Towards a Global Theory of Singularly Perturbed Dynamical Systems

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Dynamical systems with multiple time scales arise naturally in many domains. Models of neural systems provide the principal motivation for this paper. Most of the previous mathematical analysis of qualitative properties of multiple time scale systems has dealt with local phenomena that occur in low dimensions. The neural system models raise questions that lie beyond the scope of existing work. Our aim here is to outline a theory that extends the local theory to a description of qualitative features of the global dynamics for systems with two time scales. We fill in portions of this outline within the context of systems that have two slow variables and two fast variables. Even within this low dimensional setting, most basic questions about global properties remain unanswered.

The setting within which we work is a system of differential equations in R^{m+n} of the form

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= \epsilon g(x,y) \end{aligned}$$

or

$$\begin{aligned} \epsilon x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}$$

Such systems are called *singularly perturbed* differential equations or *fast-slow* vector fields. For brevity, we say that these equations define an (m, n) dimensional SP system. Here $\epsilon \geq 0$ is a small parameter, the x variables are called *fast variables*, the y variables are called *slow variables* and the two systems have time variables t and T with $dT/dt = \epsilon$. The system for $\epsilon = 0$ will be called the *singular* system or SP_0field of the corresponding SP system. We have written the equations of an SP system in two different forms to emphasize that it will be studied from two points of view. When we examine the fast time scale t, the system appears as a perturbation of a family of vector fields parametrized by y. The perturbation parameter induces a slow variation of the parameters, producing a *slowly varying system*. Bifurcations in the family of vector fields induce transitions of trajectories from the neighborhood of one family of attractors to another on the fast time scale. The asymptotic properties of the simplest types of such transitions have been studied extensively. The second system has as its limit a *differential algebraic equation* in which the differential equations for the evolution of x become algebraic equations. This limit makes good sense as a singularly perturbed system for describing trajectories that track regular portions of the 0 level set of g. In these regions, one can solve the equation $g = 0$ for x using the implicit function theorem and obtain a system of equations that describes the evolution of y. At places on the 0 level set of g where g_y is singular, the differential algebraic system may not have solutions at all. To make sense of the solutions, it is necessary to follow the evolution of the fast vector field from these points. The fast evolution of y can be viewed as "jumps" from one slow manifold to another. Piecing together trajectories from the continuous evolution along the 0 level set of g with these jumps produces a *hybrid system* of the sort studied by Back, Guckenheimer and Myers [2].

Generic properties of singularly perturbed systems have been studied previously, but much of this work has been devoted to local properties, the study of particular classes of solutions and phenomena that involve only one or two transitions between attractors of the fast subsystems. Arnold et al. [1], Grasman [7] and Mishchenko [12] give surveys of much of this theory. We note that the analysis of the forced van der Pol equation by Cartwright and Littlewood [3] can be viewed as a seminal example of a long time analysis of a system with three time scales. Takens [15, 16, 17, 18] was perhaps the first to consider the development of a systematic theory, mainly in the context of systems for which the fast subsystems had gradient structures. Rinzel [14] has used systems with two time scales as models for bursting oscillations of electrical activity in biological systems. His work does not attempt to build a comprehensive theory for singularly perturbed systems. Here, we take initial steps to construct a systematic global theory of singularly perturbed systems, including those that display oscillations on fast time scales. We sketch topological aspects of the theory and touch briefly on the asymptotic analysis of various phenomena. To make our task more manageable, we focus our attention to phenomena that occur in SP fields with $m = 2$ and $n = 2$. This allows us to draw upon the relatively complete theory of codimension two bifurcations of two dimensional vector fields and to avoid considerations of chaotic dynamics occurring within fast subsystems.

1 Preliminaries

Our goal is to characterize the qualitative properties that occur in "generic" SP systems. Our interpretation of genericity will be partly dependent upon context. Let X_ϵ be a system in R^{m+n} of the form

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= \epsilon g(x,y) \end{aligned} \tag{1}$$

with two time scales. We regard X_ϵ as a smooth (C^∞) map $X_\epsilon : R^{m+n} \to R^m \times R^n$. The range of X is written as a product to reflect the roles of the two factors in the product. Since the singular limit X_0 is a family of vector fields in R^m parametrized by R^n, the most stringent definition of equivalence of two systems will be that there is a smooth coordinate change of R^{m+n} that commutes with the projection $R^{m+n} \to R^n$ and maps one vector field to another. These coordinate changes have the form $(h(x,y), k(y))$. The local classification of normal forms given by Arnold et al. [1] is based upon this definition of equivalence. The limit $\epsilon = 0$ is indeed singular. The solutions to the equation defining an SP_0field do not approximate the solutions of an SP system for $\epsilon > 0$ since the slow variables of an SP_0 field remain constant. Therefore, we make the following definition.

Definition: Let X_ϵ be the SP vector field (1). A curve $\gamma_0 \subset R^m \times R^n$ is a trajectory of the SP_0 field X_0 if it is the limit of a one parameter family of trajectories γ_ϵ for $\epsilon > 0$. The limit is assumed to be the limit of the curves as subsets of $R^m \times R^n$ with respect to the Hausdorff metric on subsets.

The dynamics of a vector field can be decomposed into a phase portrait consisting of chain recurrent invariant sets and sets of trajectories that connect pairs of chain recurrent sets. Trajectories of the SP_0 field can be decomposed into segments during which the trajectories remain in an invariant set of the fast subsystem and segments in which the trajectories makes transitions between invariant sets. We call these segments (slow) S segments and (fast) F segments respectively. The time scales associated with S segments and F segments of the SP_0 field trajectories have an infinite separation. Trajectories that lie on equilibrium manifolds of the fast subsystem can be

Figure 1: A trajectory of a (1,2) dimensional SP system approaches a manifold of stable equilibria, leaves this manifold in an SF transition at a fold of the manifold and then approaches a second equilibrium manifold in an FS transition. The trajectory is plotted in a heavier line style than the equilibrium manifolds.

regarded as evolving on the slow time scale. If M is an equilibrium manifold on which $D_y g$ is nonsingular, then the slow vector field that describes this evolution is called the *reduced* field. The reduced field at $x \in M$ is the unique vector v tangent to M at x with the property that v projects onto $f(x)$ by the projection of $R^m \times R^n$ onto its first factor R^m. We distinguish FS transitions where an S segment precedes a F segment from SF transitions in which an S segment follows a F segment. The local theory is primarily an asymptotic analysis of the generic SF transitions that one expects to find from equilibrium manifolds of the SP_0 field to F segments.

The *long time scale* will refer to times that are long in the slow time scale t. On the long time scale, we expect that there will be trajectories that pass through many transitions. A global theory of SP systems will segment trajectories so that normal forms can be fit to each segment uniformly as $\epsilon \to 0$. Through the analysis of the normal forms, this leads to a matched asymptotic expansion for the trajectories. For the long time scale, we want to perform an *epoch analysis* in which the slow time evolution of trajectories is collapsed to a sequence of discrete maps that send points at the onset of an FS transition to the onset of the next FS transition. We call these maps induced maps. They need not be invertible like the cross-section maps of a flow, and they may be multivalued.

2 Transitions based on bifurcations of two parameter families of vector fields

Bifurcation theory of dynamical systems provides a guide to the types of FS transitions that we expect to find in SP_0 fields. Classification of FS transitions in (m, n) dimensional SP systems can therefore be based upon the classification of bifurcations of n parameter families of m dimensional vector fields. In this section, we review the classification of codimension one and two bifurcations of two parameter families of two dimensional vector fields and discuss their relationship with FS transitions of $(2, 2)$ dimensional SP systems.

Bifurcations are classified by codimension [8], and the codimension reflects the dimension of the set of trajectories undergoing an FS transition. Codimension one bifurcations of attractors give rise to FS transitions that are found in open subsets of trajectories of an SP_0 field, while higher codimension bifurcations give rise to FS transitions that are found only on nowhere dense sets of trajectories of SP_0 fields. In the setting of $(2, 2)$ dimensional SP systems, this means that codimension two bifurcations occur only at isolated trajectories of an SP_0 field. Moreover, the invariant sets of the fast subsystem are all equilibrium points, periodic orbits or polycycles composed of closed curves containing equilibrium points.

The codimension one bifurcations of two dimensional vector fields are

1. Saddle-nodes equilibria or folds: These bifurcations give rise to trajectories that make jump transitions from an equilibrium manifold of an SP_0 field. The asymptotic properties of SP trajectories near folds have been described in substantial detail [1]. The vector field

$$\begin{aligned} \dot{x} &= \epsilon \\ \dot{y} &= -y^2 - x \end{aligned}$$

is a normal form for a$(1,1)$ dimensional SP system with a fold. The matched asymptotic expansions for trajectories of this system are based upon a partition of the plane into three regions: an ϵ-dependent neighborhood U_ϵ of the origin, a neighborhood of the regular portion of the curve defined by $y^2 + x = 0$, and regions of the plane where $y >> \epsilon$. Scaling the system with the coordinate transformation $y = \epsilon^{1/3}Y$, $x = \epsilon^{2/3}X$ and $t = \epsilon^{-1/3}T$ produces the system

$$\begin{aligned} X' &= 1 \\ Y' &= -Y^2 - X \end{aligned}$$

which is independent of ϵ. Therefore, trajectories of this system can be used as a scaled model for the behavior of the vector field in U_ϵ. Explicit solutions of the equation $Y' = -Y^2 - T$ can be expressed in terms of Airy functions [7].

2. Hopf bifurcations: Hopf bifurcations are characterized by an equilibrium manifold of an SP_0 field along which complex eigenvalues of the fast field at the equilibrium cross the imaginary axis transversally. In an SP system, trajectories have difficulty tracking an emerging branch of periodic orbits at Hopf bifurcations. Consider the case of a supercritical Hopf bifurcations in which a family of stable limit cycles emerges from stable equilibria. The trajectory of the SP system comes so close to the equilibria prior to the the Hopf bifurcation that it remains close to the branch of post-critical unstable equilibria for a substantial period of time. The asymptotics of this "delayed loss of stability" have been analyzed by Nejshtadt [13]. The smoothness or analyticity of the vector field significantly affects the asymptotics of delayed loss of stability in generic SP systems. With analytic systems, Nejshtadt [13] calculates of the length of delay following bifurcation in terms of a complex-time extension of the system.

3. Saddle-connections: Homoclinic and heteroclinic bifurcations are global and have not been studied extensively within the context of SP systems. Homoclinic bifurcations appear in generic families as one mechanism for the termination of a family of stable limit cycles. Rinzel [14] observed the corresponding phenomenon in SP systems at the termination of spiking in models for bursting oscillations of neural oscillators. The

Figure 2: A trajectory of a (2,1) dimensional SP system passes through a Hopf bifurcation and then exhibits a delay in tracking the manifold of periodic orbits emerging from the Hopf bifurcation.

asymptotics of this transition have not been analyzed beyond the dynamics of the SP_0 field itself close to its homoclinic bifurcation. Here are a few preliminary comments about the analysis of the periods of oscillation in a trajectory passing through a homoclinic bifurcation. In a generic reduced system with a homoclinic orbit, there is a manifold E of equilibria for the fast system that are hyperbolic saddles. These saddles have stable and unstable manifolds that intersect transversally at a set of slow variables that form a codimension one submanifold H of E. Fenichel [5] proves that the hyperbolic structure of the manifold E persists for $\epsilon > 0$. The manifold E is the limit of a family of normally hyperbolic manifolds E_ϵ as $\epsilon > 0$. Its stable and unstable manifolds continue to intersect transversally. In a neighborhood of E, Fenichel defines coordinates for which the families of strong stable and unstable manifolds are invariant.

Periodic orbits of a reduced vector field close to a homoclinic orbit can be decomposed into a pair of segments, one lying in a neighborhood U of the manifold E and the second lying in the complement of U. The portion of trajectories lying outside U are regular orbits that vary smoothly with x and have bounded duration as $\epsilon \to 0$. The dominant portion of the increase in period of the reduced system comes from the

Figure 3: A trajectory of a (2,1) dimensional SP system passes a homoclinic orbit. A curves of sources is depicted by a dotted line and a curve of saddles is depicted by a a dashed line. The surface of periodic orbits and the stable and unstable manifolds of the saddles are shown. The homoclinic orbit is contained in the closure of all three of these surfaces.

passage of trajectories through U. In the reduced system, the length of time that a trajectory spends in U is determined by the smallest magnitude of its coordinates along the unstable manifold of the saddle point. Assume that there are coordinates for the reduced system in which the fast equations are defined by $\dot{x_i} = \lambda_i(y)x_i$. Assume that $\lambda_i(y) > 0$ for $i = 1, \ldots, u$ and that $\lambda_i(x) < 0$ for $i = u + 1, \ldots, n$ (For generic systems, we can expect that coordinate systems of this form can at best be determined with finite degrees of smoothness [10].) In each unstable direction, the x_i increase exponentially in magnitude, implying that the time a trajectory spends in U is comparable to $\max(-\ln(x_i)/\lambda_i$, $i = 1, \ldots, u$. If the slow variable evolves at a constant rate, then each cycle of the oscillation of the trajectory will carry it closer to the stable manifold of E. The cycles of the oscillation get longer, resulting in a larger change in the slow variable from one cycle to the next. Thus the growth of the period of the oscillations is faster than they would be if there were a constant change in parameter values between cycles. We conjecture that the period between cycles should be asymptotic to a function of the form $a_1 \ln(1/\epsilon) + a_2 \ln(\ln(1/\epsilon)) + a_3$.

Heteroclinic bifurcations of fast systems can be expected to lead to discontinuities in the fast transition maps that occur along fold singularities, though we are unaware of any examples of this phenomenon in the literature.

4. Saddle-nodes of periodic orbits: Manifolds of periodic orbits may have folds in generic one parameter families of vector fields. These folds are singularities for the projection map of the manifold onto the parameter space of the family. The return map for the family of periodic orbits has a discrete time saddle-node bifurcation. The asymptotic analysis of the return map is analogous to that for saddle-nodes of equilibrium points. The behavior of the "phase" variable that is ignored by the return map has not been studied. Note that the isochrons (strong stable manifolds) of a limit cycle for a planar system make an angle with the cycles that is proportional to the characteristic exponent of the cycle. Semistable limit cycles typically do not have isochrons.

The consequences of codimension two bifurcations in the fast subsystems of a $(2, 2)$ dimensional SP field have hardly been studied. The only case that gives rise to a local bifurcation is the cusp [1]. The list of codimension two bifurcations is manageable for a case by case asymptotic analysis, so we present it here as a challenge to readers.

1. Cusps: Projection of the equilibrium manifold M of a generic SP_0 field onto a space of slow variables will have singularities that are folds (codimension 1) and cusps (codimension 2). A normal form for a mapping with a cusp is $f(x_1, x_2) = (x_1^3 + x_2x_1, x_2)$. An SP_0 vector field with a cusp is

$$\begin{aligned} \dot{x} &= -x^3 + y_1x + y_2 \\ \dot{y_1} &= \epsilon c_1 \\ \dot{y_2} &= \epsilon c_2 \end{aligned}$$

As with the fold bifurcation, there is a region near the origin in which the time scales for the dynamics of x differ from those when the slow variables y_i are far from the origin. A scaling argument similar to the one used for folds indicates the size of this region. We solve the equations for $\dot{y_i}$ and substitute these solutions into the equation for $\dot{x}$. We then look for a region of the phase space in which all of the terms have comparable magnitude. Such a region exists for $x \sim O(\epsilon^{1/5})$, $y_1 \sim O(\epsilon^{2/5})$, $y_2 \sim O(\epsilon^{3/5})$, and $t \sim O(\epsilon^{-2/5})$. Trajectories that track the equilibrium manifold in this region will relax towards the stable equilibrium surface on the slower time scale $t \sim O(\epsilon^{-2/5})$ than the unit time scale for the typical evolution of x.

2. Takens-Bogdanov bifurcations: Consider the equilibrium manifold M of a generic SP_0 field with more than $n > 1$ fast variables. The linearization of the fast subsystem along M defines a smooth map of M into the space of $m \times m$ matrices. There are no constraints on this map from the equilibrium conditions, so there will be a codimension two submanifold of M at which the linearization of the fast subsystem has zero as an eigenvalue of algebraic multiplicity two. The generic points of this submanifold are called Takens-Bogdanov points. They lie on the codimension one fold surface of M at places where the number of stable and unstable eigenvalues of the linearization of the fast subsystem changes along the fold surface. The unfolding of the Takens-Bogdanov bifurcation as a codimension two bifurcation of planar vector fields is well known. There are small amplitude periodic orbits and homoclinic bifurcations associated with the Takens-Bogdanov bifurcation.

3. Degenerate Hopf bifurcations: The normal form for Hopf bifurcations contains a "nonlinear" term that determines whether the equilibrium point is weakly stable or unstable in the marginal directions. Degenerate Hopf bifurcation occurs when this quantity vanishes. A point of degenerate Hopf bifurcation in a generic two parameter family of vector fields signals the change from subcritical to supercritical Hopf bifurcation for families that cross the bifurcation curve transversally.

4. Resonant homoclinic bifurcations: In generic one parameter families of vector fields, the stability of the periodic orbits near a homoclinic orbit are determined by the properties of the saddle point that is the limit of the homoclinic orbit. In two dimensional vector fields, the trace of the linearization of the saddle point determines whether the periodic orbits are attracting or repelling. Resonant homoclinic bifurcation occurs when the trace vanishes. This occurs in generic multiparameter families of vector fields. Associated with the resonant homoclinic bifurcations in generic two parameter families are paths of saddle-node bifurcations of limit cycles that end at the codimension two point, meeting the curve of homoclinic bifurcations with a tangency that is flat.

5. Saddle-node loop: Consider the phase portrait of a two dimensional vector field near a point with a saddle-node equilibrium at which the linearization of the field has a negative eigenvalue along with its zero eigenvalue. The stable set of the equilibrium is diffeomorphic to a half plane, and the unstable set of the equilibrium is a single trajectory. If the unstable set is contained in the interior of the stable set, then the saddle-node bifurcation has been called a homoclinic saddle-node, "saddle-node in cycle" or SNIC. This is a codimension one bifurcation that persists with perturbation of the saddle-node. If the unstable set is contained in the boundary of the stable set, this is a codimension two bifurcation that has also been called a homoclinic saddle-node or saddle-node loop. Passing through the codimension two point in the parameter space of a generic two parameter family is a curve of saddle-node bifurcation, and there is a curve of homoclinic bifurcations that terminates at the saddle-node point. Algorithms for computing these codimension two bifurcations have been discussed by Schechter.

6. Triple cycles: Just as the equilibrium manifolds of SP_0 fields have projections with cusps as codimension two singularities, families of limit cycles for SP_0 fields have projections with codimension two singularities corresponding to cusps. The cusps appear by restricting the projection to the family of fixed points in a cross-section to the SP field.

7. Double saddle connections: There are several cases of transversal intersections of curves of independent codimension one bifurcations that occur in two parameter families of vector fields. The case of two intersecting curves of homoclinic bifurcations for the same saddle point leads to a more complicated situation with additional subsidiary bifurcation curves appearing. In two dimensional vector fields, this bifurcation has subsidiary homoclinic bifurcations with both "convex" and "non-convex" homoclinic orbits. In the parameter space, there are two curves of homoclinic bifurcation passing through the codimension two point for homoclinic orbits that enclose a convex angle at the saddle point. There are also two branches of homoclinic bifurcation for non-convex homoclinic orbits that terminate at the codimension two point.

3 The Long Time Scale: Induced Maps

We turn to the analysis of the long time scale dynamics of an SP system. In particular, we would like to describe the long time invariant sets of an SP system. This task is complex because the $\epsilon = 0$ limit of an SP system is indeed singular. For example, in a $(2, 2)$ dimensional system undergoing supercritical Hopf bifurcation, the transition from the equilibrium manifold of the SP_0 field to its manifold of limit cycles will have trajectories that connect every unstable point along the equilibrium manifold with every point on the limit cycle in the same fast subsystem. (This observation is a corollary of the theory of Nejshtadt [13] that analyzes the delay of SP trajectories in tracking the family the of limit cycles following the Hopf bifurcation.) The pathology represented by this example is not pursued further here. Instead, we restrict our attention to the simpler setting for $(2, 2)$ dimensional systems with only equilibrium attractors.

Manifolds of stable equilibria for $(2, 2)$ dimensional systems are two dimensional and have one dimensional boundaries at which trajectories undergo FS transitions. If γ is a curve that is in the boundary of a manifold of stable equilibria, then we follow the evolution of γ through the FS transition until γ meets the next manifold M of stable equilibria and begins flowing once more. Singular phenomena may be encountered along the F segments emanating from γ or in the relationship of γ to the slow flow along M. The induced map of the system will be a one dimensional map from γ to the boundary of M.

The Poincaré-Bendixson Theorem implies that the limit set of a bounded trajectory of a planar vector field is a periodic orbit or contains an equilibrium point. This implies that the trajectories forming the boundary of the basin of attraction of a sink for a structurally stable vector field are either in the stable manifold of saddles, tend to infinity or emerge from sources that are equilibrium points or unstable limit cycles. This observation applies to systems with saddle-nodes if the saddle-node points and their strong stable manifolds are included in the list of trajectories that form boundaries of basins of attraction. If we follow the unstable separatrices of saddle-node points in a generic (2,2) dimensional SP system, discontinuities for the induced map will be created at separatrices that lie in the stable manifold of a saddle. Transversal crossing of two saddle-node bifurcation curves also leads to discontinuity of induced maps. When equilibrium points of a system disappear in a saddle-node of equilibria, there is a discontinuous change in a basin of attraction. This change creates the possibility for new global bifurcations involving saddle connections of the fast subsystems to appear. The phenomenon may be related to a codimension two bifurcation that can be detected by bifurcation analysis. For example, if the unstable separatrix of a saddle-node p connects to a saddle point q, there may be another saddle separatrix approaching p that connects to q when it "breaks through" the saddle-node region.

Although we don't discuss systems with limit cycle attractors in detail here, we remark that new saddle connections may be created by the destruction of the cycles without the involvement of a codimension two bifurcation. If γ is a periodic orbit undergoing a saddle-node of limit cycle bifurcation, and if there are saddle separatrices forward and backwards asymptotic to γ, then there can be infinite number of heteroclinic bifurcations that accumulate at the saddle-node bifurcation point.

Most of the attention that has been directed at analysis of singularly perturbed systems examines the geometry of how trajectories leave equilibrium manifolds of an SP_0 field. There has been little analysis of how these sets of trajectories approach a new equilibrium manifold. We regard a FS transition as having as limit a map $h : \partial M \to N$, with ∂M the boundary of a manifold of attractors M, and N another manifold of attractors for the SP_0 field. The geometry of $h(\partial M)$ relative to the slow flow on N is important in determining the singularities of the induced map from ∂M to ∂N. Since the slow flows are independent of the fast subsystems, there are no implicit constraints on the geometry of ∂M relative to the flow on N. The slow time scale dynamics of trajectories that make an infinite number of transitions can be analyzed in terms of the induced maps.

We describe three phenomena that occur in generic $(2, 2)$ dimensional SP systems following an FS transition between manifolds of equilibria. In this setting, ∂M is a curve, and the points of this curve arrive on the two dimensional manifold N in an arbitrary position relative to the phase portrait of the slow flow on N. There may be points where

1. $h(\partial M)$ is tangent to the slow flow on N. This leads to turning points of induced maps.

2. $h(\partial M)$ may intersect a limit cycle of the reduced flow on N transversally. This leads to a countable sequence of discontinuities of the in-

Figure 4: The trajectories of a fold curve on one manifold of equilibria flow to another manifold of equilibria. The image of the fold curve after the FS transition is tangent to the reduced vector field on the lower manifold of equilibria at an isolated point.

duced map if the limit cycle is unstable and points near it do not remain on N forever under its slow flow.

3. $h(\partial M)$ may intersect a saddle separatrix for the slow flow. This produces discontinuities and singularities in induced maps. The singularities are described by power laws determined by the eigenvalues of the saddle.

Within the context of "hybrid systems", each of these phenomena has been discussed by Guckenheimer and Johnson [9].

Combining all of the induced maps for equilibrium manifolds of a generic $(2, 2)$ dimensional SP system yields a one dimensional mapping φ. The domain of φ is a one dimensional manifold that may well have several components. Furthermore φ can have discontinuities, singularities and critical points associated to the phenomena described above. The extensive theory of one dimensional iterations [4] can be used to characterize the types of attractors that may occur for the induced map. The examples of attractors for two dimensional hybrid systems described by Guckenheimer and Johnson [9] are readily translated into examples of SP systems whose long time dynamics display varied types of attractors.

4 Concluding Remarks

Our outline of a global theory of singularly perturbed systems is incomplete and sketchy. Much more needs to be done to describe the important geometric structures that occur in generic SP systems (and families of SP systems) and to elucidate their properties with regard to transversality. Jones and Kopell's analysis of periodic orbits in generalizations of the Hodgkin-Huxley model [11] for the action potential is a good example of what needs to be done. The importance of the applications that give rise to SP systems justifies the effort required to flesh out this outline, even though the theory will necessarily be complicated and lack the coherent elegance of singularity theory [6]. The use of singularly perturbed systems to model neural systems is compelling. Neural systems operate with a hierarchy of time scales and SP systems provide a natural way to represent this hierarchy.

Dynamical systems analysis of applications usually requires simulation that relies upon numerical integration. For SP systems, numerical integration presents the problem that "standard" numerical integration techniques require time steps that are small with respect to the fast time scale in the system. Numerical integration techniques for stiff systems and differential algebraic equations address the problem of computing slow-time trajectories of a reduced system, but the problems associated with integration along non-equilibrium attractors of an SP system and with SF transitions have not been studied numerically. We contend that effective solutions to these problems can be constructed through the further development of the global theory outlined in this paper, together with algorithms that exploit the geometric structures found in generic SP systems.

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