of torus flows. In the interior of an annulus in the parameter plane, there are two equilibrium points of the flows. Phase portaits for parameters in different regions are superimposed upon the bifurcation diagram.

Figure2: Bifurcation diagrams of saddle-node and resonant saddle bifurcations of the families with varying a and b are displayed. The ratio c/d appears to the left of each diagram.

Figure 3: The global bifurcations bounding Arnold tongues are shown. The Arnold tongues of homotopy type $(n, -(n+1))$ extend to a segment of the curve S_i. Curves of homoclinic bifurcations are labelled by H, resonant homoclinic cycle points are denoted RH, points of saddle-node loop bifurcations by s_h, and saddle-node fan points by s_f. Each saddle-node fan point lies at the end of an infinite number of Arnold tongues.

Figure 4: Phase portraits for symmetric vector fields with $c = 1$, $d = 0.1$ and $a = 0$ are shown for the values of b displayed below the phase portraits. The parameter $b = 1.3$ lies outside the resonance region and there are two periodic orbits. The parameter $b = 1.01$ lies between S_o and S_i. There is a source and a saddle on the invariant line and a stable periodic orbit. When $b = 0.997$, there are four equilibirum points and a stable periodic orbit. As b decreases, the periodic orbit disappears when the two saddle points have a heteroclinic cycle. The value $b = 0.9$ is below this bifurcation, and all trajectories tend to equilibrium points.

Figure 5: The phase portrait of the torus for parameter values inside the curve S_i. There are four equiibirium points, two of which are saddles labelled p_1 and p_2. The unstable manifold of p_1 has a branch that intersects the right hand side of the square at u_1 and a branch of its stable manifold that intersects the top of the square at s_1. The saddle p_2 has a branch of its unstable manifold that intersects the bottom of the square at u_2 and a branch of its stable manifold that intersects the left side of the square at s_2. If s_2 lies above u_1, we continue the stable manifold of p_2 until it intersects the top of the square at x_2. The segment (s_1, x_2) returns to the top of square.

Figure 6: An overview of global bifurcations in the vicinity of the positive b axis. The three Arnold tongues with homotopy types $(0, -1)$, $(1, -1)$ and $(1, 0)$ are large, with the tongue $A_{(1,-1)}$ extending into the interior of S_i. The codimension two bifurcation points along S_o are transversal intersections of saddle-nodes of equilibirums and limit cycles and saddle-node loop points. The codimension two bifurcation points on the curve of resonant saddles denoted R are resonant homoclinic bifurcations. The bifurcations on S_i are saddle-node fan points at which infinite collections of Arnold tongues terminate. Inside S_i are curves of heteroclinic bifurcations that all meet at the heteroclinic cycle point on the b axis.