Isoperimetry, volume growth and random walks.

A survey on the relationships between volume growth, isoperimetry, and the behavior of simple random walk on Cayley graphs, with examples

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Chapter 1

Introduction

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1.1 Before we start

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What is the largest area in the plane bounded by a given perimeter? This question was already considered thousand years ago. The isoperimetric inequality

$$4\pi A \leq L^2$$

where L is the perimeter and A the enclosed area shows that disks have the largest possible area given their perimeter (they are, in a sense, the only extremal sets). In dimension $n \geq 2$, any bounded set Ω with smooth boundary $\partial\Omega$ satisfies

$$n\omega_n^{1/n} V(\Omega)^{1-1/n} \leq A(\partial\Omega)$$

where ω_n is the volume of the unit ball in Euclidean n-space, A denotes the $(n-1)$-measure on the hypersurface $\partial\Omega$ and V is the Lebesgue measure in Euclidean n-space. Observe the close relationship between the growth

$$r \mapsto \omega_n r^n$$

of the volume of Euclidean n-balls and isoperimetry in Euclidean space. This relationship is one of the main theme to be developped in these notes in a more general context.

Recall that heat diffusion is modelled mathematically by the heat equation

$$\partial_t u - \sum_1^n \partial_i^2 u = 0$$

which is the prototype of a parabolic PDE. The fundamental solution of this equation (i.e., the solution $(t, y) \mapsto u_x(t, y)$ satisfying $u_x = \delta_x$ at time $t = 0$), also called the heat kernel, is provided by the Gauss kernel

$$(t, y) \mapsto \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{\|x-y\|^2}{4t}\right), \quad t > 0, \ y \in \mathbb{R}^n.$$

These objects (i.e., the heat equation and Gauss kernel) also describe the motion of a Brownian particle, i.e., the stochastic process called Brownian motion. At a naïve level, we can observe that the level sets of the Gauss kernel are Euclidean spheres, establishing a formal link between heat diffusion, isoperimetry and volume growth. Moreover, one easily comes up with the crude idea that, the more space there is, the more rapidly heat will diffuse. Indeed, the Gauss kernel indicates that if all the heat is concentrated at x at time 0 then, at time t, a good portion of the heat will be distributed somewhat uniformly in the Euclidean ball or radius $\sqrt{t}$ around x.

The notions and objects discussed above, volume growth, isoperimetry, and heat diffusion, generalize naturally to the setting of Riemannian manifolds. Indeed, the Riemannian structure provides ways to define balls, to measure volumes and to model heat diffusion through the the heat equation $(\partial_t + \Delta)u = 0$ where $\Delta = -\text{div grad}$ is the Riemannian Laplace operator.

As any n-dimensional Riemannian structure copies locally an n-dimensional Euclidean space, at small scale, many of the properties of Euclidean spaces are conserved. For instance, the volume of balls around a fixed point x grows like r^n if the radius r is small enough. For sets Ω contained in a small enough compact neighborhood K of x, we have $V(\Omega)^{1-1/n} \leq C_K A(\partial\Omega)$. The heat kernel $h(t, x, y)$ satisfies $h(t, x, x) \approx c_{n,x} t^{-n/2}$ for small enough time t and resembles locally the Gauss Euclidean kernel.

What happens at large scale is a very different matter for Euclidean geometry plays no essential role there. For instance, in the hyperbolic plane, balls have exponential volume growth as the radius goes to infinity, the linear isoperimetric inequality

$$V(\Omega) \leq A(\partial\Omega)$$

is satisfied and the heat kernel is given by

$$h(t, x, y) = \frac{\sqrt{2}}{(4\pi t)^{3/2}} e^{-t/4} \int_d^\infty \frac{s e^{-s^2/4t}}{(\cosh s - \cosh d)^{1/2}} ds$$

where $d = d(x, y)$ is the hyperbolic distance between x and y. The large time and large distances behavior of this kernel is very different from the behavior of the Euclidean Gauss kernel.

It turns out that, at large scale and for general Riemannian manifolds, only certain relations between volume growth, isoperimetry and heat diffusion behavior remain whereas many are lost. In particular, fast volume growth says little about the isoperimetric problem or the large time decay of the heat kernel $h(t, x, x)$ in general. One drastic way to forget about the irrelevant smooth n-dimensional Euclidean structure and focuss on the large scale geometry of the Riemannian manifold M under consideration is to approximate M by a discrete combinatorial structure, a graph, which one can think of as a skeleton of M. One is led to consider volume growth and isoperimetry on graphs, the meaning of which is rather clear. The role of heat diffusion is then taken by the notion of random walk. In particular, the heat kernel $h(t, x, y)$ is replaced by the probability that a random walk started at x reaches y in (discrete) time t.

This is no surprise if one remembers that the heat equation is also the equation driving Brownian motion and that random walk is the discrete analog of Brownian motion. Just as for Riemannian manifolds, on a general (symmetric, connected, locally finite) graph a fast volume growth does not imply much in terms of isoperimetry or random walk.

So the question naturally arises: does there exists a natural class of graphs (manifolds) for which volume growth has interesting implications concerning isoperimetry and/or random walk (heat kernel) behavior? One answer is that interesting results can be obtained if one assumes enough "homogeneity" of the underlying structure. In particular, in the context of Cayley graphs of finitely generated groups there is indeed strong connections between volume growth, isoperimetry and random walk behavior. Going back to Riemannian manifolds this means in particular that covers of compact manifolds enjoy similar relationships between volume growth, isoperimetry and heat kernel behavior.

This informal discussion has now taken us to the main subject treated in this book, that is, the relationships between volume growth, isoperimetry, and random walks on finitely generated groups. The next section gives precise definitions and introduces notation in that context. In order to obtain a more complete picture of the subject and to be able to stress the differences between what happens on general graphs versus what happens on Cayley graphs, we will also consider volume growth, isoperimetry and random walks on graphs. Our aim however is to concentrate on the rich and important case of Cayley graphs, providing a complete treatment of most of the results that will be mentioned as well as many illustrating examples.

1.2 Growth, isoperimetry and random walks on finitely generated groups

sec-notfgg

Let G be a finitely generated group. For any finite generating set S satisfying $S = S^{-1}$, we consider the Cayley graph (G, S) with vertex set G and an edge from x to y if and only if $y = xs$ for some $s \in S$. Thus, our edges are oriented but this is merely a convention since (x, y) is an edge if and only if (y, x) is an edge. We allow the identity element id to be in S in which case our graph has a loop at each vertex. Since edges are defined using the *set* S, there are no multiple edges. Clearly the graph (G, S) is invariant under the left action of G.

This work focuses on three functions associated with (G, S), namely,

1. the volume growth function $V_S : n \to V_S(n)$,
2. the isoperimetric profile $I_S : n \to I_S(n)$,
3. the probability of return of the simple random walk at time $2n$, $\phi_S : n \to \phi_S(n)$.

Precise definitions are given below. We will survey the relationships between the growth rate of V_S, the growth rate of I_S, and the decay rate of ϕ_S.

We will mainly be interested in these functions up to a natural equivalence relation $u \simeq v$ between positive functions defined on the positive real axis. When a function is defined only on the integers, we extend it to the positive real axis by linear interpolation. We will use the same name for the original function and its extention. In particular, we will consider V_S, I_S and ϕ_S as defined on the positive real axis.

If one of the functions u, v is monotone then $u \simeq v$ means that there exists $a, b, C > 0$ such that

$$\forall\, t, \quad u(t) \le C\, v(at) \quad \text{and} \quad v(t) \le C\, u(bt).$$

The precise meaning of $u \simeq v$ for general positive functions u, v is slightly complicated: Given two positive functions u, v defined on the positive real axis, write $u \preceq v$ if there exist $C \ge 0$ and $b > a > 0$ such that

$$\forall\, t > 0, \quad \begin{cases} u(t) \le C \sup\limits_{at \le s \le bt} v(s) \\ \inf\limits_{at \le s \le bt} u(s) \le Cv(t) \end{cases}.$$

Write $u \simeq v$ if $u \preceq v$ and $v \preceq u$. Observe that if one of the two functions u, v is monotone, then $u \preceq v$ if and only if there exist $C, c > 0$ such that $u(t) \le C\, v(ct)$ for all $t > 0$ so that our two definitions are consistent.

Let us now introduce V_S, I_S and ϕ_S in detail. As S is generating, any element g of G can be written as a word $g = s_1 \cdots s_k$ with $s_i \in S$. We define the S-length $|g|_S$ of g to be the smallest k such that g is the product of k elements of S. Clearly, for $h, g \in G$, the length $|h^{-1}g|_S$ is the same as the usual graph distance between h and g in the Cayley graph (G, S). We will write $x \sim y$ if x, y are neighbors in the graph (G, S). Define the S-boundary of a finite set $A \subset G$ by

$$\partial_S A = \{e = (x, y) \in A \times A^c : x \sim y\}.$$

Thus, the boundary of A is the set of all oriented edges from A to $A^c = G \setminus A$. This definition as several advantages over possible variants (for instance, except for orientation, A and A^c have the same boundary). An alternative definition is to define the boundary δA as the set of vertices in A that have a neighbor in A^c. Observe that $\#\delta A \le \#\partial A \le \#S \times \#\delta A$. For our purpose, this means there is no harm in choosing one or the other of these two definitions.

The volume growth function V_S of (G, S) is defined by

$$V_S(n) = \#\{g \in G : |g|_S \le n\}. \tag{1.2.1}$$ def-V

This is clearly an increasing function.

The isoperimetric profile I_S is defined by

$$I_S(n) = \inf_{\#A = n} \{\#\partial_S A\}. \tag{1.2.2}$$ def-I

Note that this function is not necessarily increasing.

Define also

$$J_S(n) = \sup_{\#A \le n} \frac{\#A}{\#\partial_S A}. \tag{1.2.3}$$
def_J

This is a non-decreasing function of n satisfying

$$\frac{n}{I_S(n)} \le J_S(n) \le n. \tag{1.2.4}$$
IJ1

In fact, group invariance shows that

$$J_S(n) = \sup_{\frac{1}{2}n < \#A \le n} \frac{\#A}{\#\partial A}$$

Indeed, if the maximum is attained at a set B with $\#B \le n/2$ then the union of two disjoint translated copies of B gives a set A with $\#A = 2\#B \le n$ and $\#\partial A \le 2\#\partial B$. Repeated use of this construction proves the claim. It follows that

$$J_S(n) \le \sup_{\frac{1}{2}n \le \ell \le n} \frac{\ell}{I_S(\ell)}. \tag{1.2.5}$$
IJ2

The reader should note that, at this writing, it is not clear that

$$J_S(n) \preceq \frac{n}{I_S(n)}.$$

This is because it is not clear if there exist positive $a < b$ and c such that

$$c \inf_{an \le \ell \le bn} J_S(\ell) \le \frac{n}{I(n)}$$

for all n. Thus, we do not know if

$$J_S(n) \simeq \frac{n}{I_S(n)}$$

in general. Similarly, whether or not I_S is $\simeq$-equivalent to the non-decreasing isoperimetric profile

$$I_S^{\uparrow}(n) = \inf_{\#A \ge n} \{\#\partial_S A\} \tag{1.2.6}$$
def-Iup

is not clear at this writing.

There is yet another way to look at the isoperimetric profile. It consists in setting

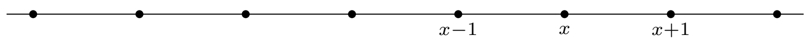
$$F_S(t) = \min\{s : \exists A \subset G \text{ such that } \#A = s \text{ and } \#\partial A < s/t\}. \tag{1.2.7}$$
def-F

This non-decreasing function F_S is related to J_S by

$$F_S(t) > k \Longleftrightarrow J_S(k) \le t. \tag{1.2.8}$$

Indeed, $F_S(t) > k$ means exactly that there are no sets A such that $\#A \le k$ and $\#\partial A < t^{-1}\#A$. Equivalently, all sets A with $\#A \le k$ satisfy $\#A/\#\partial A \le t$, i.e., $J_S(k) \le t$. Note that $F_S(t) = \infty$ for all t large enough if $J_S \simeq 1$.

Figure 1.1: The Cayley graph $(\mathbb{Z}, \{-1, +1\})$ fig-1



The third and last object we want to consider is the probability of return after $2n$ steps of the simple random walk on (G, S). The simple random walk on (G, S) is the Markov process $(X_i)_0^\infty$ with values in G which evolves as follows. If the current state is x, the next state is xs where s is chosen uniformly at random in S. To avoid parity problems, we set

$$\phi_S(n) = \mathbf{P}(X_{2n} = \mathrm{id}/X_0 = \mathrm{id}) = \mu_S^{(2n)}(\mathrm{id}) \tag{1.2.9}$$ def-phi

where

$$\mu_S(g) = \frac{1}{\#S}\mathbf{1}_S(g)$$

and $\mu^{(n)}$ is the n-fold convolution power of μ where the convolution of two functions u, v with finite support is given by

$$u \star v(x) = \sum_y u(y)v(y^{-1}x).$$

It turns out that, because S is symmetric, $\phi_S(n) = \max_g \mu_S^{(2n)}(g)$ and it follows that ϕ_S is a non-increasing function of n. See Lemma 2.1 below.

1.3 First examples

sec-firstE

Let us end this introductory chapter by describing some basic examples. Whenever the context makes it clear which set S is used to define the functions V_S, I_S, J_S, F_S and ϕ_S we will drop the reference to S in our notation.

The Cayley graph $(\mathbb{Z}, \{-1, +1\})$

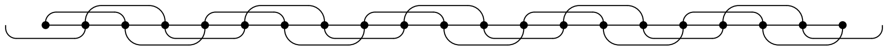
Figure 1.1 (fig-1) shows (a piece of) this Cayley graph. Obviously, $V(n) = 2n + 1$ and $I(n) = 2$, $J(n) = n/2$, $F(n) = 2n$. It is less trivials but not hard to compute

$$\phi(n) = 2^{-2n}\binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}}.$$

The Cayley graph $(\mathbb{Z}, \{-2, -1, +1, +2\})$

Figure 1.2 (fig-2) shows (a piece of) this Cayley graph. We have $V(n) = 4n + 2$, $I(n) = 4$, $J(n) = n/4$, $F(n) = 4n$. The main reason for looking at this example

Figure 1.2: The Cayley graph $(\mathbb{Z}, \{-2, -1, +1, +2\})$ fig-2



is to note that the exact computation of $\phi(n)$ in a useful closed form gets rapidly messy. An application of the local central limit theorem (in this case, a slightly tricky exercise in elementary Fourier analysis) gives $\phi(n) \sim (6\pi n)^{-1/2}$.

The Cayley graph $(\mathbb{Z}^2, \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\})$

First the reader can draw the Cayley graph of $(\mathbb{Z}^2, \{\pm e_1, \pm e_2\})$ (the square grid!) and try to find the exact volume growth (easy), the various exact isoperimetric profiles (not so easy at all), the exact formula for $\phi(2n)$ (easy with a trick!).

For a change, we look here at the generating set

$$S = \{-e_1 - e_2, -e_1 + e_2, e_1 - e_2, -e_1, -e_2, e_1, e_2, e_1 + e_2\}$$

where (e_i, e_2) is the canonical orthogonal basis in $\mathbb{R}^2$. See Figure 1.3. Balls are very nice since

$$B(n) = \{x : d(\mathrm{id}, x) \leq n\} = \{x = (x_1, x_2) : \max\{|x_1|, |x_2|\} \leq n\}.$$

Hence $V(n) = (2n+1)^2$. The boundary of these balls has cardinality $8(3n+1)$. This gives an upper bound on I and a lower bound on J. Of course, one knows that $I(n) \simeq \sqrt{n}$, $J(n) \simeq \sqrt{n}$ (dimension 2 is the only case when $I \simeq J$!), $F(n) \simeq n^2$. We appeal again to the local central limit theorem to get

$$\phi(n) \sim \frac{2}{3\pi n}.$$

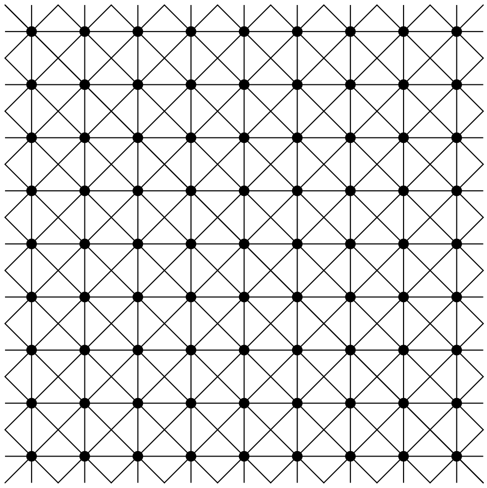
Let us set $T = S \cup \{\mathrm{id}\}$ (of course here id is just $(0,0)$). This addition of the identity element to the generating set does not change the volume and isoperimetric profiles at all (we have just added a loop to each vertex). For this generating set T, the associated simple random walk $X_n = (X_n^1, X_n^2)$ has its two coordinates X_n^1, X_n^2 evolving exactly as two independent simple random walks on $(\mathbb{Z}, \{-1, 0, +1\})$. We have $\mathbb{P}(X_{2n}^1 = 0) \sim ((4/3)\pi n)^{-1/2}$ and thus $\phi_T(n) \sim 3/(4\pi n)$.

The Heisenberg group

Here, the Heisenberg group is the group of 3 by 3 upper-triangular matrices with integer entries and one's on the diagonal, i.e.,

$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}.$$

Figure 1.3: The Cayley graph $(\mathbb{Z}^2, \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\})$ fig-3



It is generated by the four matrices obtained by setting $x = \pm 1$, $y = z = 0$ and $y = \pm 1$, $x = z = 0$. The corresponding Cayley graph is shown in Figure 1.4. figHeis It has $V(n) \simeq n^4$, $I(n) \simeq n^{3/4}$, $J(n) \simeq n^{1/4}$, $F(n) \simeq n^4$, $\phi(n) \simeq n^{-2}$. None of these results is entirely trivial. They will be proved later in these notes.

The free group $\mathbb{F}_2$

The free group on two letters $\{a, b\}$ is the set of all reduced words

$$w = \prod_1^k a^{n_i} b^{m_i}$$

where $k = 0, 1, 2, \ldots$, $n_1, m_k \in \mathbb{Z}$, $n_2, \ldots, n_k = \pm 1, \pm 2, \ldots$, $m_1, \ldots, m_{k-1} = \pm 1, \pm 2, \ldots$ in the alphabet $\{a, a^{-1}, b, b^{-1}\}$. The empty word is the identity. The product of w_1, w_2 is obtaining by writing the word $w_1 w_2$ (concatenation) and reducing it by deleting (recursively) all products $a^n a^{-n}$, $b^n b^{-n}$, $n = \pm 1, \pm 2, \ldots$.

By induction on n, one finds $V(n) = 2 \times 3^n - 1$. We claim that $I(n) \simeq n$. To prove this, for any finite set A, consider $\delta A = \{x \in A : \exists y \notin A, x \sim y\}$. Let R be the maps from $\mathbb{F}_2 \setminus \mathrm{id}$ to $\mathbb{F}_2$ which deletes the last letter of any non-empty reduced word. Let A be a finite set. If $x \notin A$ and $R(x) \in A$ then $R(x) \in \delta A$ since $x \sim R(X)$. Hence $R^{-1}(A \setminus \delta A) \subset A$. Now the sets $R^{-1}(\{x\})$, $x \in A \setminus \delta A$, form

Figure 1.4: A piece of the Cayley graph of the Heisenberg group

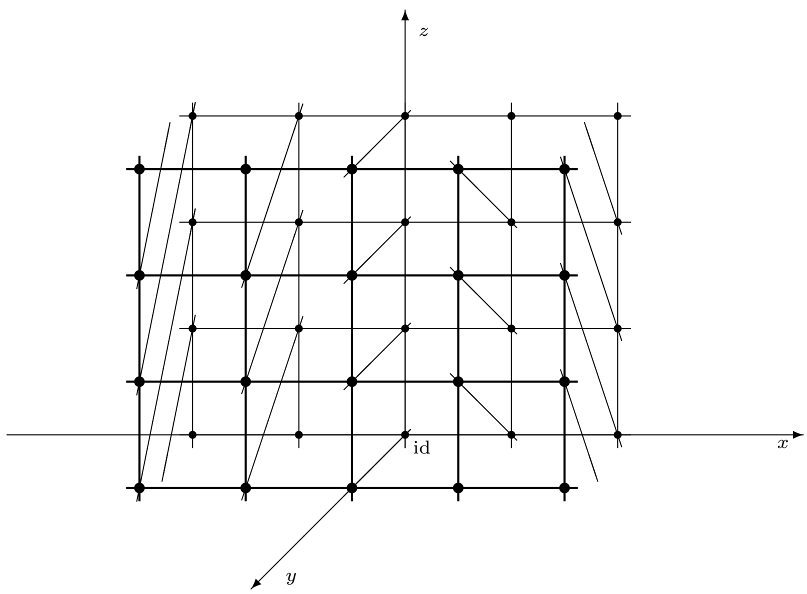


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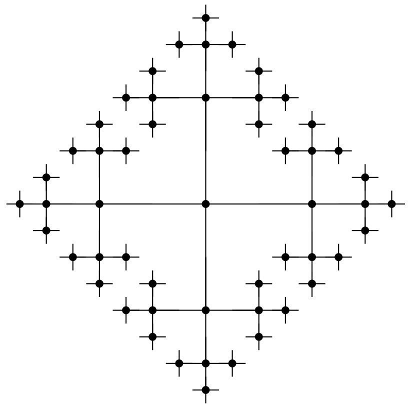
Figure 1.5: The ball of radius 4 in $(\mathbb{F}_2, \{a^{\pm 1}, b^{\pm 1}\})$



a partition of $R^{-1}(A \setminus \delta A)$ and each contains at least 3 elements (the identity element has 4 pre-images). This gives

$$\#A \geq \#R^{-1}(A \setminus \delta A) \geq 3\#(A \setminus \delta A) \geq 3(\#A - \#\delta A),$$

that is, $2\#A \leq 3\#\delta A \leq 3\#\partial A$. This yields $(2/3)n \leq I(n) \leq 4n$. Hence $J(n) \simeq 1$.

The asymptotic behavior of ϕ can be computed precisely and is given by

$$\phi(n) \sim cn^{-3/2} \left(\frac{2\sqrt{2}}{3}\right)^{2n} \text{ at infinity,}$$

for some positive finite constant c.

Chapter 2

Quasi-isometries

2.1 The notion of quasi-isometry

The notion of quasi-isometry plays a central role, both heuristically and technically, in the study of volume growth, isoperimetry, and the behavior of the return probability function ϕ. The idea is already present in the early works on volume growth where the notion of volume growth of finitely generated group was introduced in connection with the study of the (large scale) volume growth of Riemannian manifolds. For instance, it was observed that the volume growth of the universal cover $\widetilde{M}$ of a compact manifold M is comparable to the volume growth of its fundamental group $\pi_1(M)$. In fact, the relationship between $\pi_1(M)$ and $\widetilde{M}$ is very typical of what is now called a quasi-isometry (or rough-isometry).

Definition 2.1.1 *Let $(X_1, d_1), (X_2, d_2)$ be two metric spaces. A map f from X_1 to X_2 is Lipschitz if there exists a finite constant C such that*

$$\forall\, x, y \in X_1,\ \ d_2(f(x), f(y) \leq Cd(x, y).$$

It is bi-Lipschitz if there exist two positive finite constants c, C such that

$$\forall\, x, y \in X_1,\ \ cd(x, y) \leq d_2(f(x), f(y) \leq Cd(x, y).$$

def-QI **Definition 2.1.2** *Let $(X_1, d_1), (X_2, d_2)$ be two metric spaces. A map f from X_1 to X_2 is a quasi-isometry (from X_1 to X_2) if there are positive finite constants C_i, $1 \leq i \leq 5$, such that:*

1. *The set X_2 equals the C_1-neighborhood of the image of X_1 by f, that is,*

$$X_2 = \{z \in X_2 : d_2(z, f(X_1)) \leq C_1\}.$$

2. *Large distance are roughly preserved by f, that is,*

$$\forall\, x, y \in X_1,\ \ C_2^{-1}(d_1(x, y) - C_3) \leq d_2(f(x), f(y)) \leq C_4(d_1(x, y) + C_5).$$

Comparing these two definitions, we see that the requirement of being "distance preserving" is weaker for a quasi-isometry than for a bi-Lipschitz map (small distances do no matter for quasi-isometries). However, a bi-Lipschitz map does not have to be "almost surjective" as a quasi-isometry must be (condition 1 in Definition 2.1.2).

An important observation is that although a quasi-isometry need not be either injective nor surjective, it does admits a sort of inverse. Namely, Assume that f is a quasi-isometry from X_1 to X_2. For any $u \in X_2$, there is a point $v = v(u)$ such that $d_2(v, u) \leq C_1$ and $v \in f(X_1)$. This point is not uniquely defined but we pick one for each u (we use the axiom of choice here also in many case there could be an explicit procedure to pick v). Again, among the pre-image of v, we pick one, say $x = x(v) \in f^{-1}(\{v\})$ and we set $g(u) = x$. We claim that g is a quasi-isometry from X_2 to X_1 and that both $g \circ f$ and $f \circ g$ are at bounded distance from the identity map in X_1 and X_2, respectively. Consider for instance the map $g \circ f$. For $x \in X_1$, let $u = f(x)$. By definition, there exists $v \in X_2$ with $d_2(v, u) \leq C_1$ and $g(u) \in f^{-1}(\{v\})$. Thus,

$$d_1(g \circ f(x), x) = d_1(g(u), x) \leq C_4(d_2(v, u) + C_5) \leq C_4(C_1 + C_5) = C_6.$$

This also shows that $X_1 = \{x \in X_1 : d_1(x, g(X_2)) \leq C_6\}$. It also easily follows from the definition that g roughly preserves large distance as desired.

2.2 Examples of quasi-isometries

2.2.1 Coverings

As already mentioned, one of the most typical example of quasi-isometry is given by the embedding

$$f : G \mapsto \widetilde{M}, \quad f(g) = go$$

of the deck transformation group of the regular covering $\widetilde{M}$ of a compact manifold $M = \widetilde{M}/G$. Here, o is a fixed point in $\widetilde{M}$ and go is the image of o under the action of $g \in G$ on $\widetilde{M}$. More precisely, fix a Riemannian metric on M and lift it to $\widetilde{M}$. This turns $\widetilde{M}$ into a metric space $(\widetilde{M}, d)$. Let S be a finite symmetric generating set of G (it is well-known and easy to see that G must be finitely generated!). The Cayley graph structure (G, S) induces a metric d_S (the word metric) on G. We claim that f defined above is a quasi-isometry between these two metric spaces. It is plain that

$$\widetilde{M} = \{x \in \widetilde{M} : d(x, f(G)) \leq C_1\}$$

where we can take C_1 to be, say, twice the diameter of the compact manifold M. Pick $g, h \in G$ and write $h^{-1}g = s_1 \dots s_k$ with $s_i \in S$ and k minimal, that is, $k = d_S(g, h)$. Set $C_S = \max\{d(o, so) : s \in S\}$. Then

$$d(go, ho) = d(h^{-1}go, o) \quad \leq \quad \sum_{i=0}^{k-1} d(s_1 \dots s_{i+1}o, s_1 \dots s_i o)$$

$$= \sum_{i=0}^{k-1} d(s_{i+1}o, o) \leq C_S\, k = C_S\, d_S(g,h).$$

To obtain a converse inequality, let

$$U = \{z : \forall\, g \in G, d(z,o) \leq d(z,go)\}.$$

Then U is a compact fundamental domain for the action of G: $\widetilde{M} = \bigcup_G gU$ and the sets gU, $g'U$ have disjoint interiors if $g \neq g'$. The diameter of U equals the diameter of the compact manifold M and U contains the open ball $B(o,\eta) = \{z \in \widetilde{M} : d(o,z) < \eta)$ for some $\eta > 0$. Let $\Sigma = \{\sigma \in G : U \cap \sigma U \neq \emptyset\}$. The set Σ is finite. Indeed, $\sigma \in \Sigma$ implies that $d(o,\sigma o) \leq 2\mathrm{diam}(M)$ and the balls $B(\sigma o, \eta)$, $B(\sigma' o, \eta)$ are disjoint if $\sigma \neq \sigma'$. The set Σ is also symmetric because $U \cap \sigma U \neq \emptyset$ implies $\sigma^{-1}U \cap U \neq \emptyset$.

Let $U' = \{z \in \widetilde{M} : \exists\, \sigma \in \{\mathrm{id}\} \cup \Sigma,\ \ z \in \sigma U\}$. Then U' is a neighborhood of U. We claim that Σ generates G. Indeed, let γ be a distance minimizing curve joining o to go. Let $U_0, U_1, U_2, \dots U_k$ be the sequence of translates of U defined inductively as follows: $U_0 = U$. Assume that $U_0, \dots U_i$ have been constructed such that the curve γ visits each set $U_j = g_j U$, $0 \leq j \leq i$. If go belongs to U_i or to one of the sets σU_i, $\sigma \in \Sigma$, set $k = i$ and stop. If not, let U_{i+1} be the first translate of U visited by γ after it leaves $g_i U'$. Then, by definition of Σ, there exists $\sigma_i, \sigma_i' \in \Sigma$ such that $U_{i+1} = \sigma_i \sigma_i' U_i$. From this construction it follows that

$$go = \sigma_1 \sigma_i' \dots \sigma_k \sigma_k' \sigma_{k+1} o, \ \text{ that is } \ g = \sigma_1 \sigma_i' \dots \sigma_k \sigma_k' \sigma_{k+1}.$$

where $\sigma_{k+1} \in \{\mathrm{id}\} \cup \Sigma$. Moreover, if we set $\epsilon = d(U, \widetilde{M} \setminus U')$, the length of the minimizing curve γ from o to go is at least ϵk. Thus

$$d_\Sigma(\mathrm{id}, g) \leq 2k + 1 \leq 2\epsilon^{-1} d(o, go) + 1.$$

It follows that for any $g, h \in G$,

$$d_\Sigma(g,h) \leq 2\epsilon^{-1} d(go, ho) + 1.$$

Finally, we have proved that

$$(\epsilon/2)(d_\Sigma(g,h) - 1) \leq d(f(g), f(h)) \leq C_\Sigma\, d_\Sigma(g,h),$$

that is, f is a quasi-isometry from G to $\widetilde{M}$.

2.2.2 Cayley graphs

Let G be a finitely generated group and let S_1, S_2 be two finite symmetric generating sets. Then the identity map is a bi-Lipschitz map from (G, d_{S_1}) to (G, d_{S_2}). To prove this it suffices to show that

$$\forall\, x \in G,\ \ |x|_{S_1} = d_{S_1}(\mathrm{id}, x) \leq C d_{S_2}(\mathrm{id}, x) = C|x|_{S_2}.$$

This follows immediately from the definition of the word distance with

$$C = \max_{s \in S_1} \{|s|_{S_2}\}.$$

A somewhat less obvious result is the following.

pro-quasi-GH **Proposition 2.2.1** *Let G be a finitely generated group and H a subgroup of finite index. Then H is finitely generated and the inclusion map $H \to G$ is a quasi-isometry between any Cayley graphs of G and H.*

Proof: Let the right cosets of G be $Hu_0, Hu_1, \dots Hu_k$ with $u_0 = \mathrm{id}$, $u_i \in G$, $1 \le i \le k$. For any $x \in G$ let $\widetilde{x} = u_i$ if $x \in Hu_i$. Let S be a symmetric generating set for G. We claim that

$$\Sigma = \{us(\widetilde{us})^{-1} : u \in \{u_0, \dots, u_k\},\ s \in S\}$$

is a generating set of H. Observe first that, indeed, $\Sigma \subset H$. Next, let $h \in H \subset G$ and write

$$h = s_1 \dots s_n, \ \ s_i \in S.$$

Set $h_0 = \mathrm{id}$, $h_i = s_1 \dots s_i$ and write

$$h = \prod_1^n \widetilde{h_{i-1}} s_i \widetilde{h_i}^{-1}.$$

This equality is true simply because $h_0 = \mathrm{id}$ and $h_n = h \in H$ so that $\widetilde{h_n} = \mathrm{id}$. To show that Σ generates H, it suffices to show that $\widetilde{h_{i-1}} s_i \widetilde{h_i}^{-1} \in \Sigma$. But

$$\widetilde{h_i} = \widetilde{h_{i-1} s_i} = \widetilde{\widetilde{h_{i_1}} s_i}$$

because, as one easily checks, $\widetilde{xy} = \widetilde{\widetilde{x} y}$ for any $x, y \in G$. It follows that

$$\widetilde{h_{i-1}} s_i \widetilde{h_i}^{-1} = \widetilde{h_{i-1}} s_i \left(\widetilde{\widetilde{h_{i-1}} s_i} \right)^{-1} \in \Sigma$$

as desired. Not only this shows that H is finitely generated, it also yields

$$d_\Sigma(x, y) \le d_S(x, y)$$

that for any $x, y \in H \subset G$. The complementary inequality

$$d_S(x, y) \le C_1 d_\Sigma(x, y)$$

with $C = \max\{|\sigma|_S : \sigma \in \Sigma\}$ follows from writting any element in Σ as a word of minimal length in S. Finally, writting each coset representative u_i as a word of minimal length in S shows that

$$G = \{x \in G : d_S(x, H) \le C_2\}$$

with $C_2 = \max\{|u_i|_S : 1 \le i \le k\}$. Thus the inclusion maps is a quasi-isometry from (H, d_Σ) to (G, d_S). As any two Cayley graphs of a given finitely generated group are bi-Lipschitz, this proves Proposition pro-quasi-GH 2.2.1.

The next result is simpler.

Proposition 2.2.2 *Let G be a finitely generated group and H be a finite normal subgroup. Then the projection map $x \mapsto Hx$ is a quasi-isometry between any Cayley graphs of G and $H\backslash G$.*

Fix a finite generting set S and consider its projection Σ onto $H\backslash G$. By definition, we have $d_\Sigma(Hx, Hy) \leq d_S(x, y)$. Conversely, if $d_\Sigma(Hx, Hy) = k$ then there is a sequence $s_1, \dots, s_k$ of elements of S such that $Hy = Hxs_1 \dots s_k$. Hence $y = hxs_1 \dots s_k = xs_1 \dots s_k h'$ for sme $h, h' \in H$. Thus

$$d_S(x, y) \leq d_\Sigma(Hx, Hy) + \max\{|h|_S : h \in H\}.$$

This proves the proposition.

2.2.3 Trees of bounded degree at least 3

The following example is elementary but it may be surprising at first sight.

Proposition 2.2.3 *Let (X, E) be a connected tree with vertex degree $N(x)$ satisfying $3 \leq N(x) \leq M$ for some finite M. Then (X, E) is quasi-isometric to the 3-regular tree. In particular, any two regular trees of degree p and q, $p, q \geq 3$ are quasi-isometric.*

Proof: From (X, E) construct a new graph (X', E') and a map f from X to X' as follows. Enumerate the vertices of X and follow the following procedure, in order. For each vertex $x \in X$, if $N(x) = 3$, keep x and the edges around x unchanged, and set $f(x) = x$. If $N(x) = n + 3$ with $n \geq 1$, enumerate the edges at x, say, $e_0, \dots e_{n+2}$. Split x into $n + 1$ vertices $x^1, \dots, x^{n+1}$ as follows. The vertex x^1, which we identify as the image of the original vetex x by setting $f(x) = x_1$, carries the edges e_0, e_1 (and the part of the graph that comes with them) and a new edge $e_{1,2}$ leading to x^2. The vertex x_2 also carries the edge e_2 (and the part of the graph that comes with it) and another edge $e_{2,3}$ connecting to x_3. In general, for $2 \leq i \leq n$, x^i is connected to x^{i-1} by $e_{i-1,i}$, to x_{i+1} by $e_{i,i+1}$ and carries the (original) edge e_i (and the part of the graph that comes with it). Finally x_{n+1} is connected to x_n by the edge $e_{n,n+1}$ and carries e_{n+1} and e_{n+2} (and the part of the graph that comes with them). At each stage of this procedure, the graph is a connected tree. After inspection and local modification of one of the original vertex, that vertex has degree 3 and all new added vertices have degree 3. Thus after completion of this procedure we obtain a connected 3-regular tree (X', E') and a map f from X to X'. By construction,

$$X' = \{z \in X' : d'(z, f(X)) \leq M - 3\}$$

and

$$d(x, y) \leq d'(f(x), f(y)) \leq [2(M - 3) + 1]d(x, y).$$

Thus f is a bi-Lipschitz map and a quasi-isometry.

2.3 Quasi-isometry, growth and isoperimetry

2.4 invariance for Cayley graphs

For $i = 1, 2$, let (G_i, S_i) denotes the Cayley graph of the finitely generated group G_i equipped with a finite symmetric generating sets S_i. Let $d_i = d_{S_i}$ be the associated graph (i.e., word) distance. The next two sections put the following simple key result in a more general context.

Theorem 2.4.1 *Let (G_i, S_i), $i = 1, 2$, be two (locally finite) Cayley graphs of finitely generated groups as above. Assume that the metric spaces (G_1, d_1) and (G_2, d_2) are quasi-isometric. Then the volume growth functions V_1, V_2 are comparable, i.e.,*

$$V_1 \simeq V_2$$

and the isoperimetric profiles I_1, I_2, $I_1^{\uparrow}, I_2^{\uparrow}$, and J_1, J_2 are comparable, i.e.,

$$I_1^{\uparrow} \simeq I_2^{\uparrow}, \;\; J_1 \simeq J_2$$

Proving this theorem is a pretty simple exercise and will be left to the reader. Complete proofs of more general versions of the result are given below.

Is it true that I is invariant by quasi-isometry between Cayley graphs? I am not so sure.

2.4.1 Invariance of volume growth

In a metric space equipped with a Borel measure μ, set $B(x, r) = \{z : d(x, z) < r)$ and

$$V(x, r) = \mu(\{z : d(x, z) \leq r\}).$$

pro-vol-QI **Proposition 2.4.2** *Let (X_i, d_i), $i = 1, 2$, be two quasi-isometric metric spaces and let $f : X_1 \to X_2$ be a quasi-isometry. For $i = 1, 2$, assume that X_i is equipped with a Borel measure μ_i and that, for each $r \geq 1/4$, there exists a constant $C_r \in (0, \infty)$ such that*

$$\forall\, x \in X_i, \;\; i = 1, 2, \quad V_i(x, 2r) \leq C_r V_i(x, r). \tag{2.4.1}$$ loc-D

Assume further that there exist constants $c, C \in (0, \infty)$ such that

$$\forall\, x \in X_1, \;\; cV_1(x, 1/2) \leq V_2(f(x), 1/2) \leq CV(x, 1/2). \tag{2.4.2}$$ vol1-comp

Then there are constant $a, A \in (0, \infty)$ such that

$$\forall\, x \in X_1, \; \forall\, r \geq 1, \;\; aV_1(x, r) \leq V_2(f(x), r) \leq AV_1(x, r). \tag{2.4.3}$$ vol-comp

Proof: Fix M large enough and $r \geq 2M$ and $x \in X_1$. The parameter M will be chosen later: it depends only on the quasi-isometric constants of the map f. In the ball $B_1(x, r - M) = \{z : d_1(z, x) < r - M\}$, consider a maximal set of points z_i, $1 \leq i \leq k$, such that $d(z_i, z_j) > M$ for all $i \neq j$. Then the open balls

$B_1(z_i, M/2)$, $1 \le i \le k$ are disjoint and the balls $B_1(z_i, M)$ cover $B_1(x, r - M)$. It follows that the balls $B_1(z_i, 2M)$ cover $B_1(x, r)$ and, by (2.4.1),

$$V_1(x,r) \simeq \sum_1^k V_1(z_i, M/2) \simeq \sum_1^k V_1(z_i, 1).$$

Now, consider the points $f(x), f(z_i) \in X_2$. As f is a quasi-isometry, there is a constant C_1 such that $d_2(f(x), f(z_i)) \le C_1 r$. Moreover, there exists a constant $c_1 > 0$ such that, if M is chosen large enough, $d_2(f(z_i), f(z_j)) \ge c_1 M$. It follows that

$$\begin{aligned} V_2(f(x), r) &\ge a_1 V_2(f(x), 2C_1 r) \ge a_1 \sum_1^N V_2(f(z_i), c_1 M/2) \\ &\ge a_2 \sum_1^k V_2(f(z_i), 1) \ge a_3 \sum_1^k V_1(z_i, 1) \ge a_4 V_1(x, r). \end{aligned}$$

Furthermore, there exists $c_2 > 0$ such that any point in $B_2(f(x), c_2 r)$ is at bounded distance from the image by f of $B_1(x, r)$. It follows that there exists C_2 such that the balls $B_2(f(z_j), C_2 M)$ cover $B_2(x, r)$. Hence

$$\begin{aligned} V_2(x, r) &\le \sum_1^k V_2(f(z_i), C_2 M) \le A_1 \sum_1^k V_2(f(z_i), 1) \\ &\le A_2 \sum_1^k V_1(z_i, 1) \le A_3 V_1(x, r). \end{aligned}$$

This proves that $V_2(f(x), r) \simeq V_1(x, r)$ for all r large enough as desired.

2.4.2 Invariance of the isoperimetric profiles

We starts by introducing the following notion. Let (X, d, μ) be a metric measure space. For any set $A \subset X$ and $t \ge 0$, denote by $A_t = \{y : d(y, A) \le t\}$ be the t neighborhood of A. Define the boundary of a set A to be the set $\delta A = A_1 \setminus A$. Now, we can compute and compare the volume A and δA for any measurable set A (of finite measure). In full generality, these definitions are not very practical since, for instance, the volume of the boundary of A could change drastically if we replace A_1 by A_2.

Define the isoperimetric profiles $I^{\uparrow}$ and J as follows:

$$I^{\uparrow}(r) = \inf\{\mu(\delta A) : \, A \subset X, \, r \le \mu(A) < \infty\},$$

$$J(r) = \sup\left\{ \frac{\mu(A)}{\mu(\delta A)} : \, A \subset X, \, \mu(A_{1/2}) \le r \right\}.$$

Let us point out that these definitions must be used with caution if no further hypotheses are made on the metric measure space (X, d, μ). Consider the case where X is the vertex set of a connected graph and d is the natural graph distance (i.e., $d(x,y)$ is the minimal number of edges one needs to cross to go from x to y). In this case, $A_{1/2} = A$ and

$$\delta A = \{y \in A^c : \exists\, x \in A, x \simeq y\}.$$

Hence, we recover one of the usual notions of the boundary of a finite subset of the vertex set of a graph: the boundary of A is the set of points not in A having a neighboor in A (recall that for a graph with bounded degree, all the usual notions of boundary yield comparable results as far as (coarse) isoperimetry is concerned). If we now replace the graph distance d by the distance $2d$ then, for any set A, the boundary δA is the empty set! Hence, $I^{\uparrow} \equiv I \equiv 0$, $J \equiv \infty$ on $(0,\infty)$. This clearly shows that, in general, these notions are not stable under quasi-isometry. However, as we shall see, this problem is not too serious.

Let us not that if the space (X,μ) has finite volume then it follows that $I^{\uparrow} \equiv 0$ whereas $J(r) = \infty$ for $r \geq \mu(X)$.

lem-Ak1 **Lemma 2.4.3** *Assume that (X,d,μ) satisfies the mild regularity condition (loc-D 2.4.1), that is, for each $r > 1/4$ there exists C_r such that*

$$\forall\, x \in X, \quad V(x,2r) \leq C_r V(x,r).$$

Then for any fixed $\epsilon, \eta \in (1/4,\infty)$ there exists a constant $C_{\epsilon,\eta}$ such that, for any measurable compact set A,

$$\mu(A_\epsilon) \leq C_{\epsilon,\eta}\mu(A_\eta).$$

Proof: Consider a maximal collection of points z_i in A such that the balls $B(z_i,1/4)$ are disjoint. Then $\mu(A_{1/4}) \geq \sum_i V(z_i,1/4)$. Moreover, the balls $B(z_i,1/2)$ cover A. Hence the ball $B(z_i,3/2)$ cover $A_{1/2}$. It follows that

$$\mu(A_{1/2}) \leq \sum_i V(z_i,3/2) \leq C \sum_i V(z_i,1/4) \leq \mu(A_{1/4}).$$

This suffices to prove the claim.

Recall that a metric space is called a length metric space if the distance d can be computed by minimizing the length of curves between points. Namely, for any continuous curve $\gamma : [a,b] \to X$, let $L(\gamma)$ be the supremum of $\sum_1^N d(\gamma(t_{i-1}),\gamma(t_i))$ over all partitions $a = t_0 \leq t_1 \leq \ldots \leq t_N = b$ of $[a,b]$. Then (X,d) is a length metric space if $d(x,y)$ is the infimum of the length $L(\gamma)$ over all continuous curve joining x to y. If (X,d) is a locally compact complete length metric space then for any two points x,y there is a continuous curve γ joining x and y and such that $d(x,y) = L(\gamma)$.

lem-Ak **Lemma 2.4.4** *Assume that (X,d) is a locally compact complete length metric space and that the measure μ satisfies the mild regularity condition (loc-D 2.4.1), that is, for each $r \geq 1/4$ there exists C_r such that*

$$\forall\, x \in X, \quad V(x,2r) \leq C_r V(x,r).$$

Then for any fixed $k \geq 1$ there exists a constant C_k such that, for any measurable compact set A

$$\mu(A_k \setminus A) \leq C_k \mu(A_1 \setminus A).$$

It suffices to prove this for $k = 2$. Let $\{z_i\}$ be a maximal collection of points such that the balls $B(z_i, 1/4)$ are disjoint and contained in $A_1 \setminus A$ (by (2.4.1) and the boundedness of A_1, this is a finite collection). We claim that the balls $B(z_i, 2)$ cover $A_2 \setminus A_1$. If not, there is a point $y \in A_2 \setminus A_1$ such that $d(y, z_i) \geq 2$ for all z_i. As $y \in A_2$, there exists y' in A such that $d(y, y') \leq 2$. Let γ be a shortest curve from y to y'. On this curve, there exists a point z such that $d(z, A_1^c) = d(z, A) = 1/2$. Moreover, this point z is at distance at most $3/2$ of y and satisfies $d(z, z_i) \geq d(y, z_i) - d(y, z) \geq 1/2$. Hence the existence of z contradicts the maximality of the collection $\{z_i\}$.

Now, by (2.4.1), we have

$$\mu(A_1 \setminus A) \geq \sum_i V(z_i, 1/4) \geq c \sum_i V(z_i, 2) \geq c\mu(A_2 \setminus A_1).$$

From this, the claim easily follows.

pro-iso-QI **Proposition 2.4.5** *Let (X_i, d_i), $i = 1, 2$, be two locally compact complete length metric spaces. For $i = 1, 2$, assume that X_i is equipped with a Borel measure μ_i satisfying the volume regularity condition (2.4.1). Assume that there exists a quasi-isometry $f : X_1 \to X_2$ satisfying the volume comparison condition (2.4.2). Then there are constant $c_1, C_1 \in (0, \infty)$ and an integer $k \geq 1$ such that for any set $A \subset X_1$,*

$$c_1 \mu_1(A_1) \leq \mu_2([f(A)]_1) \leq C_1 \mu_1(A_1) \tag{2.4.4}$$ iso-QI-1

and

$$\mu_2([f(A)]_{k+1} \setminus [f(A)]_k) \leq C_1 \mu_1(\delta A_1). \tag{2.4.5}$$ iso-QI-2

Proof: Let $\{x_i\}$ be a maximal collection in A such that the balls $B_1(x_i, 1)$ are disjoint. Then the balls $B_1(x_i, 3)$ cover A_1 and it follows that

$$\mu_1(A_1) \simeq \sum_i \mu_1(B_1(x_i, 1)).$$

As the map $f : X_1 \to X_2$ is a quasi-isometry, there exists $R < \infty$ such that the balls $B_2(f(z_i), R)$ cover $[f(A)]_1$. Moreover there exists $M < \infty$ (depending only of the quasi-isometry constants) such that

$$\max_{z \in X_2} \#\{i : z \in B_2(f(z_i), R)\} \leq M.$$

Hence

$$\mu_2([f(A)]_1) \simeq \sum_i \mu_2(f(z_i), R) \simeq \sum_i \mu_1(B_1(z_i, 1) \simeq \mu_1(A_1).$$

This proves (2.4.4).

To prove (2.4.5), we proceed similarly. If k is picked large enough (depending on the quasi-isometry constants of f), any point in $[f(A)]_k \setminus [f(A)]_{k-1}$ is at bounded distance of a point $f(z)$ with $z \in A_r \setminus A_1$ and $r = Ck$. Let $\{z_i\}$ be a maximal collection of points in $A_r \setminus A$ such that the balls $B_1(z_i, 1)$ are disjoint. Then the balls $B_1(z_i, 2)$ cover $A_r \setminus A_1$ and

$$\mu_1(A_r \setminus A_1) \simeq \sum_I V_1(z_i, 1).$$

By construction, for C' large enough, the balls $B_2(f(z_i), C')$ cover $[f(A)]_k \setminus [f(A)]_{k-1}$. Hence we have

$$\mu_2([f(A)]_k \setminus [f(A)]_{k-1}) \leq \sum_i V_2(f(z_i), C') \simeq \sum_i V_1(z_i, 1) \simeq \mu_1(A_r \setminus A_1).$$

By Lemma 2.4.4, the desired conclusion follows.

Remark Proposition 2.4.5 is somewhat more subtle that it appears at first sight. First, it is obviously not true that $\mu_1(A) \simeq \mu_2(f(A))$ since we could have $\mu_2(f(A)) = 0$ with $\mu_1(A) > 0$. Second, and more importantly, is not true in general that $\mu_2(\delta f(A)) \leq C\mu_1(\delta A)$ as simple examples show.

th-iso-QI

Theorem 2.4.6 *Let (X_i, d_i), $i = 1, 2$, be two locally compact complete length metric spaces. For $i = 1, 2$, assume that X_i is equipped with a Borel measure μ_i satisfying the volume regularity condition(2.4.1). Assume that there exists a quasi-isometry $f : X_1 \to X_2$ satisfying the volume comparison condition (2.4.2). Then the isoperimetric profiles satisfy*

$$I_1^{\uparrow} \simeq I_2^{\uparrow}, \quad J_1 \simeq J_2.$$

Proof: We first prove the statement concerning $I_j^{\uparrow}$, $j = 1, 2$. Note that $\mu_1(X_1) < \infty$ if and only if $\mu_2(X_2) < \infty$, and that $\mu_j(X_j) < \infty$ implies $I_j^{\uparrow} \equiv 0$ (take $A = X_j$ in the definition of $I^{\uparrow}$).

Thus we can assume that $\mu_j(X_j) = \infty$. Fix $r > 0$ and let $A \subset X_1$ with $r \leq \mu_{1/2}(A) < \infty$. Then, by Lemma 2.4.3 and Proposition 2.4.5, there exists $c > 0$ such that $cr \leq \mu_2([f(A)]_{1/2}) < \infty$. Obviously, $cr \leq \mu_2([f(A)]_k) < \infty$ for any $k \geq 1$. Let $k \geq 1$ be given by Proposition 2.4.5 and set $A' = [f(A)]_k$. Then $cr \leq \mu_2(A') < \infty$ and $\mu_2(\delta A'_1) \leq C\mu_1(\delta A_1)$. Hence $I_2^{\uparrow}(cr) \leq \mu_1(\delta A_1)$. As the set A with $r \leq \mu(A) < \infty$ is arbitrary, we have $I_2^{\uparrow}(cr) \leq CI_1^{\uparrow}(r)$. As these functions are non-decreasing, the last inequality and the symmetry of the hypothesis imply $I_2^{\uparrow} \simeq I_1^{\uparrow}$ as desired.

Next we consider the profiles J_j, $j = 1, 2$. Let A be a subset of X_1 with $\mu_1(A_{1/2}) \leq r$. Let k be given by Proposition 2.4.5 and set $A' = [f(A)]_k$. Then $\mu_2(A') \simeq \mu_2([f(A)]_{1/2}) \simeq \mu_1(A_{1/2}) \geq \mu_1(A)$ and $\mu_2(\delta A') \leq C\mu_1(\delta A_1) \leq C\mu_1(\delta A)$. Hence, there is a constant C such that

$$\frac{\mu_1(A)}{\mu_1(\delta A)} \leq C\frac{\mu_2(A')}{\mu_2(\delta A')} \leq CJ_2(Cr).$$

This yields

$$J_1(r) \le C J_2(Cr).$$

By the symmetry of the hypothesis and the monotonicity of the functions J_j, $j = 1, 2$, it follows that $J_1 \simeq J_2$.

Remark One may be tempted to define another isoperimetric profile say I, by setting

$$I(r) = \inf\{\mu(\delta A) : A \subset X,\, \mu(A) = r\}.$$

This definition is problematic in many ways. First of all there may be many values of r for which there are no sets with $\mu(A) = r$. In this case, the above infimum should probably be intepreted as ∞ but this seems inadequate. A possibly better solution is to introduce the set $\mathcal{R}_\mu$ of all non-negative reals r such that there exists $A \subset X$ with $\mu(A) = r$. For $r \in \mathcal{R}_\mu$, set

$$I_0(r) = \inf\{\mu(\delta A) : A \subset X,\, \mu(A) = r\}$$

and for $r \in \overline{\mathcal{R}_\mu}$, set

$$I(r) = \liminf_{t \in \mathcal{R}_\mu, t \to r} I_0(t).$$

If $r \notin \overline{\mathcal{R}_\mu}$, set

$$I(r) = I(t), \quad t = \max\{s \in \overline{\mathcal{R}_\mu} : s \le r\}.$$

It is interesting to note that it is not always true that $I_1 \simeq I_2$ in the context of Theorem 2.4.6, even if one restricts attention to large sets, or even to sets of the form $A_{1/2}$. The isoperimetric profile I does not have enough regularity to be invariant by quasi-isometry: Let $X_1 = \mathbb{R}$ with its usual distance and let μ_1 be the Lebesgue measure. Let X_2 be $\mathbb{R} \cup N$ where $N = \cup_{i=1}^{\infty} K_i$ is a countable union of intervals $K_i \simeq (0, 1]$, each of length $1/2$ and K_i is attached to $\mathbb{R}$ at the point $x_i = i/4 \in \mathbb{R}$. The measure μ_2 on X_2 is $d\mu_2 = \sum_{n \in \mathbb{Z}} \delta_{n/4} + \sum_{i=1}^{\infty} \delta_{k'_i} + \sqrt{2} \sum_{i=1}^{\infty} \delta_{k_i}$ where $n/4$ is understood as a point on $\mathbb{R}$, k_i is the hanging extremity of K_i and k'_i is the middle point on K_i.

This gives us a metric space (see Fig. 2.1) which satisfies (2.4.1) and is obviously quasi-isometric to $X_1 = \mathbb{R}$ through the obvious isometric injection of X_1 into X_2. The condition (2.4.2) is satisfied. It follows that $J_1 \simeq J_2$, $I_1^{\uparrow} \simeq I_2^{\uparrow}$.

Now observe that, obviously, $I_1(r) \simeq 1$. Concerning I_2, we have

$$\mathcal{R}_{\mu_2} = \{\mu_2(A), A \subset X_2\} = \{n + m\sqrt{2} : n, m \in \{0, 1, 2, \dots\}\}$$

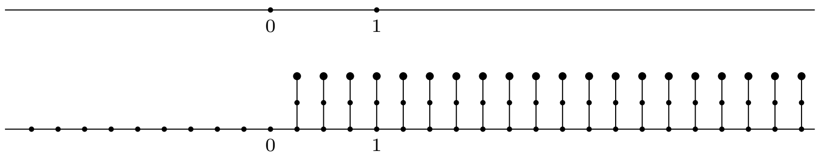
and

$$I(n + m\sqrt{2}) \simeq 1 + (m - n/2)_+$$

where $t_+ = \max\{0, t\}$. There seems to be no easy fix for this difficulty except assuming the existence of sets of measure ϵ with small boundary, for all $\epsilon \in (0, 1)$.

Technically, Proposition 2.4.5 does not apply to metric spaces such as graphs that are not length spaces. However, for graphs, this difficulty is easily overcome, for instance by considering the one-skeleton of a graph which is a nice length

Figure 2.1: The quasi-isometric spaces X_1 and X_2 fig-QI1



metric space. (all edges are isometric to a unit interval, the measure is Lebesgue measure on each edge). Let $\mathcal{G}$ be a connected locally finite graph with vertex set V and symmetric edge set E. Let d be the graph distance (i.e., $d(x, y)$ is the minimun number of edges one must cross to go from x to y). Consider the measure μ on V defined by $\mu(x) = N(x)$ is the degree of x (i.e., the number of edges $(x, y) \in E$). Let $\widetilde{V}$ be the one-skeleton of this graph equipped with its natural length distance $\widetilde{d}$ and the mesure $\widetilde{\mu}$ which is Lebesgue measure on each edge. We claim that the embedding $f : V \to \widetilde{V}$ is a quasi-isometry from (V, d, μ) to $(\widetilde{V}, \widetilde{d}, \widetilde{\mu})$ satisfying the volume condition (2.4.2). In fact,

$$\widetilde{d}(f(x), f(y)) = d(x, y), \quad \widetilde{V} = [f(V)]_{1/2},$$

and

$$\widetilde{\mu}(\widetilde{B}(f(x), 1/2)) = \frac{1}{2}\mu(B(x, 1/2)).$$

In this setting, the local regularity condition (2.4.1) amounts to assuming that there is a constant C such that

$$\forall\, (x, y) \in E, \quad N(x) \leq CN(y). \tag{2.4.6}$$ N-reg

Under this condition, for any finite set $A \subset V$, we have

$$\mu(A) \simeq \mu(A_1) \simeq \widetilde{\mu}([f(A)]_{1/2}).$$

and

$$\mu(\delta A) \simeq \widetilde{\mu}(\delta[f(A)]_{1/2}).$$

Note that $\widetilde{\mu}(\delta f(A))$ is not controled by $\mu(\delta A)$ and that $\mu(\delta A)$ is not controled by $\widetilde{\mu}(\delta[f(A)]_1)$, in general. In particular, the choice of the value $1/2$ (instead of 1) in $\mu(\delta A) \simeq \widetilde{\mu}(\delta[f(A)]_{1/2})$ is crucial.

QI-grph **Proposition 2.4.7** *Referring to the above notation, assume that the connected graph $\mathcal{G}$ satisfies (2.4.6). Let $I^{\uparrow}$, J be the isoperimetric profiles associated to $\mathcal{G}$ and $\widetilde{I}^{\uparrow}, \widetilde{J}$ those associated with its one-skeleton. Then*

$$I^{\uparrow} \simeq \widetilde{I}^{\uparrow}, \quad J \simeq \widetilde{J}.$$

Proof: For any set $A \subset V$, consider $A' = [f(A)]_{1/2}$. The remarks preceding Proposition 2.4.7 immediately give $\widetilde{I}^{\uparrow}(r) \le CI^{\uparrow}(Cr)$ and $J(r) \le C\widetilde{J}(Cr)$ for some finite constant $C \ge 1$.

Now, let A be an arbitrary set in $\widetilde{V}$ and let $A' \subset V$ be defined by $A' = A_{1/2} \cap V$. By inspection, we have

$$\widetilde{\mu}(A) \le \widetilde{\mu}(A_1) \simeq \mu(A')$$

and

$$\widetilde{\mu}(\delta A) \ge c\mu(\delta A').$$

It follows that there are constants $c, C \in (0,\infty)$ such that, for any $A \subset \widetilde{V}$ with $\widetilde{\mu}(A) \ge r$, we have

$$I^{\uparrow}(cr) \le \mu(\delta A') \le C\widetilde{\mu}(\delta A),$$

hence $I^{\uparrow}(cr) \le C\widetilde{I}^{\uparrow}(r)$. Similarly, for any $A \subset \widetilde{V}$ with $\widetilde{\mu}(A_{1/2}) \le r$, we have $\mu(A') \simeq \mu(A'_{1/2}) \simeq \widetilde{\mu}(A_1) \le Cr$ and

$$\frac{\widetilde{\mu}(A)}{\widetilde{\mu}(\delta A)} \le C\frac{\mu(A')}{\mu(\delta A')} \le CJ(Cr).$$

This yields $\widetilde{J}(r) \le CJ(r)$ and finishes the proof of Proposition 2.4.7.

The following result is a corollary of Proposition 2.4.7. Of course it is simpler and more reasonable to give a direct proof.

QI-grph1 **Proposition 2.4.8** *Assume that $\mathcal{G}_i = (V_i, E_i)$, $i = 1, 2$ are two connected graphs satisfying (2.4.6). Assume that there exists a quasi-isometry $f : V_1 \to V_2$ satisfying (2.4.2), that is, such that $N(f(x)) \simeq N(x)$ for all $x \in V_1$. Then*

$$I_1^{\uparrow} \simeq I_2^{\uparrow}, \;\; J_1 \simeq J_2.$$

Chapter 3

Nash inequalities

3.1 Denumerable Markov chains

sec-notmc

The aim of this section is to widen the scope of the definitions introduced in the context of Cayley graphs in Section 1.2.

There are several reasons why this is desirable. First, as far as random walks on groups are concerned, the notion of simple random walk on a Cayley graph is much too restrictive. More natural is the notion of random walk associated to a given measure μ on a (finitely generated) group G. By definition, this is the Markov process $(X_i)_0^\infty$ with values in G which evolves as follows. If the current state is x, the next state is xs where s is chosen uniformly at random according to μ. It is not hard to see that

$$\mathbf{P}(X_n = y/X_0 = x) = \mu^{(n)}(x^{-1}y)$$

where $\mu^{(n)}$ denotes the n-iterated convolution of μ. In analogy with (1.2.9), we set

$$\phi_\mu(n) = \mathbf{P}(X_{2n} = \mathrm{id}/X_0 = \mathrm{id}) = \mu^{(2n)}(\mathrm{id}). \tag{3.1.1}$$ def-phiq

To stick to the simplest and more relevant case, we will assume throughout that μ is symmetric, i.e.,

$$\mu(x) = \mu(x^{-1}).$$

We are also mostly interested in the case where μ is finitely supported with a support that generates G. Indeed, this is the case that is the most immediately relevant when one is interested by connections with the geometry of the Cayley graphs of the group G and the algebraic structure of G.

Second, from the viewpoint of graphs, Cayley graphs are regular graphs, i.e., graphs where all vertices have the same number of neighbors. The notion introduced above — volume growth, isoperimetry, simple random walk — easily generalize to such graphs. In this larger context, it is interesting to see what is special if anything about Cayley graphs. This is particularly relevant when one consider the relation between volume growth and isoperimetry since Cayley graphs behaves very differently than arbitrary regular graphs.

Third, from a technical view point, the treatement of the function ϕ defined at (def-phi 1.2.9) requires some machinery from reversible denumerable Markov chain theory and this machinery is better presented in its natural context which we now introduce.

Let X be a denumerable (infinite) set (e.g., $X = G$, a finitely generated group). A Markov kernel K is a non-negative function on $X \times X$ such that $\sum_y K(x,y) = 1$. In Markov chain language, a particle is hopping at random on X according to the following rule: If the current state is x, the next state is chosen at random according to $K(x,\cdot)$. The iterated kernel $K^n(x,y)$ is defined inductively by

$$K^n(x,y) = \sum_z K(x,z)K^{n-1}(z,y). \tag{3.1.2}$$

def-Kn

It represents the probability of going from x to y in exactly n steps. The kernel K is irreducible if for all $x, y \in X$ there exists an integer $n = n(x,y)$ such that $K^n(x,y) > 0$.

A measure π on X is called invariant (for K) if

$$\sum_x \pi(x)K(x,y) = \pi(y).$$

The pair (K,π) is called reversible if

$$\pi(x)K(x,y) = \pi(y)K(y,x). \tag{3.1.3}$$

In this notes, we will always assume reversibility. It is not hard to check that an irreducible Markov kernel admits at most one reversible measure, up to a constant multiplicative factor. A kernel K is symmetric if

$$\forall\, x, y \in X, \;\; K(x,y) = K(y,x).$$

This symmetry hypothesis implies that the uniform measure $\pi(A) = \#A$ is invariant and that the pair $(K,\#)$ is reversible.

Assuming that a reversible Markov kernel (K,π) is given, we will work with the (real) Hilbert space $\ell^2(\pi)$ with scalar product

$$\langle u, v\rangle = \sum_{x\in X} u(x)v(x)\pi(x).$$

The kernel K induces an operator (also denoted by K) acting on functions by

$$Ku(x) = \sum_{y\in X} K(x,y)u(y).$$

The hypothesis that (K,π) is reversible is easily seen to be equivalent to the fact that the operator K is a self-adjoint operator on $\ell^2(\pi)$. In this context, it is convenient to replace the probability of return after $2n$ steps by the quantity

$$\phi_K(n) = \max_{x\in X}\left\{\frac{K^{2n}(x,x)}{\pi(x)}\right\}.$$

We introduce a (symmetric) graph structure as follows. We say that x and y are neighbors and write $x \sim y$, if $K(x,y) > 0$ (this structure is symmetric because we assume that (K,π) is reversible). This structure is locally finite (i.e., each vertex as a finite number of neighbors) if and only if $K(x,\cdot)$ is finitely supported, for each x. The Kernel K is irreducible if and only if this graph is connected. We write $d(x,y)$ for the graph distance between x and y and set

$$V_K(n) = \inf_x \{\pi\{y : d(x,y) \le n\}\}. \tag{3.1.4}$$ def-VK

The boundary ∂A of a set A is the set of oriented edges from A to $X \setminus A$. The natural measure on edges is the measure

$$Q(B) = \sum_{(x,y)\in B} \pi(x)K(x,y). \tag{3.1.5}$$ def-Q

This allows us to define the isoperimetric functions $I_K^{\uparrow}$ and J_K by setting

$$I_K^{\uparrow}(t) = \inf_{\pi(A)\ge t} \{Q(\partial A)\}, \quad J_K(t) = \sup_{\pi(A)\le t} \left\{\frac{\pi(A)}{Q(\partial A)}\right\}. \tag{3.1.6}$$ def-IJK

In this general setting there is no good comparison between these isoperimetric profiles.

Let us compare these definitions with those given for Cayley graphs. When $X = G$ is a finitely generated group equipped with a symmetric finite generating set S, set $K(x,y) = \mu_S(x^{-1}y)$ with $\mu_S = \frac{1}{\#S}\mathbf{1}_S$. Then

$$K^{\ell}(x,y) = \mu_S^{(\ell)}(x^{-1}y)$$

If we take $\pi \equiv 1$, we have $\pi(A) = \#A$, $Q(B) = \#B/\#S$, and we recover the
sec-Cayley functions V_S, I_S, J_S and ϕ_S of Section , up to constant multiplicative factors as follows:

$$V_S = V_K, \quad I_S^{\uparrow}(n) = \#S I_K^{\uparrow}(n), \quad \#S J_S(n) = J_K(n), \quad \phi_S(n) = \phi_K(n). \tag{3.1.7}$$ =S-K

Another interesting case is when $X = G$ and

$$K(x,y) = \mu(x^{-1}y)$$

with μ a symmetric probability measure on G. Then again $K^{\ell}(x,y) = \mu^{(\ell)}(x^{-1}y)$. In this case K is locally finite if and only if μ is finitely supported.

Finally, consider a graph $\mathcal{G}$ with vertex set X and symmetric oriented edge set E (loops are allowed but not multiple edges). Here symmetry means that $(x,y) \in E$ imples $(y,x) \in E$. Assume that $\mathcal{G}$ is locally finite, i.e., $N(x) = \#\{y : (x,y) \in E\}$ is finite for all x. We say that $\mathcal{G}$ is regular if $N(x) = r$ for all x, i.e., all vertices have the same number of neighbors. The simple random walk on $\mathcal{G}$ is the markov chain on X with kernel

$$K(x,y) = \begin{cases} \frac{1}{N(x)} & \text{if } (x,y) \in E \\ 0 & \text{otherwise.} \end{cases}$$

The pair (K,N) is reversible and K is symmetric if and only if the graph is regular.

3.2 Elementary tools from analysis

Let (K,π) be a reversible Markov chain on a denumerable state space X as in Section 3.1. Consider K as an operator acting on the space $\mathcal{C}_0(X)$ of all real finitely supported functions by

$$Ku(x) = \sum_y K(x,y)u(y).$$

Because of reversibility K is self-adjoint on $\ell^2(X,\pi)$. For any $p \in [1,\infty]$, we denote by $\|f\|_p$ the norm of f in $\ell^p(\pi)$.

3.2.1 The function ϕ

To give a simple example of how this point of view can be useful, we prove

Lemma 3.2.1 *For each x, $n \to K^{2n}(x,x)$ is a non-increasing function. In particular, $n \to \phi(n)$ is non-increasing. Furthermore,*

$$\phi(n) = \sup_{x,y}\left\{\frac{K^{2n}(x,y)}{\pi(y)}\right\}.$$

Proof: Let $\mathbf{1}_x$ be the function equal to 1 at x and zero otherwise. Then,

$$\begin{aligned} K^{2n}(x,x)\pi(x) &= \langle K^{2n}\mathbf{1}_x, \mathbf{1}_x\rangle = \langle K^n\mathbf{1}_x, K^n\mathbf{1}_x\rangle \\ &= \|K^n\mathbf{1}_x\|_2^2 \le \|K^{(n-1)}\mathbf{1}_x\|_2^2 = K^{2(n-1)}(x,x)\pi(x). \end{aligned}$$

Here we have used the fact that K contracts $\ell^2(\pi)$. This follows from Jensen inequality ($|Kf|^2 \le K|f|^2$) and the fact that $\sum_x \pi(x)K(x,y) = \pi(y)$. To prove the last equality in the lemma, write

$$\begin{aligned} K^{2n}(x,y)\pi(x) &= K^{2n}(y,x)\pi(y) \\ &= \langle K^{2n}\mathbf{1}_x, \mathbf{1}_y\rangle = \langle K^n\mathbf{1}_x, K^n\mathbf{1}_y\rangle \\ &\le \|K^n\mathbf{1}_x\|_2\|K^n\mathbf{1}_y\|_2 \\ &= \left(K^{2n}(x,x)\pi(x)K^{2n}(y,y)\pi(y)\right)^{1/2}. \end{aligned}$$

It follows that

$$\frac{K^{2n}(x,y)}{\pi(y)} \le \left(\frac{K^{2n}(x,x)}{\pi(x)}\frac{K^{2n}(y,y)}{\pi(y)}\right)^{1/2} \le \sup_z\left\{\frac{K(z,z)}{\pi(z)}\right\} = \phi(n).$$

Hence

$$\phi(n) = \sup_{x,y}\left\{\frac{K^{2n}(x,y)}{\pi(y)}\right\}$$

as desired. The quantity $\phi(n)$ has many interpretations. For instance, in operator norm notation

$$\begin{aligned}\phi(n) &= \sup_x\{K^{2n}(x,x)/\pi(x)\} = \sup_{x,y}\{K^{2n}(x,y)/\pi(y)\}\\ &= \sup_x \|[K^{2n}(x,\cdot)/\pi(\cdot)]\|_2^2\\ &= \|K^n\|^2_{2\to\infty} = \|K^n\|^2_{1\to 2} = \|K^{2n}\|_{1\to\infty}. \qquad (3.2.8) \quad \boxed{\texttt{phi}}\end{aligned}$$

3.2.2 Dirichlet forms

The Dirichlet form of K is the symmetric bilinear form

$$\mathcal{E}(u,v) = \langle (I-K)u, v\rangle.$$

A simple computation shows that

$$\mathcal{E}(u,v) = \frac{1}{2}\sum_{e\in X\times X} du(e)dv(e)Q(e)$$

where $du(e) = u(y) - u(x)$ and $Q(e) = \pi(x)K(x,y)$ if $e = (x,y)$. In particular,

$$\mathcal{E}(u,u) = \frac{1}{2}\sum_{e\in X\times X}|du(e)|^2Q(e) = \frac{1}{2}\sum_{x,y}|u(x)-u(y)|^2\pi(x)K(x,y). \qquad (3.2.9) \quad \boxed{\texttt{ee}}$$

We will also use an other Dirichlet form $\mathcal{E}_*$ associated with K and defined by

$$\mathcal{E}_*(u,v) = \langle (I-K^2)u, v\rangle. \qquad (3.2.10) \quad \boxed{\texttt{ee*}}$$

Obviously, this satisfies

$$\mathcal{E}_*(u,u) = \langle (I-K^2)u,u\rangle = \|u\|_2^2 - \|Ku\|_2^2$$

since K is self-adjoint on $\ell^2(\pi)$.

It turns out that the form $\mathcal{E}$ is more convenient geometrically and the form $\mathcal{E}_*$ more convenient analytically when dealing with the discrete time semigroup K^ℓ. They satisfy $\mathcal{E}_* \le 2\mathcal{E}$. Indeed,

$$\langle Ku,u\rangle \le \frac{1}{2}\left(\langle Ku,Ku\rangle + \langle u,u\rangle\right) = \frac{1}{2}\big(\langle K^2u,u\rangle + \langle u,u\rangle\big)$$

hence

$$\mathcal{E}(u,u) = \langle u,u\rangle - \langle Ku,u\rangle \ge \langle u,u\rangle - \frac{1}{2}\big(\langle K^2u,u\rangle + \langle u,u\rangle\big) = \frac{1}{2}\mathcal{E}_*(u,u).$$

The reverse inequality $\mathcal{E}_* \ge c\mathcal{E}$ is not true in general (this is related to parity problems and to the possibility that part of the spectrum is close to -1).

We will need the following technical lemma whose content is twofold. First, it says that when K charges the diagonal uniformly, $\mathcal{E}$ and $\mathcal{E}_*$ are comparable. Second, it says that, when studying $\phi(n)$ up to $\simeq$-equivalence, one can replace K by $\frac{1}{2}(I+K)$.

lem-K+ **Lemma 3.2.2** *Define* $K_+ = \frac{1}{2}(I+K)$ *and set* $\phi_+(n) = \max_x K_+^{2n}(x,x)$. *Then*

1. *The Dirichlet form* $\mathcal{E}$ *of* K *satisfies* $\mathcal{E}(f,f) \leq 2(\|f\|_2^2 - \|K_+f\|_2^2)$.
2. $\phi(n) \leq 2\phi_+(n)$.
3. $\phi_+(4n) \leq (2/3)^{8n} + \phi(n)$.

In particular, $\phi \simeq \phi_+$.

Proof: For the first statement, observe that $\|f\|_2^2 - \|K_+f\|_2^2 = \langle (I-K_+^2)f, f\rangle = \mathcal{E}_*^+(f,f)$ is the Dirichlet form of the Markov operator K_+^2. By (2.1.2) it equals

$$\frac{1}{2}\sum_{x,y}|f(x)-f(y)|^2\pi(x)K_+^2(x,y).$$

As $K_+^2 = \frac{1}{4}(I+2K+K^2) \geq \frac{1}{2}K$ pointwise, it follows that $\mathcal{E} \leq 2\mathcal{E}_*^+$ which is the desired result.

To prove the second statement, fix x and write

$$\begin{aligned} K_+^{2n}(x,x) &= 2^{-2n}\sum_0^{2n}\binom{2n}{i}K^i(x,x) \\ &\geq 2^{-2n}\sum_0^{n}\binom{2n}{2i}K^{(2i)}(x,x) \\ &\geq \frac{1}{2}K^{(2n)}(x,x). \end{aligned}$$

The last inequality uses $\sum_0^n\binom{2n}{2i} = 2^{2n-1}$ and the fact that $i \to K^{(2i)}(x,x)$ is non-increasing.

Finally, we prove the last statement using spectral theory. Let E_λ be a spectral resolution of K so that

$$K = \int_{-1}^{1}\lambda dE_\lambda.$$

Then, for any function $u \in \ell^2$,

$$\begin{aligned} \langle (K_+)^{8n}u,u\rangle &= \int_{-1}^{1}\left(\frac{1+\lambda}{2}\right)^{8n}\langle dE_\lambda u,u\rangle \\ &= \int_{-1}^{1/3}\left(\frac{1+\lambda}{2}\right)^{8n}\langle dE_\lambda u,u\rangle + \int_{1/3}^{1}\left(\frac{1+\lambda}{2}\right)^{8n}\langle dE_\lambda u,u\rangle \\ &\leq (2/3)^{8n}\|u\|_2^2 + \int_{1/3}^{1}\lambda^{2n}\langle dE_\lambda u,u\rangle \\ &\leq (2/3)^{8n}\|u\|_2^2 + \int_{-1}^{1}\lambda^{2n}\langle dE_\lambda u,u\rangle = (2/3)^{4n}\|u\|_2^2 + \langle K^{2n}u,u\rangle \end{aligned}$$

Here we have used the elementary inequality $1+\lambda \leq 2\lambda^{1/4}$ for $1/3 \leq \lambda \leq 1$. The desired result clearly follows from $\langle (K_+)^{8n}u,u\rangle \leq (2/3)^{8n}\|u\|_2^2 + \langle K^{2n}u,u\rangle$ by taking $u = \mathbf{1}_x$.

3.3 Nash inequalities and the behavior of ϕ

We now collect a number of results that will be essential in relating volume growth, isoperimetry, and the behavior of ϕ.

3.3.1 The technique of John Nash

subsec-Nashtech

The theorem that follows is named after J. Nash because its roots are in a celebrated 1958 paper of Nash where he studies solutions of uniformly elliptic equations. The specific statement presented below has been obtained through a long chain of improvements and modifications due to different authors. Nash's argument is used by N. Varopoulos in [41] to study the decay of continuous time Markov kernels. Its first clean use for discrete time Markov chains is in Carlen et al [5]. This was extended and polished in [11, 12]. The useful version presented here is due to T. Coulhon [8].

def-Nprofil

Definition 3.3.1 *The Nash profile of the chain (K,π) is the smallest positive non-decreasing function N_K such that*

$$\forall f \in \mathcal{C}_0(X),\ \ \|f\|_2^2 \le N_K(\|f\|_1^2/\|f\|_2^2)\left(\|f\|_2^2 - \|Kf\|_2^2\right).$$

th-N

Theorem 3.3.2 (J. Nash) *Set $\pi_* = \inf \pi$. Assume that there is a non-decreasing positive continuous function N defined on $[\pi_*, \infty)$ such that*

$$\|f\|_2^2 \le N(\|f\|_1^2/\|f\|_2^2)\left(\|f\|_2^2 - \|Kf\|_2^2\right)$$

for all $f \in \mathcal{C}_0(X)$. Then

$$\phi(n) \le \psi(n)$$

where ψ is the decreasing C^1 function solution of

$$\begin{cases} \psi(t) = -\psi'(t)N(1/\psi(t)) \\ \psi(0) = 1/\pi_* \end{cases}$$

In particular, for $\alpha \ge 0$, $-\infty < \beta < \infty$, we have

1. *If $N(t) \preceq (1+t)^{1/\alpha}$ then $\phi(n) \preceq (1+n)^{-\alpha}$.*
2. *If $N(t) \preceq [\log_*(t)]^\alpha$ then $\phi(n) \preceq \exp\left(-n^{1/(\alpha+1)}\right)$.*
3. *If $N(t) \preceq [\log_*(t)]^\alpha[\log_* \log_*(t)]^{-\beta}$ then $\phi(n) \preceq \exp\left(-\left[n(\log_* n)^\beta\right]^{1/(\alpha+1)}\right)$.*

Here and in the rest of the paper $\log_*(t) = \log(2+t)$.

Remarks 1. The ratio $\|f\|_1^2/\|f\|_2^2$ is bounded below by π_* and for $t > \pi_*$ we must have $N(t) \ge 1$. In particular, if $\pi_* = 0$ then $\psi \equiv \infty$.

2.Of course, ψ is also defined implicitely by

$$t = \int_{\pi_*}^{1/\psi(t)} N(s)\frac{ds}{s}.$$

Proof of Theorem 3.3.2: Fix $f \in \mathcal{C}_0(X)$ satisfying $\|f\|_1 = 1$. Set $U(n) = \|K^n f\|_2^2$. The hypothesis implies

$$U(n) \leq N(1/U(n))[U(n) - U(n+1)]$$

because $\|K^n f\|_1 \leq \|f\|_1 \leq 1$ and N is non-decreasing. Observe that $n \to U(n) - U(n+1)$ is non-increasing because

$$\begin{aligned} U(n) - U(n+1) &= \langle K^n f, K^n f\rangle - \langle K^{n+1} f, K^{n+1} f\rangle \\ &= \langle (I - K^2)K^n f, K^n f\rangle \\ &= \|(I - K^2)^{1/2} K^n f\|_2^2 \\ &\leq \|(I - K^2)^{1/2} K^{n-1} f\|_2^2 = U(n-1) - U(n). \end{aligned}$$

It follows that we also have

$$U(n+1) \leq N(1/U(n+1))[U(n) - U(n+1)].$$

Thus, the linear extention of U to the positive real axis satisfies

$$U(t) \leq -N(1/U(t))U'(t)$$

where, for t an integer, $U'(t)$ can be taken to be equal to either the left or right derivative. It follows easily from the definition of ψ that

$$U(t) \leq \psi(t)$$

because $U(0) \leq 1 = \psi(0)$. This implies

$$\langle K^{2n} f, f\rangle = \|K^n f\|_2^2 \leq \psi(n)$$

for all non-negative functions $f \in \mathcal{C}_0(X)$ with $\|f\|_1 = 1$. Taking f to be the function equal to $\pi(x)^{-1}\mathbf{1}_x$ yields

$$\phi(n) = \sup_x \left\{K^{2n}(x,x)/\pi(x)\right\} \leq \psi(n)$$

which is the desired result. The remaining statements follow by calculus.

The next statement gives a simple but remarkably general result based on Theorem 3.3.2.

1/2 **Theorem 3.3.3** *Let (K,π) be a reversible irreducible Markov chain on an infinite countable set X. Assume that $Q_* = \inf\{\pi(x)K(x,y) : x, y,\ x \sim y\} > 0$. Then (K,π) satisfies a Nash inequality with $N(t) \leq 8Q_*^{-2}t^2$. In particular, there exists a finite constant C such that*

$$\forall\, n, \quad \phi(n) \leq Cn^{-1/2}.$$

Proof For any function $f \in \mathcal{C}_0$ and any $x \in X$ there is a finite sequence of distinct elements x_i, $i = 0, 1, \dots, N$, such that $x = x_0$, $x_{i+1} \sim x_i$, $f(x_N) = 0$. Hence

$$|f(x)|^2 \le \sum_i |f(x_{i+1})^2 - f(x_i)^2| \le Q_*^{-1} \sum_e |df^2(e)|Q(e).$$

Next, note that

$$\begin{aligned}
\sum_e |df^2(e)|Q(e) &= \sum_{x,y:x\sim y} |f(x) - f(y)||f(x) + f(y)|Q((x,y)) \\
&\le \left(\sum_{x,y:x\sim y} |f(x) - f(y)|^2 Q((x,y)) \right)^{1/2} \left(\sum_{x,y:x\sim y} |f(x) + f(y)|^2 Q((x,y)) \right)^{1/2} \\
&\le 2\sqrt{2}\mathcal{E}(f,f)^{1/2}\|f\|_2.
\end{aligned}$$

Hence

$$\|f\|_2^2 \le \|f\|_\infty \|f\|_1 \le (2\sqrt{2}Q_*^{-1})^{1/2}\mathcal{E}(f,f)^{1/4}\|f\|_1\|f\|_2^{1/2},$$

that is

$$\|f\|_2^2 \le 8Q_*^{-2} \left(\|f\|_1^2/\|f\|_2^2\right)^2 \mathcal{E}(f,f)$$

as desired.

3.3.2 The converse statement

We now present a partial converse to Theorem th-N 3.3.2

th-Nconv **Theorem 3.3.4** *Associate to*

$$\phi : n \mapsto \phi(n) = \sup_x \{K^{2n}(x,x)/\pi(x)\}$$

the non-decreasing function $\mathcal{N} = \mathcal{N}_\phi$ defined by

$$\mathcal{N}(t) = \inf_{k:\ t\phi(k)<1} \left(1 + \frac{k}{-\log(t\phi(k))}\right). \tag{3.3.1}$$ N1

Then the chain K satisfies the Nash inequality

$$\|f\|_2^2 \le \mathcal{N}(\|f\|_1^2/\|f\|_2^2)\left(\|f\|_2^2 - \|Kf\|_2^2\right).$$

In particular, for $\alpha \ge 0$, $-\infty < \beta < \infty$, we obtain the following estimates.

1. *If $\phi(n) \preceq (1+n)^{-\alpha}$ then $\mathcal{N}(t) \preceq (1+t)^{1/\alpha}$.*
2. *If $\phi(n) \preceq \exp\left(-n^{1/(\alpha+1)}\right)$ then $\mathcal{N}(t) \preceq [\log_*(t)]^\alpha$.*
3. *If $\phi(n) \preceq \exp\left(-\left[n(\log_* n)^\beta\right]^{1/(\alpha+1)}\right)$ then*

$$\mathcal{N}(t) \preceq [\log_*(t)]^\alpha [\log_* \log_*(t)]^{-\beta}.$$

Proof: This result rests on the following observation: reversibility and the Cauchy-Schwarz inequality imply that the function

$$i \to \frac{\|K^i f\|_2}{\|K^{i-1} f\|_2}$$

is non-decreasing for any function $f \in \mathcal{C}_0(X)$. Now, fix f such that $\|f\|_1 = 1$ and set $U(i) = \|K^i f\|_2^2$. Then $U(i) \le \phi(i)$ (see (3.2.8)). Since $i \to U(i)/U(i-1)$ is non-decreasing and $\log x \le x - 1$, $x > 0$, we obtain

$$\begin{aligned} \log[\|f\|_2^2/\phi(k)] &\le \log[U(0)/U(k)] \le k \log[U(0)/U(1)] \\ &\le k \left(\|f\|_2^2 - \|Kf\|_2^2\right) \|Kf\|_2^{-2}. \end{aligned}$$

Rewrite this as

$$\|Kf\|_2^2 \le \frac{k}{\log[\|f\|_2^2/\phi(k)]} \left(\|f\|_2^2 - \|Kf\|_2^2\right).$$

Using $\|f\|_2^2 = \|Kf\|_2^2 + (\|f\|_2^2 - \|Kf\|_2^2)$, we get

$$\|f\|_2^2 \le \left(1 + \frac{k}{\log[\|f\|_2^2/\phi(k)]}\right) \left(\|f\|_2^2 - \|Kf\|_2^2\right). \tag{3.3.2}$$

N1*

This proves the desired Nash inequality.

The specific upper bounds on $\mathcal{N}$ stated in the theorem follow from the definition by inspection and the remarks below.

Remarks 1. The function ϕ satisfies $\phi(k) \le \phi(0) = 1/\pi_*$. For $t < \pi_*$, one can take $k = 0$ in the definition of $\mathcal{N}$ and we get $\mathcal{N}(t) = 1$ on $(0, \pi_*)$.

2. Set $A = \phi(0)/\phi(1)$ and recall that $i \to \phi(i)/\phi(i+1)$ is non-increasing. Define $\widetilde{\mathcal{N}}_\phi = \widetilde{\mathcal{N}}$ by

$$\widetilde{\mathcal{N}}(t) = \mathcal{N}(1/\phi(n)) \text{ if } t = 1/\phi(n)$$

and define $\widetilde{\mathcal{N}}$ by linear interpolation on the non-negative real axis. Then

$$\widetilde{\mathcal{N}}(t/A) \le \mathcal{N}(t) \le \widetilde{\mathcal{N}}(At).$$

In particular $\widetilde{\mathcal{N}} \simeq \mathcal{N}$.

3. Given $t \ge 1/\phi(0) = \pi_*$, we have $t\phi(k) \ge \phi(k)/\phi(0) \ge [\phi(1)/\phi(0)]^k = A^{-k}$ since $i \to \phi(i)/\phi(i+1)$ is non-increasing. Hence,

$$\inf_{\substack{k \\ \phi(k) < 1/t}} \frac{k}{-\log(t\phi(k))} \ge \frac{1}{\log A}.$$

It follows that, for $t \ge \pi_*$,

$$\mathcal{N}(t) \le 2 \log A \inf_{\substack{k \\ \phi(k) < 1/t}} \frac{k}{-\log(t\phi(k))}.$$

4. Let ψ be a continuous decreasing function such that $\phi \le \psi$. Then

$$\inf_{\substack{k \\ \phi(k)<1/t}} \frac{k}{-\log(t\phi(k))} \le \inf_{\substack{k \\ \psi(k)<1/t}} \frac{k}{-\log(t\psi(k))}.$$

For $t > 1/\psi(0)$ and $t = 1/\psi(u)$, $u \ge 1$, we can take $k = [2u+1]$ to obtain

$$\mathcal{N}(t) \le \frac{A_1\, u}{-\log(\psi(2u)/\psi(u))}$$

where $A_1 = 8\log A$, $A = \phi(0)/\phi(1)$. This formula is useful for computing explicit examples. For instance, if $\psi(t) \simeq (1+t)^{-\alpha}$, we get $\psi(2u)/\psi(u) \simeq 1$ and $\mathcal{N}(t) \preceq u \preceq (1+t)^{1/\alpha}$. If instead $\psi(t) \simeq \exp(-t^{1/(1+\alpha)})$, we get $\psi(2u)/\psi(u) \simeq \psi(u)$ and $\mathcal{N}(t) \preceq u/[-\log\psi(u)]$ with $u \simeq [\log_* t]^{1+\alpha}$ which gives $\mathcal{N}(t) \preceq [\log_* t]^{\alpha}$.

5. Let $\psi \ge \phi$ be as above. Assume further that ψ is (piecewise) smooth and satisfies

$$\frac{\log(\psi(u)) - \log(\psi(2u))}{u} \ge \varepsilon\, \frac{-\psi'(u)}{\psi(u)}$$

for some $\varepsilon > 0$. This is satisfied if

$$(\diamond) \qquad \forall\, t, s,\ \ s \in [t, 2t],\ \ \frac{-\psi'(s)}{\psi(s)} \ge \varepsilon \frac{-\psi'(t)}{\psi(t)}.$$

Then, for $t > 1/\psi(0)$ and $t = 1/\psi(u)$, we get

$$\mathcal{N}(t) \le \frac{A_2\,\psi(u)}{-\psi'(u)} = \frac{A_2}{-t\,\psi' \circ \psi^{-1}(1/t))}$$

where ψ^{-1} is the inverse of ψ (not $1/\psi$).

The last remark above shows that Theorem 3.3.4 is close to be a sharp converse of Theorem 3.3.2. In fact, Theorems 3.3.2, 3.3.4 and the remarks above give the following result.

th-N=phi

Theorem 3.3.5 *Fix a reversible chain (K, π) and assume that $\pi_* > 0$. Let N is a positive non-decreasing continuous function. Let ψ be a positive decreasing smooth function with $\psi(0) = 1/\pi_*$. Assume that ψ and N are related by $-\psi'(t)N(1/\psi(t)) = \psi(t)$, $\psi(0) = 1/\pi_*$. Assume further that ψ satisfies $(\diamond)$ for some $\varepsilon > 0$. Then the following properties are equivalent.*

1. *The chain K satisfies*

$$\|f\|_2^2 \le \widetilde{N}(\|f\|_1^2/\|f\|_2^2)\ \left(\|f\|_2^2 - \|Kf\|_2^2\right)$$

where $\widetilde{N}$ is an non-decreasing function satisfying $\widetilde{N} \simeq N$

2. *The chain K satisfies $\phi \le \tilde{\psi}$ for some non-increasing $\tilde{\psi} \simeq \psi$.*

Proof: Assume 1. Then Theorem 3.3.2 shows that $\phi(n) \le \widetilde{\psi}(n)$ where $\widetilde{\psi}$ is defined implicitely by

$$t = c_1 \int_{\pi_*}^{1/\widetilde{\psi}(t)} N(c_2 s)\frac{ds}{s} = c_1 \int_{c_2\pi_*}^{c_2/\widetilde{\psi}(t)} N(s)\frac{ds}{s}$$

for some finite constants $c_1, c_2 \ge 1$. This implies that $\widetilde{\psi}(t) = c_2\psi(t')$ with $t' = c_1^{-1}(t+t_0)$, $t_0 = \int_{\pi_*}^{c_2\pi_*} N(s)\frac{ds}{s}$. Thus, for $t > t_0$, we have

$$c_2\psi(2c_1^{-1}t) \le \widetilde{\psi}(t) \le c_2\psi(c_1^{-1}t).$$

On the interval $[0, 1/\pi_*]$, we have $\psi \simeq \widetilde{\psi} \simeq 1$. Thus $\phi \le \widetilde{\psi}$ with $\widetilde{\psi} \simeq \psi$ as desired.

Assume 2. The there are constants $c_1, c_2 \ge 1$ such that $\phi(n) \le c_1\psi(n/c_2)$. By hypothesis, the function $t \to \overline{\psi}(t) = c_1\psi(t/c_2)$ satisfies $(\diamond)$. Hence Theorem 3.3.4 and Remarks 3 above yields a Nash inequality with Nash function

$$\overline{N}(t) = \frac{1}{-t\overline{\psi}' \circ \overline{\psi}^{1}(1/t)} = \frac{c_2}{-c_1 t\psi' \circ \psi^{-1}(1/(c_1 t))} = c_2 N(c_1 t).$$

This finishes the proof of Theorem 3.3.5.

3.4 Nash inequality and volume growth

We now discuss what the behavior of $\phi(n)$ says about volume growth. First we establish a lower bound on the volume growth V in terms of the Nash profile N. By Theorem 3.3.4, this implies lower bounds on V in terms of ϕ.

th-NV **Theorem 3.4.1** *Assume that (K, π) satisfies the Nash inequality*

$$\forall\, f \in \mathcal{C}_0(X), \quad \|f\|_2^2 \le N(\|f\|_1^2/\|f\|_2^2)\,\left(\|f\|_2^2 - \|Kf\|_2^2\right)$$

where N is a positive increasing continuous function. Then, for all $x \in X$ and all integers n, m with $m \le n$,

$$V(x,n) \ge \min\{N^{-1}((m/2)^2), 2^{(n/m)-1}\pi_*\}.$$

In particular, for $\alpha \ge 0$, $-\infty < \beta < \infty$,

1. *If $N(t) \preceq (1+t)^{2/\alpha}$ then $V(x,n) \succeq (1\wedge\pi_*)[(1+n)/\log_*(n)]^\alpha$*
2. *If $N(t) \preceq \log_*(t)^{2\alpha}$, this yields $V(x,n) \succeq (1\wedge\pi_*)\exp(n^{1/(\alpha+1)})$.*
3. *If $N(t) \preceq [\log_* t)]^{2\alpha}[\log\log_* t]^{-2\beta}$ then*

$$V(x,n) \succeq (1\wedge\pi_*)\exp\left([n(\log_* n)^{2\beta}]^{1/(\alpha+1)}\right).$$

Furthermore, in case 1 where $N(t) \preceq (1+t)^{2/\alpha}$ with $\alpha > 0$ the result stated above can be improved to $V(x,n) \succeq (1 \wedge \pi_*)(1+n)^\alpha$.

cor-NV **Corollary 3.4.2** *Fix $\alpha \geq 0$ and $-\infty < \beta < \infty$.*

1. *If $\phi(n) \preceq (1+n)^{-\alpha/2}$ then $V(x,n) \succeq (1 \wedge \pi_*)(1+n)^\alpha$.*
2. *If $\phi(n) \preceq \exp(-n^{1/(\alpha+1)})$ then $V(x,n) \succeq (1 \wedge \pi_*) \exp(n^{2/(\alpha+2)})$.*
3. *If $\phi(n) \preceq \exp(-[n(\log_* n)^\beta]^{1/(\alpha+1)})$ then*

$$V(x,n) \succeq (1 \wedge \pi_*) \exp\left([n(\log_* n)^\beta]^{2/(\alpha+2)}\right).$$

Proof: For all integers r, ℓ, apply the hypothesis to $f = \max\{(\ell+r) - d(x,y), 0\}$. On $B(x,\ell)$, this function is greater or equal to r. Hence

$$r^2 V(x,\ell) \leq 2N(V(x,\ell+r))V(x,\ell+r).$$

Here we used the facts that

$$\|f\|_2^2 - \|Kf\|_2^2 = \frac{1}{2}\sum_{y,z} |f(z)-f(y)|^2 K^2(z,y)\pi(z)$$

and that $|f(z)-f(y)| \leq 2$ if $K^2(z,y) \neq 0$ for our choice of f. We write this inequality as

$$V(x,\ell+r) \geq N^{-1}\left((r^2/2)[V(x,\ell)/V(x,\ell+r)]\right).$$

Now, fix m, n satisfying $0 < m \leq n$ and set $a = [n/m]$, $\ell_i = im$, $r = m$ where $i = 0, \dots, a$. Then, either there is a $i \in \{0, \dots, a-1\}$ such that $V(x,\ell_i)/V(x,\ell_{i+1}) \geq 1/2$ and it follows that

$$\begin{aligned} V(x,n) &\geq V(x,\ell_{i+1}) \\ &\geq N^{-1}\left((m/2)^2\right), \end{aligned}$$

or for all $i \in \{0, \dots, a-1\}$ $V(x,\ell_i)/V(x,\ell_{i+1}) < 1/2$ and it follows that

$$V(x,n) \geq 2^a V(x,0) \geq 2^{(n/m)-1}\pi_*.$$

This proves the first announced result. The specific results stated in the theorem easily follow. For instance, if $N(t) \preceq [\log_*(t)]^\alpha$ then $N^{-1}(t) \succeq \exp t^{1/\alpha}$ and $V(x,n) \succeq \min\{e^{m^{2/\alpha}}, 2^{n/m}\pi_*\}$. For $m = n^{\alpha/(\alpha+2)}$ this yields the desired result.

When $N(t) \preceq (1+t)^{2/\alpha}$ with $\alpha > 0$, the above iteration must be improved to obtain the announced lower bound. Proceeding as above, we have

$$\ell^2 V(x,\ell) \leq 2N(V(x,2\ell))V(x,2\ell).$$

For $\ell = 2^{n-1}$, this gives

$$V(x,2^n) \geq \left(c4^n V(x,2^{n-1})\right)^\theta$$

where $\theta = \alpha/(2+\alpha)$. Hence

$$V(x, 2^n) \quad \geq \quad \left[\prod_1^n c^{\theta^i} 4^{\theta^i(n-i+1)}\right] \pi(x)^{\theta^n} \geq \varepsilon 2^{n\alpha}(1 \wedge \pi_*)$$

because $0 < \theta < 1$, $\frac{\theta}{1-\theta} = \alpha/2$, and

$$n \sum_1^n \theta^i = n\left(\frac{1-\theta^{n+1}}{1-\theta} - 1\right) = n\frac{\theta(1-\theta^n)}{1-\theta} \geq -C + \alpha n/2.$$

Hence, $V(x, \ell) \succeq (1 \wedge \pi_*)(1+\ell)^\alpha$.

Chapter 4

The volume and ϕ

The aim of this chapter is to present very general results relating volume growth and the behavior of ϕ. One of the reasons to present such results is to emphasize the contrast between what happens for general reversible Markov chains and for random walks on groups.

4.1 General volume upper bounds on ϕ

Let (K,π) be an irreducible reversible Markov chain on a countable set X. We will make only one assumption here, namely,

$$Q_* = \inf_{e=(x,y):x\sim y}\{Q(e)\} = \inf_{(x,y):x\sim y}\{\pi(x)K(x,y)\} > 0. \tag{4.1.1}$$ Q*

Note that by summing of all $y \sim x$, this implies that K is locally finite and that

$$\pi_* = \inf_X \pi > 0. \tag{4.1.2}$$ pi*

Under this mild assumption we prove an upper bound on ϕ in terms of the volume growth. The proof illustrates the Nash inequality technique of Theorem 3.3.2.

th-FK **Theorem 4.1.1** *Let (K,π) be an irreducible reversible Markov chain on an infinite countable set X satisfying (4.1.1). Set*

$$w(t) = \inf\{n : V(n) > t\} \text{ and } N(t) = 128 Q_*^{-1} t w(4t).$$

Then (K,π) satisfies the Nash inequality

$$\|f\|_2^2 \le N(\|f\|_1^2/\|f\|_2^2)\mathcal{E}(f,f).$$

and

$$\phi(n) \le \psi(n)$$

where ψ is the decreasing function defined implicitely by

$$t = 128 Q_*^{-1} \int_{\pi_*}^{1/\psi(t)} w(s)ds.$$

In particular, for $\alpha > 0$, we have

1. *If $V(n) \succeq (1+n)^\alpha$ then $\phi(n) \preceq (1+n)^{-\alpha/(\alpha+1)}$*
2. *If $\log V(n) \succeq (1+n)^\alpha$ then $\phi(n) \preceq (1+n)^{-1}(\log_* n)^{1/\alpha}$.*

Proof Let $A \subset X$ be a finite set. Let $\lambda(A)$ be defined by

$$\lambda(A) = \inf \left\{ \frac{\mathcal{E}(f,f)}{\|f\|_2^2} : f \neq 0, \ \mathrm{support}(f) \subset A. \right\}. \tag{4.1.3}$$

Thus $\lambda(A)$ can be understood as the lowest eigenvalue of $(I-K)$ with Dirichlet (i.e. vanishing) boundary condition in A. The next result gives a general lower bound on $\lambda(A)$ in terms of volume growth. Such inequalities are often called Faber-Krahn inequalities.

Let f be a function with support in Ω normalized by $\|f\|_\infty = 1$. We have

$$\|f\|_2^2 = \sum |f(x)|^2 \pi(x) \leq \pi(A). \tag{4.1.4}$$ FK1

Let x_0 be a point such that $|f(x_0)| = 1$. Let r_0 be the radius of the largest ball centered at x_0 and contained in A. Then there is a sequence of points $x_1, \dots, x_{n+1}$, $r_0 = n$, such that $x_i \sim x_{i+1}$, $i = 0, 1, \dots, n-1$, $x_1, \dots, x_{n-1} \in A$, $x_n \notin A$. In particular, we have $f(x_n) = 0$. It follows that

$$\begin{aligned} 2\mathcal{E}(f,f) &= \sum |f(x) - f(y)|^2 \pi(x) K(x,y) \\ &\geq \sum_0^n |f(x_i) - f(x_{i+1})|^2 \pi(x) K(x,y) \\ &\geq \frac{Q_*}{n+1} \left(\sum_0^n |f(x_i) - f(x_{i+1})| \right)^2 \\ &\geq \frac{Q_*}{n+1} |f(x_0) - f(x_{n+1})|^2 = \frac{Q_*}{n+1}. \end{aligned} \tag{4.1.5}$$

It follows that $\lambda(A) \geq \frac{Q*}{2(n+1)\pi(A)}$. By the definition of $r_0 = n$, we also have $\pi(A) \geq \pi(B(x_0, n))$, i.e., $\pi(A) \geq V(n)$. This yields

$$\lambda(A) \geq \frac{Q*}{4\pi(A) w(\pi(A))}.$$

By the definitions of $\lambda(A)$ and of the Nash function N in Theorem th-FK 4.1.1, this inequality gives

$$\|f\|_2^2 \leq (1/8) N(\pi(\mathrm{supp}(f))/4) \mathcal{E}(f,f)$$

for any function $f \in \mathcal{C}_0$. Let $t > 0$ to be chosen later and write, for any $f \geq 0$,

$$\|f\|_2^2 \leq 4\|(f-t)_+\|_2^2 + \|f\mathbf{1}_{\{f \leq 2t\}}\|_2^2 \leq 4\|(f-t)_+\|_2^2 + 2t\|f\|_1$$

where $u_+ = \max\{u, 0\}$. Observe that

$$\|(f-t)_+\|_2^2 \leq (1/8)N(\pi(f \geq t)/4)\mathcal{E}(f,f) \leq (1/8)N((4t)^{-1}\|f\|_1)\mathcal{E}(f,f).$$

Hence

$$\|f\|_2^2 \leq (1/2)N((4t)^{-1}\|f\|_1)\mathcal{E}(f,f) + 2t\|f\|_1.$$

Picking $4t = \|f\|_2^2/\|f\|_1$ gives

$$\|f\|_2^2 \leq N(\|f\|_1^2/\|f\|_2^2)\mathcal{E}(f,f).$$

This is the desired Nash inequality (because $\mathcal{E}(|f|,|f|) \leq \mathcal{E}(f,f)$ working with non-negative functions suffices).

4.2 General volume lower bounds on ϕ.

This section presents a result essentially due to F. Lust-Piquard which gives a lower bounds on $\phi(n)$ in terms of V. These bounds complement the results stated in Corollary 2.2.5. We also borrow ideas from a recent paper of T. Coulhon and A. Grigory'an.

Lower bounds on $K^{2n}(x,x)$ start with the following observation.

$$K^{2n}(x,x) \geq \frac{\pi(x)}{V(x,m)}\left(1 - \sum_{z \in B(x,m)^c} K^n(x,z)\right)^2.$$

To see this let $k^n(x,z) = K^n(x,z)/\pi(z)$ be the kernel of K^n with respect to the measure π. Write

$$\begin{aligned}
\frac{K^{2n}(x,x)}{\pi(x)} &= k^{2n}(x,x) = \sum_z |k^n(x,z)|^2\pi(z) \\
&\geq \sum_{z \in B(x,m)} |k^n(x,z)|^2\pi(z) \\
&\geq \frac{1}{V(x,m)}\left(\sum_{z \in B(x,m)} k^n(x,z)\pi(z)\right)^2 \\
&= \frac{1}{V(x,m)}\left(1 - \sum_{z \in B(x,m)^c} K^n(x,z)\right)^2. \qquad (4.2.1)
\end{aligned}$$

lb1

Hence a lower bound on $\phi(n)$ in terms of V is obtained if we can find $m = m(n)$ such that $K^n(x, B(x,m)^c) \leq 2/3$. A trivial solution worth noting is that $n = m$ always works since $K^n(x, B(x,n)^c) = 0$. This leads to the bound

$$K^{2n}(x,x) \geq \frac{\pi(x)}{V(x,n)}$$

which is improved in the following theorem.

th-phi-V **Theorem 4.2.1** *Fix $x \in X$. Assume that there exists a positive increasing function v such that $V(x,2r) \le \pi(x)v(r)$, $v(0) \ge 2$, and $r \to r^2/\log v(r)$ is increasing. Then, for all n,*

$$K^{2n}(x,x) \ge \frac{\pi(x)}{9V(x,m)}$$

where m is the smallest integer such that $8n \le m^2/\log v(m)$. In particular,

1. *For $\alpha \ge 0$, if $V(x,n) \preceq \pi(x)(1+n)^\alpha$ then*
$$K^{2n}(x,x) \succeq \pi(x)[(1+n)\log_* n]^{-\alpha/2}.$$

2. *For $0 < \beta < 2$, if $V(x,n) \preceq \pi(x)\exp(n^\beta)$ then*
$$K^{2n}(x,x) \succeq \pi(x)\exp(-n^{\beta/(2-\beta)}).$$

Proof: Define $P = K + i(I - K^2)^{1/2}$. This make sense because $I - K^2$ is a positive operator on $\ell^2(\pi)$. Further, P is an isometry because $PP^* = P^*P = I$. By inspection, we have

$$K^n = \sum_0^n a(n,k)\Re(P^k)$$

with

$$a(n,k) = \begin{cases} 2^{-n+1}\binom{n}{(n-k)/2} & \text{if } n-k \text{ is even and } k>0 \\ 2^{-n}\binom{n}{n/2} & \text{if } n \text{ is even and } k=0 \\ 0 & \text{if } n-k \text{ is odd.} \end{cases}$$

The number $a(n,k)$ is the probability of going from 0 to k or $-k$ in exactly n steps of a nearest-neighbor random walk on $\mathbb{Z}$. We will need the following facts.

1. $\Re(P^n)$ is a polynomial in K of degree atmost n (in fact $\Re(P^n) = P_n(K)$ where P_n is the n^{th} Tchebychev polynomial).

2. $\sum_{k \ge m} a(n,k) \le 2\exp\left(-\frac{m^2}{2n}\right)$

For details, see K. Carne paper C [6].

Now, fix $m \le n$ and set $S(k) = B(x, 2^{k+1}m) \setminus B(x, 2^k m)$, $k = 0,1,2,\ldots$. We will use the above machinery to estimate $\sum_{z \in S(k)} K^n(x,z)$. Write

$$\begin{aligned} \pi(x)\sum_{z \in S(k)} K^n(x,z) &= \langle K^n \mathbf{1}_{S(k)}, \mathbf{1}_x \rangle \\ &= \sum_\ell a(n,\ell)\langle \Re(P^\ell)\mathbf{1}_{S(k)}, \mathbf{1}_x \rangle \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\ell \geq 2^k m} a(n,\ell)\langle \Re(P^\ell)\mathbf{1}_{S(k)}, \mathbf{1}_x\rangle \\
&\leq \sum_{\ell \geq 2^k m} a(n,\ell)\|\mathbf{1}_{S(k)}\|_2\|\mathbf{1}_x\|_2 \\
&\leq 2\pi(x)^{1/2}\pi(S(k))^{1/2}\exp\left(-\frac{4^k m^2}{2n}\right).
\end{aligned}
$$

The third equality is the crucial step. It uses the fact that $\Re(P^k)$ is a polynomial of degree at most k in K which implies that $\Re(P^k)(x,y) = 0$ if $d(x,y) \geq k$. The first inequality uses the fact that P is a contraction on $\ell^2(\pi)$.

Recall that we want to estimate $K^n(x, B(x,m)^c)$. We have

$$
\begin{aligned}
K^n(x, B(x,m)^c) &= \sum_k K^n(x, S(k)) \\
&\leq 2\pi(x)^{-1/2}\sum_k \pi(S(k))^{1/2}\exp\left(-\frac{4^k m^2}{2n}\right) \\
&\leq 2\sum_k \exp\left(-\frac{4^k m^2}{2n} + \frac{1}{2}\log\frac{V(x,2^{k+1}m)}{\pi(x)}\right) \\
&\leq 2\sum_k \exp\left(-\frac{4^k m^2}{2n} + \frac{1}{2}\log v(2^k m)\right)
\end{aligned}
$$

If we pick m so that $8n \leq m^2/[\log v(m)]$, the first term and the ratios of two consecutive terms in this series are all bounded by $\exp\left(-\frac{5}{2}\log v(m)\right)$. It follows that

$$K^n(x, B(x,m)^c) \leq \frac{2}{v(m)^{5/2} - 1} \leq \frac{2}{3}$$

because $v(m) \geq v(0) \geq 2$. Hence (4.2.1) yields

$$K^{2n}(x,x) \geq \frac{\pi(x)}{9V(x,m)}$$

where m is the smallest integer such that $8n \leq m^2/[\log v(m)]$. This ends the proof of Theorem 4.2.1.

T. Coulhon and A. Grigory'an have found a very nice simple proof of the following complementary result.

th-phi-V-doub **Theorem 4.2.2** *Assume that $V(x,2n) \leq CV(x,n)$ for all n and some $x \in X$. Then*

$$\forall\, n \geq 8C, \quad \phi(n) \geq \frac{e^{-4C}}{V(x, 4n^{1/2})}.$$

In particular, if for some $\alpha \geq 0$, $V(x,n) \simeq (1+n)^\alpha$ then $\phi(n) \succeq (1+n)^{-\alpha/2}$.

Proof: Recall again that $i \to \|K^i f\|_2/\|K^{i-1}f|_2$ is non-decreasing. It follows that

$$\frac{\|K^\ell f\|_2^2}{\|f\|_2^2} \geq \left(\frac{\|Kf\|_2^2}{\|f\|_2^2}\right)^\ell .$$

For any finite set A define

$$\lambda(A) = \sup_{\substack{\mathrm{supp}(f)\subset A\\ f\not\equiv 0}} \frac{\|Kf\|_2^2}{\|f\|_2^2}.$$

Then,

$$\begin{aligned}
\phi(n) = \|K^n\|_{1\to 2}^2 &\geq \sup_A \sup_{\substack{\mathrm{supp}(f)\subset A\\ \|f\|_1=1}} \|f\|_2^2 \left(\frac{\|Kf\|_2^2}{\|f\|_2^2}\right)^n \\
&\geq \sup_A \frac{\lambda(A)^n}{\pi(A)}. \qquad (4.2.2)\ \boxed{\texttt{lb2}}
\end{aligned}$$

Now we use the volume growth hypothesis to estimate $\lambda(A)$ from below when A is a ball. Namely, write

$$\begin{aligned}
\lambda(A) &= 1-(1-\lambda(A)) \\
&= 1 - \inf_{\substack{\mathrm{supp}(f)\subset A\\ f\not\equiv 0}} \frac{\|f\|_2^2 - \|Kf\|_2^2}{\|f\|_2^2} \qquad (4.2.3)\ \boxed{\texttt{lb2'}}
\end{aligned}$$

To obtain a lower bound on $\mu(A)$ it suffices to pick a test function f. If $A = B(x,2\ell)$ set $f(y) = \max\{(2\ell - d(x,y)), 0\}$. Then, $\|f\|_2^2 \geq \ell^2 V(x,\ell)$ and

$$\begin{aligned}
\|f\|_2^2 - \|Kf\|_2^2 &= \frac{1}{2}\sum_{y,z} |f(z)-f(y)|^2 K^2(y,z)\pi(y) \\
&\leq 4 \sum_{y\in B(x,2\ell)} \sum_{z\in X)} K^2(y,z)\pi(y) \leq 4V(x,2\ell).
\end{aligned}$$

Hence

$$\lambda(B(x,2\ell)) \geq 1 - \frac{4C}{\ell^2}.$$

For $n \geq 8C$, we can choose $\ell = [n^{1/2}]+1$ and use $(1-x) \geq e^{-2x}$ for $0 < x \leq 1/2$ to obtain

$$\begin{aligned}
\phi(n) &\geq \frac{1}{V(x,2\ell)}\left(1-\frac{4C}{\ell^2}\right)^n \\
&\geq \frac{\exp(-8Cn/\ell^2)}{V(x,2\ell)} \geq \frac{e^{-8C}}{V(x,4n^{1/2})}.
\end{aligned}$$

This proves Theorem 4.2.2.

Remark: The proof of Theorem 4.2.2 is simple when compared to that of Theorem 4.2.1 but it must be emphasized that the hypothesis $V(x,2r) \leq CV(x,r)$ is a strong assumption. In practice, it is hard to verify that V satisfies this doubling condition because it requires matching polynomial upper and lower bounds on V. In principle, the same proof could be used in other situations, under the hypothesis that $V(x, \ell + r(\ell)) \le CV(x, \ell)$ for some positive non-decreasing function r. The conclusion is then that $\lambda(B(x, \ell + r(\ell))) \le 1 - \frac{4C}{r(\ell)^2}$ and $\phi(n) \succeq 1/V(r^{-1}(n))$ where r^{-1} is the inverse function of r. For instance, assume that there exists $c, C > 0$, $a > 0$, $0 < \alpha \le 1$ such that $c \exp(an^\alpha) \le V(x, n) \le C \exp(an^\alpha)$. Then the above condition is satisfied with $r(n) = n^{1-\alpha}$ and it follows that $\phi(n) \succeq \exp(-n^{\alpha/2(1-\alpha)})$. Theorem 4.2.1 gives a better result under a much weaker hypothesis in this case.

4.3 The Viscek graphs

This section presents an example showing that Theorem 4.1.1 is sharp. Namely, we show that there are graphs $\mathcal{G} = (X, E)$ such that the volume growth function satisfies $V(n) \simeq n^d$ for some values of d (including arbitrary large values of d) and the return probability function ϕ satisfies $\phi(n) \simeq n^{-d/(d+1)}$. For such examples with $d > 1$, the lower bounds from Theorems 4.2.1 and 4.2.2 are far off whereas the very general upper bound given by Theorem 4.1.1 is sharp.

Let us start by noting that the Cayley graph $(\mathbb{Z}, \{\pm 1\})$ gives the desired example for $d = 1$. For simplicity, we will discuss only one case $d > 1$.

The Viscek graph $\mathcal{G} = \mathcal{G}_4$ (the parameter 4 will be explained below) is constructed as the increasing limit of finite planar graphs $\mathcal{G}(n)$, $n = 0, 1, 2, \dots$ where $\mathcal{G}(n)$ is obtained from $\mathcal{G}(n-1)$ by a simple procedure that we now describe. See Figures 4.1, 4.2.

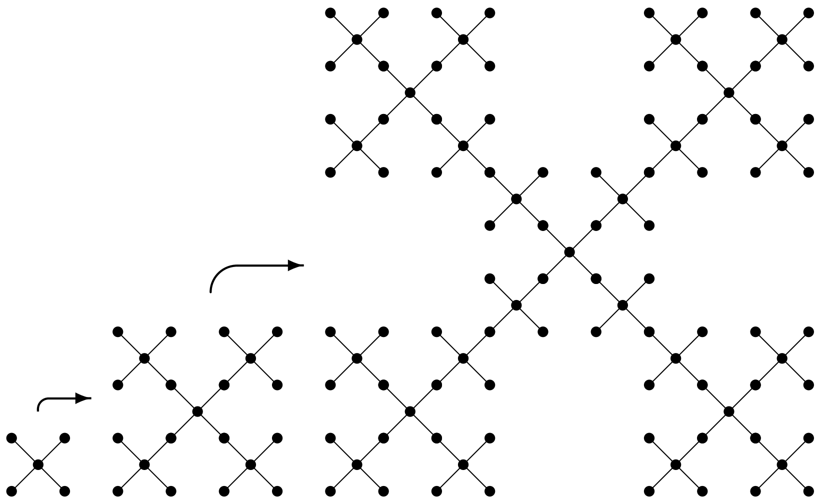
Start with $\mathcal{G}(1)$ being a four branched star around a central vertex. Observe that any two distinct peripheral vertices in $\mathcal{G}(1)$ are at distance $2 = \operatorname{diam}[\mathcal{G}(1)]$ of each other. Suppose that $\mathcal{G}(n-1)$ has been constructed and that it contains 4 points x_1, x_2, x_3, x_4 such that $d(x_i, x_j) = \operatorname{diam}[\mathcal{G}(n-1)]$ for all $i \ne j$. To obtain $\mathcal{G}(n)$, pick 5 copies $\mathcal{G}^i(n-1)$, $i = 0, \dots, 4$, of $\mathcal{G}(n-1)$ and identify x_i^0 with x_i^i for $i = 1, 2, 3, 4$ where x_k^i, $1 \le k \le 4$, are the four special points in $\mathcal{G}^i(n-1)$. Thus $\mathcal{G}(n)$ is made of a "central" copy of $\mathcal{G}(n-1)$ attached to 4 "peripheral" copies of $\mathcal{G}(n-1)$. For any $i \in \{1, 2, 3, 4\}$, pick $j(i) \in \{1, 2, 3, 4\} \setminus \{i\}$ and observe that, by construction, the points $x^i_{j(i)}$, $1 \le i \le 4$, give four points in $\mathcal{G}(n)$ such that

$$d(x^i_{i+2}, x^j_{j+2}) = \operatorname{diam}[\mathcal{G}(n)] = 3 \operatorname{diam}[\mathcal{G}(n-1)]$$

for all $1 \le i \ne j \ne 4$.

This construction generalizes in an obvious way if we replace the parameter 4 by any integer $N \ge 2$, yielding the Viscek graph $\mathcal{G}_N$ (for $N = 2$, we obtain the usual doubly infinite path (the Cayley graph of $(\mathbb{Z}, \{\pm 1\})$). More precisely, let $\mathcal{G}(1)$ be a star with $N + 1$ vertices. To construct $\mathcal{G}(n)$, use $N + 1$ copies $\mathcal{G}^i(n-1)$, $0 \le i \le N$, of $\mathcal{G}(n-1)$, each containing N marked elements $x_1^i, \dots x_N^i$ such that $d(x_k^i, x_\ell^i) = \operatorname{diam}[\mathcal{G}^i(n-1)]$, $1 \le k \ne \ell \le N$. Attach the peripheral

Figure 4.1: The Viscek graph $\mathcal{G}(n), n = 1, 2, 3$ fig-V1



copies $\mathcal{G}^i(n-1)$, $1 \le i \le N$, to the central copy $\mathcal{G}^0(n-1)$ by identifying x_i^0 and x_i^i. For each $i \in \{1, \dots, N\}$, pick $j(i) \in \{1, \dots, N\} \setminus \{i\}$ and set $x_i' = x_{j(i)}^i$. Then, in $\mathcal{G}(n)$, $d(x_k', x_\ell') = \mathrm{diam}[\mathcal{G}(n)] = 3\,\mathrm{diam}[\mathcal{G}(n-1)]$ so that the construction can be repeated Note that, by construction, the Viscek graphs are trees.

We want to prove the following result.

pro-Vis **Proposition 4.3.1** *For any fixed integer N, the Viscek graph $\mathcal{G}_N$ and the associated simple random walk satisfy*

$$V(n) \simeq n^d, \quad \phi(n) \simeq n^{-d/(d+1)}$$

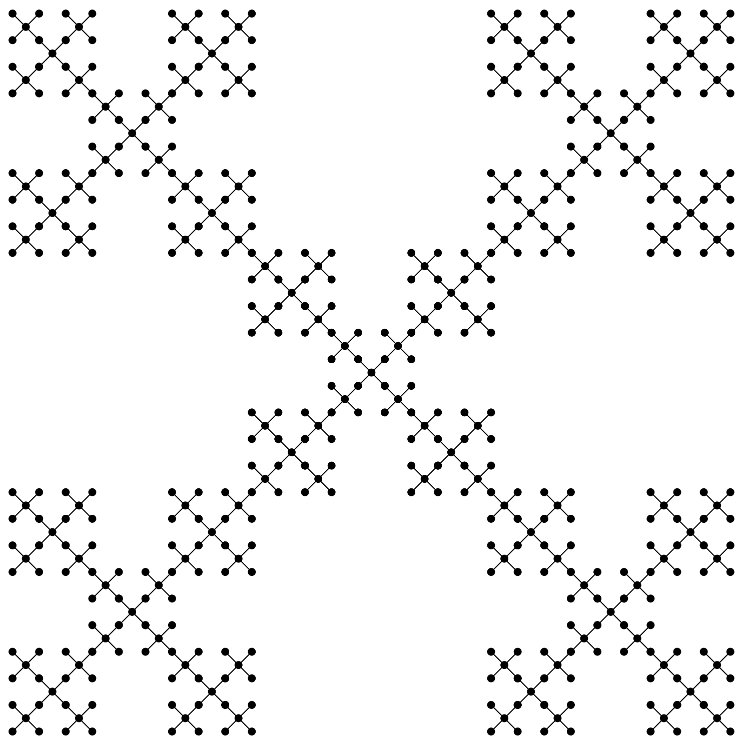
where $d = \log(N+1)/\log 3$.

Proof: We work with a fixed parameter N. First we observe that, by construction, $\#\mathcal{G}(n) = (N+1)\#\mathcal{G}(n-1) - N$, $\#\mathcal{G}(1) = N+1$. Thus $\#\mathcal{G}(n) = 1 + N(N+1)^{n-1} \simeq (N+1)^n$.

Next, for any k, define a k-block to be a subgraph isomorphic to the k-generation finite graph $\mathcal{G}(k)$. Fix $x \in \mathcal{G}$, $r \ge 3$ and consider the integer m such that $3^m \le r < 3^{m+1}$. The vertex x belongs to some m-block B and any m-block has diameter $2 \times 3^{m-1}$. Hence $B(x,r) \supseteq B$. Hence $\#B(x,r) \ge \#\mathcal{G}(m) \simeq (N+1)^m$. The point x also belongs to some $(m+2)$-block, call it A. Let y be a point in $B(x,r)$. Suppose that $y \notin A$. Then the shortest path from x to y passes through one of the N "corners" z of A. Let A' be the $(m+2)$-block adjacent to A and containing z. Any point at distance at most $2 \times 3^{m+1}$ of z belongs either to A or to A'. By construction, $d(z,y) \le 3^{m+1}$ and $y \notin A$. Hence $y \in A'$. That

Figure 4.2: The Viscek graph $\mathcal{G}(4)$

fig-V2



is, $B(x,r)$ is contained in the union of A and its N adjacent $(m+2)$-blocks. It follows that $\#B(x,r) \le (N+1)\#\mathcal{G}(m+2) \simeq (N+1)^m$. Now, the reversible measure π of the simple random walk on $\mathcal{G}_N$ has $\pi(x)$ proportional to the degree of x. Since the degree of a vertex varies from 1 to N, we obtain

$$V(x,r) \simeq (N+1)^m \simeq r^d, \quad d = \log(N+1)/\log 3.$$

This proves the first assertion in Proposition 4.3.1.

Applying Theorem 4.1.1, we obtain the upper bound

$$\phi(n) \preceq n^{-d/(d+1)}.$$

To obtain a matching lower bound, we use (4.2.2) with A being a m-block. This gives, for any n and m,

$$\phi(n) \ge \frac{\lambda(A)^n}{\pi(A)}.$$

We know that, for a m-block A, $\pi(A) \simeq (N+1)^m$. We need to estimate $\lambda(A)$ from below. Recall (see 4.2.3) that

$$\lambda(A) = 1 - \inf_{\substack{\operatorname{supp}(f)\subset A \\ f\neq 0}} \frac{\|f\|_2^2 - \|Kf\|_2^2}{\|f\|_2^2} \tag{4.3.4}$$

where (K,π) is the simple random walk on $\mathcal{G}$.

We now construct a test function f. By definition, the block A has N corner vertices $x_1,\dots,x_N$. Call the shortest path from x_i to x_j, $i \neq j$, a diagonal. These diagonals meet at a unique point o, the center of A. Define f along each half-diagonal from o to x_i so that it varies linearly with $f(o) = 1$, $f(x_i) = 0$. Thus at a vertex x on the half-diagonal from o to x_i and at distance k from the center o, we have $f(x) = 1 - k3^{1-m}$ (the distance from the center to any of the corner is 3^{m-1}). Now, for any vertex y in A, there exists a unique vertex x on one of the diagonals such that the shortest path from o to y leave the diagonals at x (i.e., the graph A is made of the diagonals together with sub-trees hanging from those diagonals). Define f at y by setting $f(y) = f(x)$. Thus, f varies only along the diagonals and stays constant when one wanders away from the diagonals. See Figure 4.3.

For this function f, we have

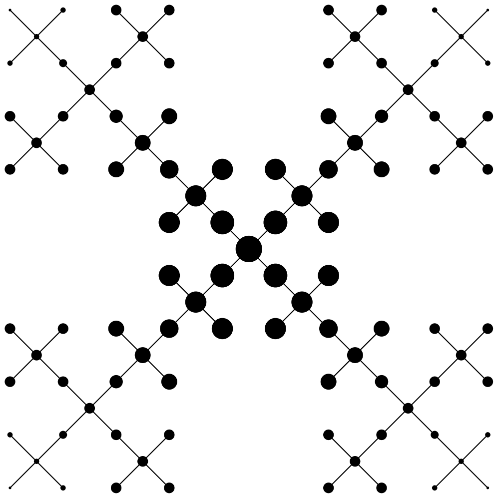
$$\lambda(A) \ge 1 - \frac{\|f\|_2^2 - \|Kf\|_2^2}{\|f\|_2^2}.$$

Thus is suffice to estimate $\|f\|_2^2$ from below and $\|f\|_2^2 - \|Kf\|_2^2$ from above. The m-block A contains a central $(m-1)$-block A'. This block A' is exactly the set of those vertices in A at distance at most 3^{m-2} from the center o of A. By the definition of f, $f(x) \ge 1 - 1/3 = 2/3$ o A'. Thus

$$\|f\|_2^2 \ge (4/9)\pi(A') \succeq (N+1)^m.$$

fig-Vf

Figure 4.3: The function f on a 3-block:
f is proportional to the radius of the black disks



We also have (Recall that K^2 is the step-2 iterated kernel of the simple random on $\mathcal{G}$)

$$\|f\|_2^2 - \|Kf\|_2^2 = \frac{1}{2}\sum_{x,y} |f(x) - f(y)|^2 K^2(x,y)\pi(x).$$

Now, f is constant along subtrees hanging off the diagonals of A and is decreasing linearly along the N half-diagonals. Hence

$$\sum_{x,y} |f(x) - f(y)|^2 K^2(x,y)\pi(x) \leq 4N \times (2 \times 3^{1-m})^2 3^m \preceq 3^{-m}.$$

It follows that there is a constant $C = C(N)$ such that

$$\lambda(A) \geq 1 - C(3(N+1))^{-m}.$$

This, the volume estimate $\pi(A) \geq c(N+1)^m$ and (lb2'' 4.3.4) yield

$$\phi(n) \geq c(N+1)^{-m}(1 - C(3(N+1))^{-m})^n.$$

If we pick m such that $n \simeq (3(N+1))^m$ then $(1 - C(3(N+1))^{-m})^n \simeq 1$ and

$$(N+1)^{-m} \simeq (3(N+1))^{-m\frac{\log(N+1)}{\log[3(N+1)]}} \simeq n^{-d/(d+1)}.$$

Hence $\phi(n) \succeq n^{-d/(d+1)}$ as desired.

Chapter 5

Consequences of isoperimetric inequalities

This chapter shows how the isoperimetric profile I can be used to bound the volume growth function V from below and ϕ from above.

5.1 Isoperimetry and volume lower bound

pro-IV **Proposition 5.1.1** *For $t \geq 0$, define $v(t)$ to be such that*

$$t = \int_{\pi_*}^{v(t)} \frac{ds}{I(s)},$$

i.e., v is the solution of $v'(s) = I(v(s))$, $v(0) = \pi_*$ with $\pi_* = \inf \pi$. Then

$$\forall t \geq 0, \;\; V(t) \geq v(t).$$

In particular, for $d \geq 1$, $0 \leq \alpha < \infty$, and $-\infty < \beta < \infty$,

1. *If $I(n) \succeq (1+n)^{(d-1)/d}$ then $V(n) \succeq (1+n)^d$.*
2. *If $I(n) \succeq n(\log_* n)^{-\alpha}$ then $V(n) \succeq \exp\left(n^{1/(\alpha+1)}\right)$.*
3. *If $I(n) \succeq n(\log_* \log_* n)^{\beta}(\log_* n)^{-\alpha}$ then $V(n) \succeq \exp\left(\left[n(\log_* n)^{\beta}\right]^{1/(\alpha+1)}\right)$.*

Here $\log_* n = \log(2+n)$.

Proof: For any finite set A such that $\pi(A) = n$, $I(n) \leq Q(\partial A)$. For the balls $B(x,\ell) = \{y \in X : d(x,y) \leq \ell\}$, this inequality yields

$$I(V(x,\ell)) \leq Q(\partial B(x,\ell)) \leq \begin{cases} V(x,\ell) - V(x,\ell-1) \\ V(x,\ell+1) - V(x,\ell) \end{cases}$$

where $V(x,\ell)=\pi(B(x,\ell))$. Indeed, by reversibility,

$$\begin{aligned} Q(\partial B(x,\ell)) &\le \sum_{\substack{y,z:\\ d(x,y)=\ell\\ d(x,z)=\ell+1}} K(y,z)\pi(y) \\ &\le \begin{cases} \sum_{y:d(x,y)=\ell}\pi(y) \\ \sum_{z:d(x,z)=\ell+1}\pi(z). \end{cases} \end{aligned}$$

With our convention concerning the extention of functions defined on the integers to the whole positive axis by linear interpolation, this gives

$$I(V(x,t)) \le V'(x,t)$$

(when t is an integer, both the left and right derivatives satisfy this inequality). The first assertion follows. The explicit results given in the proposition follow by inspection and somewhat tedious calculus.

5.2 Isoperimetry, Nash profile and ϕ

We will now show that the isoperimetric function J can be used to bound the Nash profile N and then ϕ from above.

The isoperimetric profile J is related to Nash inequalities in a simple but usually non-optimal way as stated in the following result.

th-NJ **Theorem 5.2.1** *Let $J=J_K$ be the isoperimetric profile of the chain(K,π). Assume (K,π) satisfies the Nash inequality*

$$\forall\, f\in\mathcal{C}_0(X),\ \ \|f\|_2^2 \le N(\|f\|_1^2/\|f\|_2^2)\left(\|f\|_2^2-\|Kf\|_2^2\right) \tag{5.2.1}$$ NI-J

where N is non-decreasing. Then

$$J \le 2N. \tag{5.2.2}$$ NJ

To prove that J is bounded by $2N$ apply (5.2.1) to $f=\mathbf{1}_A$ where A is a finite set. Observe that $\|\mathbf{1}_A\|_2^2=\#A=\|\mathbf{1}_A\|_1$ and that

$$\begin{aligned} \|K\mathbf{1}_A\|_2^2 &= \sum_{x\in X}\left|\sum_y K(x,y)\mathbf{1}_A(y)\right|^2\pi(x) \\ &= \sum_{x\in X}\left|\sum_y K(x,y)\mathbf{1}_X(y)-\sum_y K(x,y)\mathbf{1}_{A^c}(y)\right|^2\pi(x) \\ &\ge \sum_{x\in A}\left|1-\sum_y K(x,y)\mathbf{1}_{A^c}(y)\right|^2\pi(x) \\ &\ge \pi(A)-2Q(\partial A). \end{aligned}$$

Hence (NI-J 5.2.1) yields

$$\pi(A) \leq 2N(\pi(A))Q(\partial A).$$

This gives the desired inequality $J \leq 2N$ because N is non-decreasing.

The next lemma is a clean discrete version of the classical co-area formula of geometric measure theory.

Lemma 5.2.2 (Co-area formula) *For any non-negative function f*

$$\sum_e |df(e)|Q(e) = 2\int_0^\infty Q(\partial F_t)dt$$

where $F_t = \{x : f(x) > t\}$.

Proof: Write

$$\begin{aligned}
\sum_e |df(e)|Q(e) &= 2 \sum_{(x,y):f(x)>f(y)} (f(x)-f(y))Q((x,y)) \\
&= 2 \sum_{(x,y):f(x)>f(y)} \int_{f(y)}^{f(x)} dtQ((x,y)) \\
&= 2 \int_0^\infty \sum_{(x,y):f(x)>t\geq f(y)} Q(x,y)dt \\
&= 2 \int_0^\infty Q(\partial F_t)dt.
\end{aligned}$$

The following two results depends on this co-area formula.

th-JN **Theorem 5.2.3** *For any denumerable Markov chain, the Nash inequality*

$$\forall\, f \in \mathcal{C}_0(X),\ \ \|f\|_2^2 \leq 4\, J^2(4\|f\|_1^2/\|f\|_2^2)\, \mathcal{E}(f,f)$$

is satisfied.

th-Jphi **Theorem 5.2.4** *For any denumerable, locally finite, Markov chain*

$$\phi(n) \leq 2\psi(n)$$

where ψ is the non-increasing derivable function solution of

$$\psi(t) = -\psi'(t)N(1/\psi(t)), \quad \psi(0) = 1/\pi_*$$

with $N(t) = 32\, J^2(4t)$ and $\pi_* = \inf \pi$. Of course, ψ is also defined implicitely by

$$t = \int_{\pi_*}^{1/\psi(t)} N(s)\frac{ds}{s}.$$

In particular, for $\alpha > 0$, $-\infty < \beta < \infty$, we have

1. *If* $J(t) \preceq (1+t)^{1/\alpha}$ *then* $\phi(n) \preceq n^{-\alpha/2}$.
2. *If* $J(t) \preceq [\log_*(t)]^{\alpha}$ *then* $\phi(n) \preceq \exp\left(-n^{1/(2\alpha+1)}\right)$.
3. *If* $J(t) \preceq [\log_*(t)]^{\alpha}[\log_* \log_*(t)]^{-\beta}$ *then*

$$\phi(n) \preceq \exp\left(-\left[n(\log_* n)^{2\beta}\right]^{1/(2\alpha+1)}\right).$$

Proof of Theorem 5.2.3 For any finite set B and $A \subset B$, we have

$$\pi(A) \leq J(\pi(B))Q(\partial A). \tag{5.2.3}$$ NJ1

Fix a non-negative function f, set $F_t = \{x : f(x) > t\}$ and write $f(x) = \int_0^\infty \mathbf{1}_{F_t}(x)dt$. Using (5.2.3) with $A = F_t$, $B = \mathrm{supp(f)}$, and Lemma 3.2.1, we get

$$\begin{aligned}
\|f\|_1 &= \sum_x f(x)\pi(x) = \int_0^\infty \pi(F_t)dt \\
&\leq J(\pi(\mathrm{supp}(f)))\int_0^\infty Q(\partial F_t)dt \\
&\leq \frac{1}{2}J(\pi(\mathrm{supp}(f)))\sum_e |df(e)|Q(e).
\end{aligned}$$

Replacing f by f^2 yields

$$\|f\|_2^2 \leq \frac{1}{2}J(\pi(\mathrm{supp}(f)))\sum_e |d(f^2)(e)|Q(e).$$

Now

$$\begin{aligned}
\left(\sum_e |d(f^2)(e)|Q(e)\right)^2 &= \left(\sum_{x,y}|f(x)-f(y)||f(x)+f(y)|Q((x,y))\right)^2 \\
&\leq \left(\sum_{x,y}|f(x)-f(y)|^2Q((x,y))\right)\left(\sum_{x,y}|f(x)+f(y)|^2Q((x,y))\right) \\
&\leq 8\mathcal{E}(f,f)\|f\|_2^2
\end{aligned}$$

it follows that

$$\|f\|_2^2 \leq 2J^2(\pi(\mathrm{supp}(f)))\,\mathcal{E}(f,f). \tag{5.2.4}$$ NJ2

To finish the proof, set $f_t^+ = \max\{f-t, 0\}$. Then $f^2 \leq f_+^2 + 2tf$. Hence

$$\begin{aligned}
\|f\|_2^2 &\leq \|f_t^+\|_2^2 + 2t\|f\|_1 \\
&\leq 2\,J^2(\pi(\{f \geq t\}))\,\mathcal{E}(f,f) + 2t\|f\|_1.
\end{aligned}$$

The last inequality uses (5.2.4), $\mathrm{supp}(f_t^+) \subset \{f \geq t\}$ and $\mathcal{E}(f_t^+, f_t^+) \leq \mathcal{E}(f,f)$. If we pick $4t = \|f\|_2^2/\|f\|_1$ and observe that $\pi(\{f \geq t\}) \leq t^{-1}\|f\|_1$, we obtain

$$\|f\|_2^2 \leq 2\,J^2(4\|f\|_1^2/\|f\|_2^2)\mathcal{E}(f,f) + \frac{1}{2}\|f\|_2^2.$$

Hence

$$\|f\|_2^2 \leq 4\,J^2(4\|f\|_1^2/\|f\|_2^2)\mathcal{E}(f,f).$$

which is the desired inequality. The restriction that f is non-negative is easily removed since $\mathcal{E}(|f|,|f|) \leq \mathcal{E}(f,f)$. This ends the proof of Proposition 3.2.3.

Proof of Theorem 5.2.4 We now show that Theorem 5.2.3, Lemma 3.2.2 and Theorem 3.3.2 give Theorem 5.2.4. First we use Theorem 5.2.3 and Lemma 3.2.2 to see that the auxiliary chain $K_+ = \frac{1}{2}(I+K)$ satisfies the Nash inequality

$$\|f\|_2^2 \leq N(\|f\|_1^2/\|f\|_2^2)\left(\|f\|_2^2 - \|K_+f\|_2^2\right)$$

with $N(t) = 8\,J^2(4t)$. Theorem 3.3.2 then shows that

$$\phi_+(n) \leq \psi(n)$$

where ψ is the solution of $\psi(t) = -\psi'(t)N(1/\psi(t))$, $\psi(0) = 1/\pi_*$. Finally, Lemma 3.2.2 yields

$$\phi(n) \leq 2\psi(n)$$

which is the desired result.

Remark Theorem 3.3.5 establishes an almost satisfactory equivalence between upper bounds on ϕ and Nash inequalities. In contrast, the bound

$$J_K \leq 2\mathcal{N}_\phi$$

with

$$\mathcal{N}_\phi(t) = \inf_{\substack{k \\ \phi(k)<1/t}} \left(1 + \frac{k}{-\log(t\phi(k))}\right)$$

established by Theorems 3.3.4 and 5.2.1 is rather weak when compared to the fact that the Nash inequality

$$\|f\|_2^2 \leq 4J^2(4\|f\|_1^2/\|f\|_2^2)\mathcal{E}(f,f)$$

always holds as stated in Theorem 5.2.3. In view of this inequality one could hope to establish $J^2 \preceq \mathcal{N}_\phi$ instead of the weaker inequality $J \preceq \mathcal{N}_\phi$. However, Ledoux and Coulhon have shown that the inequality $J \preceq \mathcal{N}_\phi$ is sharp in the context of denumerable reversible Markov chains.

Chapter 6

Bounding J and ϕ using volume growth on Cayley graphs

This chapter presents a number of results that are specific to invariant Markov chains. For simplicity and clarity, we will work in the context of Cayley graphs. Thus, let G be a finitely generated group with a fixed generating set S satisfying $S = S^{-1}$. We consider the invariant Markov kernel

$$K(g,h) = \mu_S(g^{-1}h) = \frac{1}{\#S}\mathbf{1}_S(g^{-1}h)$$

which defines the simple random walk on the Cayley graph (G,S). Since K is symmetric, we take $\pi \equiv 1$. We thus have

$$Q(g,h) = \frac{1}{\#S}\mathbf{1}_S(g^{-1}h)$$

and

$$\mathcal{E}(f,f) = \frac{1}{2\#S}\sum_{g\in G,\ s\in S}|f(g)-f(gs)|^2.$$

In this setting, the definitions set down in the introduction and in Section 2 coincide except for the unimportant multiplicative factor $\#S$ in $I_S(n) = \#S\, I_Q(n)$, $\#S\, J_S(n) = J_Q(n)$.

6.1 A Poincaré type inequality and its consequences

The results presented in this section are all based on the following very simple lemma.

lem-PP **Lemma 6.1.1** *Let G be a finitely generated group with symmetric generating set S. For any function $f \in \mathcal{C}_0(G)$, and $n = 1, 2, \dots$, let*

$$f_n(x) = \frac{1}{V_S(n)} \sum_{z \in x B_S(n)} f(z)$$

where $B_S(n)$ is the ball of radius n around id in the Cayley graph (G, S). Thus, $f_n(x)$ is the mean of f in the ball of radius n around x. Then

$$\|f - f_n\|_1 \leq n \sum_{x \in G, s \in S} |f(x) - f(xs)|.$$

Proof: Fix $x, y \in G$ with $|y|_S \leq n$ and write $y = s_1 \cdots s_k$ with $s_i \in S$, $k \leq n$. Then set $s_0 = \text{id}$ and observe that

$$|f(x) - f(xy)| = \left| \sum_1^k f(xs_0 \cdots s_{i-1}) - f(xs_0 \cdots s_i) \right|.$$

Hence

$$|f(x) - f(xy)| \leq \sum_1^k |f(xs_0 \cdots s_{i-1}) - f(xs_0 \cdots s_i)|.$$

Summing over all $x \in G$ we get

$$\begin{aligned} \sum_{x \in G} |f(x) - f(xy)| &\leq \sum_1^k \sum_{x \in G} |f(x) - f(xs_i)| \\ &\leq k \sum_{x \in G, s \in S} |f(x) - f(xs)|. \end{aligned}$$

Summing over all y such that $|y|_S \leq n$ and dividing by $V_S(n)$, we get

$$\begin{aligned} \sum_{x \in G} |f(x) - f_n(x)| &\leq \frac{1}{V_s(n)} \sum_{y : |y|_S \leq n} \sum_{x \in G} |f(x) - f(xy)| \\ &\leq n \sum_{x \in G, s \in S} |f(x) - f(xs)|. \end{aligned}$$

This is the desired result.

The next theorem gives an upper bound on the isoperimetric profile in terms of the volume growth.

th-CSC **Theorem 6.1.2** *Let (G, S) be a Cayley graph. Define $w = w_S : (0, \infty) \to (0, \infty)$ by*

$$w(t) = \inf\{n : V_S(n) > t\}.$$

Then $J(t) \leq 2\, w(2t)$, i.e.,

$$\#A \leq 2\, w(2\#A) \times \#\partial A.$$

In particular, for $d \geq 0$, $0 < \gamma \leq 1$, we have

1. *If* $V(n) \succeq (1+n)^d$ *then* $J(t) \preceq (1+t)^{1/d}$.
2. *If* $V(n) \succeq \exp(n^\gamma)$ *then* $J(t) \preceq [\log_*(t)]^{1/\gamma}$.

Proof: We will prove the following statement: for any non-negative function $f \in \mathcal{C}_0(G)$,

$$\#\{x : f(x) \geq \lambda\} \leq 2\lambda^{-1}\, w\left(2\|f\|_1/\lambda\right) \sum_{x\in G, s\in S} |f(x) - f(xs)|.$$

Taking $f = \mathbf{1}_A$ and $\lambda = 1$ will then yield the announced result. To prove this inequality, for some n to be chosen later, write

$$\#\{x : f(x) \geq \lambda\} \leq \#\{x : |f(x) - f_n(x)| \geq \lambda/2\} + \#\{x : f_n(x) \geq \lambda/2\}$$

and observe that $\#\{x : |f(x) - f_n(x)| \geq \lambda/2\} \leq 2\lambda^{-1}\|f - f_n\|_1$. Hence, by Lemma 3.1,

$$\#\{x : f(x) \geq \lambda\} \leq \#\{x : f_n(x) \geq \lambda/2\} + 2n\lambda^{-1} \sum_{x\in G, s\in S} |f(x) - f(xs)|.$$

Inspecting the definition of f_n, we see that $f_n \leq V(n)^{-1}\|f\|_1$. Thus, the choice $n = w(2\|f\|_1/\lambda)$ yields $\#\{f_n(x) \geq \lambda/2\} = 0$ and

$$\#\{x : f(x) \geq \lambda\} \leq 2\lambda^{-1}\, w\left(2\|f\|_1/\lambda\right) \sum_{x\in G, s\in S} |f(x) - f(xs)|.$$

cor-CSCphi

Corollary 6.1.3 *Let* (G,S) *be a Cayley graph. Define* $w = w_S : (0,\infty) \to (0,\infty)$ *by*

$$w(t) = \inf\{n : V_S(n) > t\}.$$

Then

$$\phi(n) \leq 2\psi(n)$$

where ψ *is the non-increasing derivable function solution of*

$$\psi(t) = -\psi'(t) N(1/\psi(t)), \quad \psi(0) = 1$$

with $N(t) = 16(\#S)^2\, w^2(8t)$. *Of course,* ψ *is also defined implicitely by*

$$t = \int_1^{1/\psi(t)} N(s)\frac{ds}{s}.$$

In particular, for $d \geq 0$, $0 < \gamma \leq 1$, *we have*

1. *If* $V(n) \succeq (1+n)^d$ *then* $\phi(n) \preceq n^{-d/2}$.
2. *If* $V(n) \succeq \exp(n^\gamma)$ *then* $\phi(n) \preceq \exp\left(-n^{\gamma/(\gamma+2)}\right)$.

Proof: Use Theorem 6.1.2 and Theorem 5.2.4. Note that the function N defined in Corollary 6.1.3 is a Nash function for (G,S) (i.e (G,S) satisfies the Nash inequality $\|f\|_2 \leq N(\|f\|_1^2/\|f\|_2^2)\mathcal{E}(f,f)$, $f \in \mathcal{C}_0(G)$).

6.2 A direct comparison between V and ϕ

This section is based on an idea of W. Hebisch [19] which is also used in [20]. We need the following useful result.

lem-Heb **Lemma 6.2.1** *Let G be a finitely generated group with symmetric generating set S. Set $\mu = \varepsilon \mathbf{1}_{\mathrm{id}} + (1-\varepsilon)(\#S)^{-1}\mathbf{1}_S$, $\varepsilon \in (0,1)$. Then*

$$\left|\mu^{(2n+m)}(x) - \mu^{(2n+m)}(\mathrm{id})\right| \le \sqrt{2\,\varepsilon\,\#S/m}\;|x|\,\mu^{(2n)}(\mathrm{id})$$

for all $x \in G$ and all integers n, m.

Proof: Recall that $df(e) = f(xs) - f(x)$ if $e = (x, xs)$, $x \in G, s \in S$ is a edge, and set $\|df\|_\infty = \max\{|df(e)| : e = (x,xs), x \in G, s \in S\}$. Then write

$$\left|\mu^{(2n+m)}(x) - \mu^{(2n+m)}(id)\right| \le |x|\;\|d\mu^{(2n+m)}\|_\infty .$$

This reduces the proof to the claim that

$$\|d\mu^{(2n+m)}\|_\infty \le \sqrt{2\,\varepsilon\,\#S/m}\;\mu^{(2n)}(id).$$

To prove this claim, write

$$\begin{aligned}
\left|\mu^{(2n+m)}(xs) - \mu^{(2n+m)}(x)\right| \;&\le\; \left(\sum_{\sigma\in S}\left|\mu^{(2n+m)}(x\sigma) - \mu^{(2n+m)}(x)\right|^2\right)^{1/2}\\
&=\; \left(\sum_{\sigma\in S}\left|\sum_{y\in G}\left[\mu^{(n+m)}(y^{-1}x\sigma) - \mu^{(n+m)}(y^{-1}x)\right]\mu^{(n)}(y)\right|^2\right)^{1/2}\\
&\le\; \sum_{y\in G}\left(\sum_{\sigma\in S}\left|\mu^{(n+m)}(y^{-1}x\sigma) - \mu^{(n+m)}(y^{-1}x)\right|^2\right)^{1/2}\mu^{(n)}(y).
\end{aligned}$$

Using the Cauchy-Schwarz inequality and the fact that $\sum_G |\mu^{(n)}(x)|^2 = \mu^{(2n)}(\mathrm{id})$, we obtain

$$\begin{aligned}
\|d\mu^{(2n+m)}\|_\infty \;&\le\; \left[\mu^{(2n)}(\mathrm{id})\right]^{1/2}\left(\sum_{y\in G}\sum_{\sigma\in S}\left|\mu^{(n+m)}(y\sigma) - \mu^{(n+m)}(y)\right|^2\right)^{1/2}\\
&\le\; (\varepsilon\,\#S)^{1/2}\left[\mu^{(2n)}(\mathrm{id})\right]^{1/2}\left(\sum_{y\in G}\sum_{z\in G}\left|\mu^{(n+m)}(yz) - \mu^{(n+m)}(y)\right|^2\mu^{(2)}(z)\right)^{1/2}.
\end{aligned}$$

Let K be the Markov operator defined by $Kf = f \star \mu$. Then K is reversible with respect to $\pi \equiv 1$ because μ is symmetric. In terms of the modified Dirichlet form

$$\begin{aligned}
\mathcal{E}_*(f,f) &= \|f\|_2^2 - \|Kf\|_2^2 = \langle (I-K^2)f, f\rangle \\
&= \frac{1}{2}\sum_x\sum_z |f(xz)-f(x)|^2\mu^{(2)}(z) \\
&= \|(I-K^2)^{1/2}f\|_2^2
\end{aligned}$$

the last inequality becomes

$$\begin{aligned}
\|d\mu^{(2n+m)}\|_\infty &\le \sqrt{2\,\varepsilon\,\#S}\,\left[\mu^{(2n)}(\mathrm{id})\right]^{1/2}\mathcal{E}_*(\mu^{(n+m)},\mu^{(n+m)})^{1/2} \\
&\le \sqrt{2\,\varepsilon\,\#S}\,\left[\mu^{(2n)}(\mathrm{id})\right]^{1/2}\|(I-K^2)^{1/2}K^m\mu^{(n)}\|_2 \\
&\le \sqrt{2\,\varepsilon\,\#S}\,\|(I-K^2)^{1/2}K^m\|_{\to 2}\mu^{(2n)}(\mathrm{id})
\end{aligned}$$

The last inequality uses $\|\mu^{(n)}\|_2^2 = \mu^{(2n)}(\mathrm{id})$. The desired claim follows from this and the next lemma.

lem-analytic **Lemma 6.2.2** *For any reversible Markov chain (K,π)*

$$\|(I-K^2)^{1/2}K^m\|_{2\to 2} \le m^{1/2}.$$

Proof: Let $K^2 = \int_0^1 \lambda dE\lambda$ be a spectral decomposition of the positive self-adjoint contraction K^2 on $\ell^2(\pi)$. Then, for any $f \in \ell^2(\pi)$,

$$\|(I-K^2)^{1/2}K^{2\ell}f\|_2^2 = \int_0^1 (1-\lambda)\lambda^{2\ell}dE_\lambda(f,f).$$

When $m = 2\ell$, the desired result follows from

$$\max_\lambda \in [0,1]\{\lambda^{2\ell}-\lambda^{2\ell+1}\} \le (2s+1)^{-1}.$$

For $m = 2\ell+1$,

$$\|(I-K^2)^{1/2}K^m\|_{2\to 2} \le \|(I-K^2)^{1/2}K^{2\ell}\|_{2\to 2} \le (2\ell+1)^{1/2} \le m^{1/2}.$$

Lemma lem-Heb 6.2.1 implies the following result.

pro-rnm **Proposition 6.2.3** *Let G be a finitely generated group with symmetric generating set S and let $\mu = \varepsilon\mathbf{1}_{\mathrm{id}} + (1-\varepsilon)(\#S)^{-1}\mathbf{1}_S$, $\varepsilon \in (0,1)$. Then, for all integers n, m,*

$$\mu^{(2n+m)}(\mathrm{id}) \le 2V(r(n,m))^{-1}$$

where

$$r(n,m) = m^{1/2}\frac{\mu^{(2n+m)}(\mathrm{id})}{2\sqrt{2\,\varepsilon\,\#S}\;\mu^{(2n)}(\mathrm{id})}.$$

Proof: It follows from Lemma 6.2.1 that

$$\mu^{(2n+m)}(x) \geq \frac{1}{2}\mu^{(2n+m)}(id) \quad \text{when} \quad |x| \leq r(n,m).$$

Summing over the ball of radius $r(n,m)$ yields the desired result.

Corollary 6.2.4 *Let G be a finitely generated group with symmetric generating set S. Let $d > 0$. If $\phi(n) \simeq (1+n)^{-d/2}$ then $V(n) \preceq (1+n)^d$.*

Proof: Consider $\mu_S = (\#S)^{-1}\mathbf{1}_S$ and $\mu = \frac{1}{2}(\mathbf{1}_{\mathrm{id}} + \mu_S)$. By Lemma 3.2.2, we have $\phi_+(n) = \mu^{(2n)}(\mathrm{id}) \simeq \phi(n) \simeq (1+n)^{-d/2}$. Proposition 6.2.3 with $m = n = 2\ell^2$ yields

$$\phi_+(3\ell^2) = \mu^{(6\ell^2)}(\mathrm{id}) \leq 2V(c\ell)^{-1}$$

for some constant $c > 0$. The desired result follows.

cor-HSCphi **Corollary 6.2.5** *Let G be a finitely generated group with symmetric generating set S. Let $d > 0$, $0 < \gamma \leq 1$.*

(1) If $V(n) \succeq (1+n)^d$ then $\phi(n) \preceq (1+n)^{-d/2}$.

(2) If $V(n) \succeq \exp(n^\gamma)$ then $\phi(n) \preceq \exp\left(-n^{\gamma/(\gamma+2)}\right)$.

Given Proposition 6.2.3, the technique of proof is similar to that used for Theorem 3.4.1. Details can be found in [20]. We do not give the details here since this result has already been proved by another method in Theorem 6.1.3.

th-V=phi-d **Theorem 6.2.6** *Let G be a finitely generated group with symmetric generating set S. Let $d > 0$. The following two properties are equivalent.*

1. $V(n) \simeq (1+n)^d$;
2. $\phi(n) \simeq (1+n)^{-d/2}$.

Proof: Assume that that $V(n) \simeq (1+n)^d$. Then Corollary 6.1.3 (or Corollary 6.2.5) gives $\phi(n) \preceq (1+n)^{-d/2}$ whereas Theorem **??** gives $\phi(n) \succeq (1+n)^{-d/2}$.

Assume instead that $\phi(n) \simeq (1+n)^{-d/2}$. Then Corollary 3.4.2 gives $V(n) \succeq (1+n)^d$ whereas Corollary 6.2.5 gives $V(n) \preceq (1+n)^d$.

Next we prove a Gaussian lower bound that will be needed later on.

th-Gausslb **Theorem 6.2.7** *Let G be a finitely generated group. Let S be a symmetric generating set containing the identity. Let $|x|$ denote the length of $x \in G$ with respect to S and set Let $\mu = (\#S)^{-1}\mathbf{1}_S$. Assume that $V_S(n) \simeq (1+n)^{-d/2}$ for some $d \geq 0$. Then there exists $c_1, c_2 > 0$ such that*

$$\mu^{(n)}(x) \geq c_1(1+n)^{-d/2}e^{-c_2|x|^2/n}$$

for all x, n satisfying $|x| \leq n$.

Proof: By lemma 6.2.1, there exists a constant $C > 0$ such that for all integers s, t, and all $x \in G$,

$$\mu^{(2t+s)}(x) \geq \mu^{(2t+s)}(\mathrm{id}) - C\sqrt{s}\,|x|_S \mu^{(2t)}(\mathrm{id}).$$

Now, by Theorem 6.2.6, the hypothesis $V_S(n) \simeq (1+n)^d$ implies that $\phi(n) \simeq (1+n)^{-d/2}$. Using the fact that id $\in S$, this shows that there exist $c > 0$ and $\eta > 0$ such that

$$\mu^{(2t+s)}(x) \geq c(1+t)^{-d/2}$$

for all integers t, s and all $x \in G$ satisfying $t \leq s \leq 2t$ and $|x| \leq \eta\sqrt{4t}$. Thus, for any integer $n \geq 6$ and all $x \in G$ satisfying $|x| \leq \eta\sqrt{n}$,

$$\mu^{(n)}(x) \geq c(1+n)^{-d/2}. \tag{6.2.1}$$ mu-n>

Indeed, it suffices to write $n = 2t+s$ with $t \leq s \leq 2t$ to obtain the desired result. Of course (6.2.1) also holds true for $n \leq 6$ by inspection.

Now, fix x and n such that

$$\eta\sqrt{n} < |x| \leq \frac{\eta n}{24}.$$

Let m be the smallest integer such that

$$|x| \leq \frac{\eta}{12}\sqrt{mn}.$$

Observe that $4m \leq n$ because we assume $|x| \leq \eta n/24$. Obviously, one can find points

$$\mathrm{id} = y_0, y_1, \ldots, y_{m-1}, y_m = x$$

along a geodesic path from id to x in (G, S) such that

$$|y_i^{-1}y_{i+1}|_S \leq 1 + |x|/m \leq 1 + \frac{1}{12}\eta\sqrt{n/m}, \quad i = 0, \ldots, m-1.$$

Set

$$B = B(\mathrm{id}, \rho), \quad B' = B(\mathrm{id}, 1+3\rho) \quad \text{with} \quad \rho = \frac{\eta}{12}\sqrt{n/m}.$$

Then write

$$n = \sum_0^{m-1} n_i \quad \text{with} \quad 4 \leq [n/m] \leq n_i \leq [n/m] + 1$$

and

$$\begin{aligned} \mu^{(n)}(x) &= \mu^{(n_0)} * \cdots * \mu^{(n_{m-1})}(x) \\ &\geq \sum_{z_0 \in B}\sum_{z_1 \in B} \cdots \sum_{z_{m-1} \in B} \mu^{(n_0)}(z_0)\mu^{(n_1)}(z_0^{-1}z_1)\cdots\mu^{(n_{m-1})}(z_{m-2}^{-1}x). \end{aligned} \tag{6.2.2}$$ mu-chain1

Now observe that there exists $c_0 > 0$ such that

$$\min_{z_{i-1} \in B}\min_{z_i \in B} \mu^{(n_i)}(z_{i-1}^{-1}z_i) \geq \min_{z \in B'} \mu^{(n_i)}(z) \geq c_0(1+n/m)^{-d/2} \tag{6.2.3}$$ mu-chain2

because B' has radius

$$1+3\rho = 1+\frac{\eta}{4}\sqrt{n/m} \leq \eta\sqrt{n_i}$$

and thus we can apply (6.2.1). From (6.2.2), (6.2.3) and the hypothesis that $V_S(n) \simeq (1+n)^d$, it follows that there exist c_1, c_2, c_3 such that

$$\begin{aligned}
\mu^{(n)}(x) &\geq [c_0(1+n/m)^{-d/2}]^m (\#B)^m - 1 \\
&\geq c_1^m(1+n/m)^{-d/2} \geq c_2(1+n)^{-d/2}e^{-c_3|x|^2/n}
\end{aligned}$$

for all x, n satisfying $|x| \leq \eta n/24$. Here we have used the fact that m is of order $|x|^2/n$. This estimate is easily extended to the full range $|x| \leq n$ by inspection.

It is worth noting that the Gaussian lower bound of Theorem 6.2.7 can be complemented by a matching upper bound.

Theorem 6.2.8 *Let G be a finitely generated group. Let S be a symmetric generating set containing the identity. Let $|x|$ denote the length of $x \in G$ with respect to S and set Let $\mu = (\#S)^{-1}\mathbf{1}_S$. Assume that $V_S(n) \simeq (1+n)^{-d/2}$ for some $d \geq 0$. Then there exists c_3, c_4 such that*

$$\mu^{(n)}(x) \leq c_3(1+n)^{-d/2}e^{-c_4|x|^2/n}$$

for all x, n.

We refer the reader to [20] for the proof of this result. Theorem 6.2.7 and 6.2.8 are stated and proved in [20] in greater generality for any probability measure μ on G with finite symmetric generating support containing the identity.

Chapter 7

A collection of explicit statements

It might be useful to collect some of the specific results obtained so far. This is done in the following two statements. The first theorem collects result for general graphs (or countable Markov chains), the second collects results for Cayley graphs.

Let (K, π) be a reversible Markov chain on a countable space X. We assume that K is locally finite, irreducible.

th-coll **Theorem 7.0.9** *Let V, J, ϕ be as defined in Section* sec-notmc *3.1 and let N be the Nash profile of (K, π) (Section* subsec-Nashtech *3.3.1).*

1. *For $\alpha \geq 0$, we have*
   - $\phi(t) \preceq (1+t)^{-\alpha/2} \iff N(t) \preceq (1+t)^{2/\alpha}$.
   - $J(t) \preceq (1+t)^{1/\alpha} \implies \begin{cases} V(t) & \succeq & (1+t)^{\alpha} \\ \phi(t) & \preceq & (1+t)^{-\alpha/2}. \end{cases}$
   - $\phi(t) \preceq (1+t)^{-\alpha/2} \implies \begin{cases} V(t) & \succeq & (1+t)^{\alpha} \\ J(t) & \preceq & (1+t)^{2/\alpha}. \end{cases}$
   - $V(t) \succeq (1+t)^{\alpha} \implies \phi(t) \preceq (1+t)^{-\alpha/(\alpha+1)}$.
2. *For $\alpha \geq 0$ and $-\infty < \beta < \infty$, we have*
   - $\phi(t) \preceq \exp(-[t(\log_* t)^{\beta}]^{1/(\alpha+1)}) \iff N(t) \preceq [\log_* t]^{\alpha}[\log_* \log_* t]^{-\beta}$.
   - $J(t) \preceq [\log_* t]^{\alpha/2}[\log_* \log_* t]^{-\beta/2}$

$$\implies \begin{cases} V(t) & \succeq & \exp([t(\log_* t)^{\beta/2}]^{2/(\alpha+2)}) \\ \phi(t) & \preceq & \exp(-[t(\log_* t)^{\beta}]^{1/(\alpha+1)}). \end{cases}$$

- $\phi(t) \preceq \exp(-[t(\log_* t)^{\beta}]^{1/(\alpha+1)})$

$$\Longrightarrow \begin{cases} V(t) & \succeq & \exp([t(\log_* t)^{\beta/2}]^{2/(\alpha+2)}) \\ J(t) & \preceq & [\log_* t]^{\alpha}[\log_* \log_* t]^{-\beta}. \end{cases}$$

- $\log V(t) \succeq (1+t)^{\alpha} \implies \phi(t) \preceq t^{-1}(\log_* t)^{1/\alpha}$.

Proof The references given below are for both part (1) and part (2). The first statement is from Theorem 3.3.5. The second follows from (??), Proposition 5.1.1 and Theorem 5.2.4. The third follows from Corollary 3.4.2 and Theorems 3.3.5, 5.2.1. The fourth statement is from Theorem 4.1.1.

th-coll-CG **Theorem 7.0.10** *Let (G,S) be a Cayley graph, $K(x,y)=\mu_S(x^{-1}y)$ and $\pi \equiv 1$. For $d \geq 0$ and $0<\gamma\leq 1$, we have*

- $V(t) \succeq (1+t)^d \implies \begin{cases} J(t) & \preceq & (1+t)^{1/d} \\ \phi(t) & \preceq & (1+t)^{-d/2}. \end{cases}$

- $\log V(t) \succeq (1+t)^{\gamma} \implies \begin{cases} J(t) & \preceq & [\log_* t]^{1/\gamma} \\ \phi(t) & \preceq & \exp(-t^{\gamma/(\gamma+2)}). \end{cases}$

Proof The first implication follows from Theorem 6.1.2 and the second from Corollary 6.1.3. Both take advantage of the group structure to improve very significantly on the fourth statments of part 1 and 2 of Theorem 7.0.9.

The sharpness of these results will be discussed later, in particular in Chapter 9.2 but let us note here that for any Cayley graph and any $d>0$ Theorems 7.0.9 and 7.0.10 give the equivalence

$$V(t) \succeq (1+t)^d \iff J(t) \preceq (1+t)^{1/d} \iff \phi(t) \preceq (1+t)^{-d/2}.$$

When the comparison function is not a power function, the results are more subtle. Indeed, for any Caylay graph (G,S) and $\gamma \in (0,1)$, we have

$$\log V(t) \succeq (1+t)^{\gamma} \Longrightarrow J(t) \preceq [\log_* t]^{1/\gamma}$$
$$J(t) \preceq [\log_* t]^{1/\gamma} \Longrightarrow \log V(t) \succeq (1+t)^{\gamma/(\gamma+1)}$$

and

$$\log V(t) \succeq (1+t)^{\gamma} \Longrightarrow \phi(t) \preceq \exp(-t^{\gamma/(\gamma+2)})$$
$$\phi(t) \preceq \exp(-t^{\gamma/(\gamma+2)}) \Longrightarrow \log V(t) \succeq (1+t)^{\gamma/(\gamma+1)}$$

....

This leaves open the following question

For Cayley graphs, is the behavior of ϕ tightly related to that of J?

A more technical but more precise way of asking the same question is

For Cayley graphs, does one have $J^2 \preceq \mathcal{N}_\phi$*?*

In support of a positive answer, we have Kesten's theorem which states that

$$\phi(n) \simeq e^{-n} \Longleftrightarrow J \simeq 1$$

and Varopoulos' result

$$\phi(n) \simeq n^{-\alpha/2} \Longleftrightarrow J(n) \simeq n^{1/\alpha}$$

$\alpha = 1, 2, \ldots$. However,

....

Chapter 8

Non-amenable groups and groups of intermediate growth

This chapter covers briefly two distinct topics: Non-amenability and intermediate growth.

8.1 Non-amenable groups

Let G be a discrete group. One says that G is amenable if there exists a continuous linear functional μ defined on the space $\mathbf{B}$ of all bounded functions on G which preserves positivity, preserves the constant function $\mathbf{1}$ and is invariant by left and right translations (requiring bi-invariance is the same as requiring left or right invariance). This is a well studied class of groups. It contains all finite groups, all Abelian groups, all subgroups of amenable groups, all groups that contains an amenable normal subgroup whose quotient is amenable. In particular any solvable group is amenable. We refer to Wagon [43] for motivations and to Pier [27] for a book length treatment of locally compact amenable groups.

A group which is not amenable is called non-amenable. This notion is relevant here because of celebrated results of Følner (1955) [14] and Kesten (1959) [22].

Theorem 8.1.1 *For a finitely generated Cayley graph the following properties are equivalent.*

1. *G is non-amenable.*
2. *(Følner)* $I(n) \simeq (1+n)$ *(i.e., $J \simeq 1$).*
3. *(Ketsen)* $\phi(n) \simeq \exp(-n)$.

We will not prove this theorem here but we observe that the equivalence between 2) and 3) follows from Theorems 2.4 and 2.12 and are elementary. It follows from the theorem that finitely generated non-amenable groups have exponential volume growth. As we shall see shortly there are many examples of amenable groups having exponential volume growth.

8.2 Intermediate growth

In 1968, J. Milnor and J. Wolf [24, 44] proved that finitely generated solvable groups (see the beginning of the next chapter) either have exponential volume growth or contains a nilpotent subgroup of finite index. Together with a result of H. Bass [3] this shows that, for a finitely generated solvable group, either $V(n) \simeq \exp(n)$ or $V(n) \simeq (1+n)^d$ for some integer d. For a long time the question of whether or not there exist groups with V growing faster than any polynomial but slower than any exponential was left open. It was settled by R. Grigorchuk who proved that there exists a rich class of such finitely generated groups. These groups are called groups of intermediate growth. See the survey of R. Grigorchuk [15] where further references can be found. The examples of Grigorchuk satisfy $\exp(n^{1/2}) \preceq V(n) \preceq \exp(n^\alpha)$ with $1/2 \le \alpha < 1$.

Groups of intermediate growth must be amenable. There are several classes of groups in which intermediate growth does not appear. These include the class of solvable groups (J. Wolf [44], J. Milnor [24]) and the class of finitely generated subgroups of connected Lie groups (J. Tits [36]).

The known results concerning I, J, and ϕ that are relevant for groups of intermediate growth are as follows:

Theorem 8.2.1 *Let (G,S) be a Cayley graph of a finitely generated group.*

(1) If $V(n) \succeq \exp(n^\gamma)$ for some $0 < \gamma \le 1$ then

$$I(n) \succeq \frac{n}{[\log_* n]^{1/\gamma}}, \quad J(n) \preceq [\log_* n]^{1/\gamma}, \quad \phi(n) \preceq e^{-n^{\gamma/(\gamma+2)}}.$$

(2) If either $I(n) \succeq n/[\log_* n]^{1/\gamma}$ or $\phi(n) \preceq \exp\left(-n^{\gamma/(\gamma+2)}\right)$, then

$$V(n) \succeq \exp\left(n^{\gamma/(1+\gamma)}\right).$$

Whether or not these results are sharp is not known except for (1) with $\gamma = 1$ (i.e., exponential growth) in which case we will present several examples that shows that (1) is sharp for certain groups and not sharp for others.

Chapter 9

Polycyclic groups.

This chapter focuses on polycyclic groups and, more generally, on groups that contains a polycyclic subgroup of finite index. We shall see that for polycyclic groups the behavior of and the relationships between V, I, J and ϕ is well understood. Recall the following classical definitions (see [33, 34]):

A group G is solvable (or soluble) if it admits a series of subgroups

$$G = G_1 \supset G_2 \cdots \supset G_k = \{\mathrm{id}\}$$

such that G_{i+1} is a normal subgroup of G_i and G_i/G_{i+1} is abelian for $1 \leq i < k$.

A group is polycyclic if it admits a series of subgroups

$$G = G_1 \supset G_2 \cdots \supset G_k = \{\mathrm{id}\}$$

such that G_{i+1} is a normal subgroup of G_i and G_i/G_{i+1} is cyclic for $1 \leq i < k$.

Given two subgroups H, L of a group, let $[H, K]$ be the subgroup generated by the commutators $[h, k] = hkh^{-1}k^{-1}$. The lower central series of a group H is the non-increasing sequence

$$H = H_1 \supset \cdots \supset H_i \supset \cdots$$

defined inductively by $H_i = [H_{i-1}, H]$. A group is nilpotent if and only if its lower central series terminates, i.e., there exists an integer c called the class of H such that $H_c \neq \{\mathrm{id}\}, H_{c+1} = \{\mathrm{id}\}$.

Any finitely generated nilpotent group is polycyclic. Any polycyclic group is solvable and finitely generated. Any subgroup of a polycyclic group is polycyclic. We will encounter examples of solvable non-polycyclic groups later on.

9.1 The polynomial realm

This section contains results in which the hypothesis and the conclusion involve power functions only. We do not assume that G is polycyclic nor contains a polycyclic group of finite index.

We first recal a celebrated result of M. Gromov. See [16, 37]. This theorem explains why this section is included in a chapter about polycyclic groups.

Theorem 9.1.1 (M. Gromov) *Let (G, S) be a Cayley graph. Assume that there exists a constant $\alpha \geq 0$ and an increasing sequence of integers $(n_i)_1^\infty$ such that*

$$V(n_i) \preceq n_i^\alpha.$$

Then there exists a nilpotent group $H \subset G$ of finite index in G. Further,

$$V(n) \simeq (1+n)^d \quad \textit{where} \quad d = \sum_1^c \ell \, \mathbf{rk}(H_\ell/H_{\ell+1})$$

where c is the nilpotency class of H, H_i, $1 \leq i \leq c$, is the lower central series of H and $\mathbf{rk}(H_i/H_{i+1})$ is the torsion-free rank of the Abelian group H_i/H_{i+1}. Furthermore, H can be taken to be torsion free.

The growth of nilpotent groups was first computed by H. Bass [3]. The special case of the above theorem when one assumes that G is solvable follows from work of H. Bass, J. Milnor and J. Wolf [3, 24, 44] and is easier than the general case. In this section as well as in future sections, we will distinguish between the results for which a proof is known which does not use any structure theorem and those for which all known proofs use some structure theorem. The results that depend on structure theorems are marked with **(*)**. For instance, the proof of Theorem 7.1.2 below does not use any structure theorem whereas the proof of Theorem 5.1.3 uses one, namely Gromov's theorem.

Theorem 9.1.2 *Let (G, S) be a finitely generated Cayley graph and fix $\alpha \geq 0$. The following properties are equivalent.*

(1) $V(n) \succeq (1+n)^\alpha$.

(2) $I(n) \succeq (1+n)^{(\alpha-1)/\alpha}$ *(i.e.,* $J(n) \preceq (1+n)^{1/\alpha}$*).*

(3) $\phi(n) \preceq (1+n)^{-\alpha/2}$.

Proof: It suffices to apply Theorem 5.0.1. Observe that the equivalence between (2)and (3) is proved by showing that they are equivalent to (1).

Theorem 9.1.3 (*) *Let (G, S) be a finitely generated Cayley graph. Fix $\alpha \geq 0$. The following properties are equivalent.*

(1) $\exists\, C,\ \exists\, n_i \nearrow \infty$ *such that* $V(n_i) \leq C(1+n_i)^\alpha$.

(2) $\exists\, C,\ \exists\, n_i \nearrow \infty$ *such that* $I(n_i) \leq C(1+n_i)^{(\alpha-1)/\alpha}$.

(3) $\exists\, c > 0,\ \exists\, n_i \nearrow \infty$ *such that* $J(n_i) \geq c(1+n_i)^{1/\alpha}$.

(4) $\exists\, c > 0,\ \exists\, n_i \nearrow \infty$ *such that* $\phi(n_i) \geq c(1+n_i)^{-\alpha/2}$.

(5) G contains a nilpotent subgroup H of finite index and

$$d = \sum \ell \ \mathbf{rk}(H_\ell/H_{\ell+1}) \leq \alpha$$

where H_i is the lower central series of H.

Furthermore, the last property implies that $V(n) \simeq (1+n)^d$, $I^{\uparrow}(n) \simeq (1+n)^{(d-1)/d}$, $J(n) \simeq (1+n)^{1/d}$, and $\phi(n) \simeq (1+n)^{-d/2}$.

Proof: Fix $\alpha > 0$. Assume that (2) or (3) or (4) is satisfied. Then theorem 5.0.1 shows that certainly, for any $\beta > \alpha$, there exists $n_i \nearrow \infty$ such that $V(n_i) \leq C(1+n_i)^\beta$ (if not, we would have a contradiction). This shows that if any one of (1), (2), (3) or (4) is satisfied we can use Gromov's Theorem 7.1.1 and conclude that G contains a nilpotent subgroup H of finite index. Let

$$d = \sum_{\ell \geq 1} \ell \ \mathbf{rk}(H_\ell/H_{\ell+1}).$$

Then, we have $V_G(n) \simeq V_H(n) \simeq (1+n)^d$. It follows that $d \leq \alpha$. Furthermore, by Theorem 4.1.2, Corollary 4.1.3 and Theorem 2.3.2, $V_G(n) \simeq (1+n)^d$ implies $\phi(n) \simeq (1+n)^{-d/2}$ and $I(n) \succeq (1+n)^{(d-1)/d}$. Thus, to finish the proof, we only have to show that $V_G(n) \simeq (1+n)^d$ implies $I^{\uparrow}(n) \preceq (1+n)^{(d-1)/d}$ and $J(n) \succeq (1+n)^{1/d}$.

Lemma 9.1.4 *Let (G, S) be a Cayley graph. Assume that $V_S(n) \simeq (1+n)^d$. Then there exists a sequence of finite sets $F_i \subset F_{i+1} \subset G$ such that, for all $i = 1, 2, \ldots,$*

1. *$\#F_i \simeq (1+i)^d$*
2. *$\#\partial F_i \simeq (1+i)^{d-1}$.*
3. *If $2^{r-1} < i \leq 2^r$, then F_i is a ball of radius j with $2^{r-1} < j \leq 2^r$.*

In particular, $I^{\uparrow}(n) \preceq (1+n)^{(d-1)/d}$ and $J(n) \succeq (1+n)^{1/d}$.

Proof: First, under the hypothesis of the lemma, Theorem 4.1.2 implies that the ball $B(\mathrm{id}, j)$ of radius j satisfies $\partial B(\mathrm{id}, j) \succeq (1+j)^{d-1}$. Now, fix i and let r be the integer such that $2^{r-1} < i \leq 2^r$. Then $B(\mathrm{id}, 2^r) \supset \bigcup_{2^{r-1} < \ell \leq 2^r} \{z : |z| = \ell\}$ and

$$\begin{aligned} C2^{rd} &\geq \#B(\mathrm{id}, 2^r) \\ &\geq |S|^{-1} \sum_{2^{r-1} < \ell \leq 2^r} \#\partial B(\mathrm{id}, \ell) \\ &\geq |S|^{-1} 2^{r-1} \min_{2^{r-1} < \ell \leq 2^r} \#\partial B(\mathrm{id}, \ell). \end{aligned}$$

Hence there exists j such that $2^{r-1} < j \leq 2^r$ and

$$\partial B(\mathrm{id}, j) \leq 2C\,|S|\,2^{r(d-1)} \simeq (1+j)^{d-1} \simeq (1+i)^{d-1}.$$

This ends the proof of the Lemma.

Theorem 9.1.5 *Let (G,S) be a finitely generated Cayley graph and fix $d \geq 0$. The following properties are equivalent.*

(1) $V(n) \simeq (1+n)^d$.

(2) $J(n) \simeq (1+n)^{1/d}$.

(3) $\phi(n) \simeq (1+n)^{-d/2}$.

Proof: (1) implies (3) by Theorem 2.3.2 and Corollary 4.1.3. (1) implies (2) by Theorem 4.1.2 and Lemma 7.1.4.

We now prove that (2) implies (1). Proposition 3.1.1 and (1.1.1) show that

$$J(n) \preceq (1+n)^{1/d} \quad \text{implies} \quad V(n) \succeq (1+n)^d.$$

Hence it suffices to prove that $J(n) \succeq (1+n)^{1/d}$ implies $V(n) \preceq (1+n)^d$. This follows from Theorem 4.1.2. Indeed, we have

$$(1+n)^{1/d} \preceq J(n) \preceq w(n)$$

where $w(n) = \inf\{k : V(k) > n\}$. Thus

$$V\left((1+\ell)^{1/d}\right) \preceq 1+\ell.$$

This shows that $V(n) \preceq (1+n)^d$.

The fact that (3) implies (1) follows from Corollary 2.2.5 and Corollary 4.2.4.

Remark: Stating (2) in the above theorem in terms of the isoperimetric profile I requires some care. In particular, we are not able to prove that

(2') $$I(n) \simeq (1+n)^{(d-1)/d}$$

is equivalent to (2). What can be shown is that (2) is equivalent to

$$I(n) \succeq (1+n)^{(d-1)/d} \quad \text{and} \quad c \inf_{an \leq \ell \leq bn} I(\ell) \leq (1+n)^{(d-1)/d}$$

for some $a,b,c > 0$. One can also ask whether or not (2) can be replaced by

(2") $$I^{\uparrow}(n) \simeq (1+n)^{(d-1)/d}.$$

The answer is yes but the proof uses Gromov's theorem. We will show that (1) is equivalent to (2"). "(1) implies (2")" follows from Theorem 4.1.2. To prove "(2") implies (1)", assume that there is a $\gamma > d$ such that $V(n) \succeq (1+n)^{\gamma}$. Then, Theorem 4.1.2 contradicts (2"). Thus there exists an increasing sequence of integers (n_i) such that $V(n_i) \leq An_i^{\gamma}$. By Gromov's theorem, it follows that $V(n) \simeq (1+n)^a$ for some integer a. Theorem 6.1.3. and (2") then imply that $a = d$.

Corollary 9.1.6 (*) *Let $G,S)$ be a finitely generated Cayley graph. Fix $d \geq 0$. The following property is equivalent to each of the properties (1), (2) and (3) of Theorem 7.1.5. and to (2") above.*

(4) G contains a nilpotent subgroup H of finite index and $d = \sum \ell\ \mathbf{rk}(H_\ell/H_{\ell+1})$ where H_i is the lower central series of H.

9.2 Polycyclic groups having exponential growth

chap-poly

This section describes what is known about the isoperimetric profiles I, J and the return probability ϕ when G is polycyclic and $V(n) \simeq \exp(n)$.

The following theorem gather results obtained by G. Alexopoulos, T. Coulhon and L. Saloff-Coste, W. Hebisch, C. Pittet and N. Varopoulos.

Theorem 9.2.1 *Let (G, S) be a Cayley graph of a finitely generated group G having exponential volume growth. Assume that G contains a polycyclic subgroup of finite index. Then*

1. *The isoperimetric profile satisfies $J(n) \simeq \log_* n$, $I^\uparrow(n) \simeq n/\log_*(n)$.*
2. *The probability of return after $2n$ steps satisfies $\phi(n) \simeq \exp(-n^{1/3})$.*

Example: The simplest example of a polycyclic group with exponential volume growth is given by the semidirect product $G = \mathbb{Z} \propto_\tau \mathbb{Z}^2$ where the action τ is given by matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Proof of Theorem 7.2.1: Upper bounds. Since $V(n) \simeq \exp(n)$, Theorem 4.1.2 shows that $J(n) \preceq \log_*(n)$ (also $I^\uparrow(n) \succeq n/\log_* n$) and Corollary 4.1.3 yields $\phi(n) \preceq \exp(-n^{1/3})$. The proofs of the matching lower bounds are given in Sections 7.3 and 7.4 below. Note that, by invariance of $J, I^\uparrow$ and ϕ under quasi-isometry, It suffices to prove the theorem for polycyclic groups.

The following theorem states that the results collected above in Theorems 7.1.5, 7.2.1 and in Section 6.1 are sufficient to describe precisely the possible behaviors of $V, I^\uparrow, J$ and ϕ for those finitely generated groups that can be realized as discrete subgroups of connected Lie groups.

Theorem 9.2.2 (*) *Assume that G is a discrete subgroup of a Lie group having finitely many connected components. Then*

- **either** *G is non-amenable in which case, if G is finitely generated, then $V(n) \simeq \exp(n)$ and*

$$\phi(n) \simeq \exp(-n), \quad J(n) \simeq 1, \quad I^\uparrow(n) \simeq (1+n);$$

- **or** *G is amenable, finitely generated, and*

  (a) *either $V(n) \simeq \exp(n)$, in which case*

$$\phi(n) \simeq e^{-n^{1/3}}, \quad J(n) \simeq \log_*(n), \quad I^\uparrow(n) \simeq \frac{1+n}{\log_*(n)};$$

  (b) *or $V(n) \not\simeq \exp(n)$, in which case there exists an integer d such that $V(n) \simeq (1+n)^d$ and*

$$\phi(n) \simeq (1+n)^{-d/2}, \quad J(n) \simeq (1+n)^{1/d}, \quad I^\uparrow(n) \simeq (1+n)^{1-1/d}.$$

Theorem 7.2.2 is a corollary of Theorems 7.1.5, 7.2.1, and of the following result

Theorem 9.2.3 *Let G be a discrete subgroup of a Lie group having finitely many connected components. Then either G contains a free group on two generators or G is finitely generated and contains a polycyclic subgroup of finite index.*

Theorem 7.2.3 is proved in Section 7.5 below.

9.3 Følner sets

This section contains the proof of the following result.

Theorem 9.3.1 *Let (G, S) be a Cayley graph of a polycyclic group G. Then the isoperimetric profile of (G, S) satisfies*

$$J(n) \succeq \log_* n, \quad I^{\uparrow}(n) \preceq n/\log_* n.$$

Since J and $I\uparrow$ are invariant under quasi-isometry, Theorem 7.3.1 implies similar bounds for groups containing a polycyclic group of finite index. Theorem 7.2.1(1) follows. The proof of Theorem 7.3.1 requires a number of remarks and algebraic lemmas. We start with the following simple lemma.

Lemma 9.3.2 *Let N be a group generated by a finite symmetric set B. Let H be a finitely generated group of automorphisms of N with finite symmetric generating set A. Then there exists an integer q such that*

$$\forall\, h \in H,\ \forall\, x \in N, \quad |h(x)|_B \le q^{|h|_A}\, |x|_B.$$

Proof: This follows trivially by induction if we set $q = \sup_{a\in A,\, b\in B}\{|a(b)|_B\}$. We will apply Lemma 7.3.2 when N is a normal subgroup of a larger group G, H is another subgroup of G generated by a finite set A and H acts on N by restriction to N of the inner automorphisms $g \to hgh^{-1}$, $h \in H$.

Next, we use some classical results to reduce slightly the complexity of the group G we need to consider in Theorem 7.3.1. Since G is polycyclic, there exists an integer m such that G^m is torsion free and G/G^m is finite. See [18, 33] Thus, passing to a subgroup of finite index, we can assume that G is torsion free. According to a result of Malcev ([34], 15.1.4), any finitely generated polycyclic group contains a (finitely generated) subgroup of finite index whose derived group is nilpotent. Thus, passing to a subgroup of finite index, we can assume that we have an exact sequence

$$0 \to N \to G \to \mathbb{Z}^r$$

where N is a finitely generated nilpotent group. We take advantage of (part of) this special structure in the following elementary algebraic lemma.

Lemma 9.3.3 *Let G be a finitely generated group. Assume that there is an exact sequence*

$$0 \to N \to G \to \mathbb{Z}^r$$

with N finitely generated. Let B be a finite symmetric generating set of N. Let a_i, $i = 1, \ldots, r$, be elements of G which project on the canonical basis of $\mathbb{Z}^r$ and set $A = \{a_i^\epsilon; i = 1, \ldots, r,\ \epsilon = \pm 1\}$. Let $S = A \cup B$. Then S generates G, any x in G can be written uniquely as

$$x = y\, a_1^{k_1} \cdots a_r^{k_r}, \quad y \in N, \ \ k = (k_1, \ldots, k_r) \in \mathbb{Z}^r,$$

and there exists an integer q such that

$$|y|_B \le q^{|x|_S}, \ \ |k| \le |x|_S$$

where $|k| = \sum_1^r |k_i|$.

Proof: The fact that S generates G is obvious. We will use the following notation. For any $k = (k_1, \ldots, k_r) \in \mathbb{Z}^r$, set

$$\mathbf{k} = a_1^{k_1} \cdots a_r^{k_r}.$$

By hypothesis, the set

$$K = \{\mathbf{k} : k \in \mathbb{Z}^r\} \subset G \tag{9.3.1}$$

is a section of $\mathbb{Z}^r$ in G. Thus, any $x \in G$ can be written uniquely as $x = y\,\mathbf{k}$ with $y \in N$ and $\mathbf{k} \in K$. The point of the Lemma 7.3.3 is to yield an estimate on $|y|_B$ and $|k|$ in terms of $|x|_S$. For the proof, we need the following lemma.

Lemma 9.3.4 *Referring to the setting of Lemma 7.3.3, there exists an integer q with the following property. For each $\epsilon = \pm 1$, each r-tuple $k = (k_1, \ldots, k_r) \in \mathbb{Z}^r$ and each $i \in \{1, \ldots, r\}$ there exists $x \in N$ such that*

$$a_1^{k_1} \cdots a_r^{k_r} a_i^\epsilon = x a_1^{k_1} \cdots a_i^{k_i+\epsilon} \cdots a_r^{k_r} \ \ \textit{and} \ \ |x|_B \le q^{|k|}.$$

Proof: We start with an auxiliary result. Using the notation introduced at (7.3.1), we claim that there exists an integer q such that for any $\mathbf{k} \in K$, $\epsilon = \pm 1$ and $i \in \{1, \ldots, r\}$, there exists $y \in N$ satisfying

$$\mathbf{k}\, a_i^\epsilon\, \mathbf{k}^{-1} = y\, a_i^\epsilon \text{ and } |y|_B \le q^{|k|}. \tag{9.3.2}$$

The proof is by induction on $|k| = \sum_1^r |k_\ell|$. The claim is trivial if $|k| = 0$, i.e., $\mathbf{k} = \mathrm{id}$. Assume that the claim is true if $|k| = \ell$ and fix a k with $|k| = \ell + 1$. Let $j = \max\{\ell : k_\ell \neq 0\}$ and set

$$k_j' = \begin{cases} k_j - 1 & \text{if } k_j > 0 \\ k_j + 1 & \text{if } k_j < 0, \end{cases}$$

and $k' = (k_1, \ldots, k_{j-1}, k_j', 0, \ldots, 0)$, $\epsilon' = k_j - k_j'$. Set also

$$z = [a_j^{\epsilon'}, a_i^\epsilon] = a_j^{\epsilon'} a_i^\epsilon a_j^{-\epsilon'} a_i^{-\epsilon}.$$

Then,

$$\begin{aligned}\mathbf{k}\, a_i^{\epsilon}\, \mathbf{k}^{-1} &= \mathbf{k}'\, a_j^{\epsilon'}\, a_i^{\epsilon} a_j^{-\epsilon'}\, (\mathbf{k}')^{-1} \\ &= \mathbf{k}'\, z\, a_i^{\epsilon}\, (\mathbf{k}')^{-1} = \mathbf{k}'\, z\, (\mathbf{k}')^{-1}\, \mathbf{k}'\, a_i^{\epsilon}\, (\mathbf{k}')^{-1}.\end{aligned}$$

Letting the group generated by A act on N by restriction of inner automorphisms in G, Lemma 7.3.2 yields

$$|\mathbf{k}'\, z\, (\mathbf{k}')^{-1}|_B \leq |z|_B\, q^{|k'|} \leq M q^{|k|-1} \tag{9.3.3}$$

where

$$M = \max\{|[\alpha, \alpha']|_B : \alpha, \alpha' \in A\}.$$

Furthermore, by the inductive hypothesis,

$$\mathbf{k}'\, a_i^{\epsilon}\, (\mathbf{k}')^{-1} = y'\, a_i^{\epsilon} \text{ with } |y'|_B \leq q^{|k'|} = q^{|k|-1}. \tag{9.3.4}$$

Thus, $\mathbf{k}\, a_i^{\epsilon}\, \mathbf{k}^{-1} = y\, a_i^{\epsilon}$ with $y = \mathbf{k}'\, z\, (\mathbf{k}')^{-1}\, y'$. By (7.3.3), (7.3.4), it follows that $|y|_B \leq (M+1)q^{|k|-1}$. This finish the proof of the claim (7.3.2) if $q \geq (M+1)$, which we can assume is satisfied from the beginning.

To finish the proof of Lemma 7.3.4, we write $a_1^{k_1} \cdots a_r^{k_r} a_i^{\epsilon} = \mathbf{k}_i\, \mathbf{k}^i\, a_i^{\epsilon}$ with

$$\mathbf{k}_i = a_1^{k_1} \cdots a_{i-1}^{k_{i-1}} \quad \mathbf{k}^i = a_i^{k_i} \cdots a_r^{k_r}.$$

Set $\ell = \sum_{j=1}^{i-1} |k_j|$, $\ell' = \sum_{j=i}^{r} |k_j|$. Then, applying (7.3.2), there exists $y \in N$ with $|y|_B \leq q^{\ell'}$ such that

$$\mathbf{k}_i\, \mathbf{k}^i\, a_i^{\epsilon} = \mathbf{k}_i\, y\, a_i^{\epsilon} \mathbf{k}^i.$$

Now, Lemma 7.3.2 yields

$$\mathbf{k}_i\, y = x\, \mathbf{k}_i$$

with $x \in N$ and $|x|_B \leq |y|_B q^{\ell} \leq q^{\ell+\ell'} = q^{|k|}$. Finally, we get

$$\begin{aligned}a_1^{k_1} \cdots a_r^{k_r} a_i^{\epsilon} &= \mathbf{k}_i\, \mathbf{k}^i\, a_i^{\epsilon} \\ &= \mathbf{k}_i\, y\, a_i^{\epsilon}\, \mathbf{k}^i = x \mathbf{k}_i\, a_i^{\epsilon}\, \mathbf{k}^i \\ &= x a_1^{k_1} \cdots a_i^{k_i+\epsilon} \cdots a_r^{k_r}\end{aligned}$$

with $x \in N$ and $|x|_B \leq q^{|k|}$, as desired.

Proof of Lemma 7.3.3: Fix an integer $q \geq 2$ so large that the conclusion of Lemma 7.3.2 (with $H = \langle A \rangle$ acting on N by restriction of the inner automorphisms in G) and Lemma 7.3.4 are satisfied. We will prove Lemma 7.3.3 for this q. Only the estimates on $|y|_B$ and $|k|$ need to be proved. We proceed by induction on $|x|_S$. If $|x|_S = 0$, the result is trivial. Assume that we have proved the desired estimates if $|x|_S \leq \ell$. Let x be such that $|x|_S = \ell + 1$ and write $x = x'\, s$ with $|x'|_S = \ell$, $s \in S = A \cup B$. By the induction hypothesis,

$$x' = y'\, \mathbf{k}' \;\text{ with }\; |y'|_B \leq q^{\ell}, \;\; |k'| \leq \ell.$$

We consider two cases, depending on whether $s \in A$ or $s \in B$.

If $s \in A$, Lemma 7.3.4 yields

$$\mathbf{k}' s = z \mathbf{k} \text{ with } |z|_B \leq q^{|k'|} \leq q^\ell \text{ and } |k| = |k'| + 1 \leq \ell + 1.$$

Thus $x = y \mathbf{k}$ with $y = y' z$ and $\mathbf{k}$ satisfying

$$|y|_B \leq 2 q^\ell \leq q^{\ell+1} = q^{|x|_S}, \quad |k| \leq |x|_S$$

since $q \geq 2$.

If $s \in B$, we can write

$$x = y' \xi \mathbf{k}$$

with

$$\mathbf{k} = \mathbf{k}' \text{ and } \xi = \mathbf{k}' s (\mathbf{k}')^{-1}.$$

By Lemma 7.3.2, $|\xi|_B \leq q^\ell$. Hence $y = y' \xi$ satisfies $|y| \leq 2q^\ell \leq q^{|x|_S}$. This ends the proof of Lemma 7.3.3.

We will need two refinements of Lemma 7.3.3. The first of these two results is crucial for constructing Følner sequences. The second result will be used only in next sextion, in the proof of the lower bound on the probability of return ϕ.

Lemma 9.3.5 *Referring to the setting of Lemma* 7.3.3*, there exists an integer q such that, for any $x \in G$ and any $s \in S$, if $x = y \mathbf{k}$ with $y \in N$, $\mathbf{k} \in K$, then*

$$xs = z \mathbf{k}' \text{ with } z \in N, \ \mathbf{k}' \in K \text{ and } |z|_B \leq |y|_B + q^{|k|}, \ |k'| \leq |k| + 1.$$

Proof: Let q be an integer so large that Lemma 7.3.2 and 7.3.4 are satisfied. If $s \in A$, apply Lemma 7.3.4 to write

$$x = y \mathbf{k} s = y z \mathbf{k}'$$

with $|k'| = |k| + 1$ and $|z|_B \leq q^{|k|}$. If $s \in B$, write

$$x = y \mathbf{k} s = y z \mathbf{k}$$

with $z = \mathbf{k} s \mathbf{k}^{-1}$ and use Lemma 7.3.2 to see that $|z|_B \leq q^{|k|}$.

Lemma 9.3.6 *Referring to the setting of Lemma* 7.3.3*, let ψ be the canonical projection $G \mapsto G/N = \mathbb{Z}^r$. Given a sequence $\sigma = (s_i)_1^\ell \in S^\ell$, let*

$$k(i) = \psi(s_1 \cdots s_i) = (k_1(i), \dots, k_r(i))$$

and set

$$t(\sigma) = \max_{1 \leq i \leq \ell} |k(i)|.$$

Then there exists an integer q such that, for any integer ℓ and any sequence $\sigma = (s_i)_1^\ell \in S^\ell$, we have

$$s_1 \dots s_\ell = y \mathbf{k} \text{ with } y \in N, \ \mathbf{k} \in K$$

and

$$|y|_B \leq \ell q^{t(\sigma)}, \ |k| \leq t(\sigma).$$

Proof: We proceed by induction on ℓ. If $\ell = 0$ there is nothing to prove. Assuming that we have the desired result for any sequence of length ℓ of elements of S, we prove it for $\sigma = (s_1, \dots, s_{\ell+1})$, $s_i \in S$. To this end, write

$$x = s_1 \cdots s_{\ell+1} = x' s_{\ell+1}$$

with $x' = s_1 \cdots s_\ell$. By the induction hypothesis, $x' = y'\,\mathbf{k}'$ with $|y'|_B \le \ell\, q^{t(\sigma')}$, $|k'| \le t(\sigma')$, $\sigma' = (s_1, \dots, s_\ell)$. By Lemma 7.3.5, we have

$$x = x' s_{\ell+1} = y'\,\mathbf{k}'\, s_{\ell+1} = y\;\mathbf{k}$$

with

$$|y|_B \le |y'|_B + q^{|k'|} \le \ell\, q^{t(\sigma')} + q^{|k'|} \le (\ell+1) q^{t(\sigma')}$$

and $\mathbf{k} = \psi(s_1 \cdots s_{\ell+1})$ thus $|k| \le t(\sigma)$. The proof is complete since $t(\sigma') \le t(\sigma)$.

Remark: The difference between Lemma 7.3.6 and Lemma 7.3.3 is rather subtle. On the one hand observe that $t(\sigma)$ is bounded by ℓ for any sequence of length ℓ, but that it can in fact be much smaller. On the other hand, $|s_1 \cdots s_\ell|_S$ can be much smaller than ℓ.

Finaly, we need a variant of the construction used in Lemma 7.1.4 to produce Følner sequences in groups of polynomial growth.

Lemma 9.3.7 *Let (N, B) be the Cayley graph of a finitely generated group N. Assume that $V_B(n) \simeq (1+n)^d$ for some $d > 0$. Then for any integer $R \ge 2$, there exists an increasing sequence of finite sets $D(i)$ such that*

1. *For each i, $D(i) = B(\mathrm{id}, \rho(i))$ for some $\rho(i)$ satisfying*

$$R^{2i} \le \rho(i) \le R^{2(i+1)} - (R^2 - 1)R^i.$$

   In particular there exist c_0, C_0 such that $c_0 R^{2di} \le \#D(i) \le C_0 R^{2di}$.

2. *There exists C_1 such that, for each i and $\rho(i)$ as above, the set $C(i) = B(\mathrm{id}, \rho(i) + (R^2-1)R^i) \setminus B(\mathrm{id}, \rho(i))$ satisfies $\#C(i) \le C_1 R^{(2d-1)i}$.*

Proof: Consider the annulus

$$A(i) = B(\mathrm{id}, R^{2(i+1)}) \setminus B(\mathrm{id}, R^{2i}).$$

Subdivise this annulus into the union of the annulii

$$A(i,\ell) = B(\mathrm{id}, R^{2i} + (\ell+1)R^i(R^2-1)) \setminus B(\mathrm{id}, R^{2i} + \ell(R^2-1)R^i)$$

where ℓ varies from 0 to $R^i - 1$. Then

$$\begin{aligned} \#A(i) &= \sum_{\ell=0}^{R^i-1} \#A(i,\ell) \\ &\ge R^i \min_{\ell \in \{0,\dots,R^i-1\}} \{\#A(i,\ell)\}. \end{aligned}$$

It follows that there exist a constant C_1 and an integer $\ell_0 \in \{0, \dots, R^i - 1\}$ such that

$$\#A(i, \ell_0) \le C_1 R^{(2d-1)i}.$$

Setting

$$D(i) = B(\mathrm{id}, \rho(i)), \;\; \rho(i) = R^{2i} + \ell_0 (R^2 - 1) R^i,$$

we find that $C(i) = A(i, \ell_0)$ and thus $\#C(i) \le C_1 R^{(2d-1)i}$ as desired. This ends the proof of Lemma 7.3.7.

Proof of Theorem 7.3.1: We will prove the following more precise result.

Lemma 9.3.8 *Let G be a finitely generated group. Assume that there is an exact sequence*

$$0 \to N \to G \to \mathbb{Z}^r$$

with N a nilpotent finitely generated group. Let B be a finite symmetric generating set of N. Let A be as in Lemma 7.3.3 *and consider the Cayley graph (G, S) where $S = A \cup B$. Then there exist a constant C, an integer M, and an increasing sequence of finite sets $F(i)$ with the following properties.*

1. *For each positive integer j there is at least one i such that*

$$M^j \le \#F(i) \le M^{j+1}.$$

2. *For each integer i,*

$$\#\partial F(i) \le \frac{C \#F(i)}{\log_*[\#F(i)]}.$$

Before embarking with the proof, observe that this yields

$$J(n) \succeq \log_*(n), \;\; I^{\uparrow}(n) \preceq \frac{n}{\log_* n}$$

for any group G as in the lemma. By passing to a subgroup of finite index, this proves Theorem 7.3.1.

Returning to the proof of Lemma 7.3.8, we observe that N being nilpotent, there exists a $d \ge 0$ such that the volume growth function of (N, B) satisfies $V_B(n) \simeq (1+n)^d$. Thus we can apply Lemma 7.3.7 to the Cayley graph (N, B). Let q be the integer given by Lemma 7.3.5. Fix $R = q$ and let $D(i) \subset N$ be the sets given by Lemma 7.3.7. In G, consider the finite subsets

$$F(i) = \{x = y\,\mathbf{k} \in G : y \in D(i) \subset N, \;\; \mathbf{k} \in K, |k| \le i\}. \tag{9.3.5}$$

These sets clearly form an increasing sequence and there exists two positive constants c_2, C_2 such that

$$c_2 i^r R^{2di} \le \#F(i) \le C_2 i^r R^{2di}. \tag{9.3.6}$$

Hence

$$\sup_i \frac{\#F(i+1)}{\#F(i)} < \infty$$

which shows that one can pick M large enough so that conclusion 1 of Lemma 7.3.8 is satisfied.

The main part of the proof is to estimate the size of the boundary of $F(i)$. Here it is convenient to work with the boundary $\delta F(i)$ instead of $\partial F(i)$. Recall that $\delta F(i)$ is the set of point in $F(i)$ which have at least one neighbor in $F(i)^c$. See Chapter 1, Section 1.

Fix an integer i. Let $\rho(i)$, $R^{2i} \leq \rho(i) < R^{2(i+1)}$ be the integer given by Lemma 7.3.7 so that $D(i) = B_N(\mathrm{id}, \rho(i)) \subset N$. By Lemma 7.3.5, if $x \in F(i)$ and $s \in S$, $xs = z\,\mathbf{k}'$ with $|z|_B \leq \rho(i) + R^i$ and $|k'| \leq i+1$. It follows that $\partial F(i)$ is contained in the union of the sets

$$\{x = y\,\mathbf{k} \in G : y \in N,\ \mathbf{k} \in K,\ |y|_B \leq \rho(i),\ |k| = i\}$$

and

$$\{x = y\,\mathbf{k} \in G : y \in C(i),\ \mathbf{k} \in K,\ |k| \leq i\}$$

with $C(i) \subset N$ as in Lemma 7.3.7. Of course,

$$\#\{\mathbf{k} \in K,\ |k| = i\} \simeq (1+i)^{r-1}.$$

Hence there exist constants C_3, C_4 such that

$$\#\partial F(i) \leq C_3 \left((1+i)^{r-1} R^{2di} + (1+i)^r R^{(2d-1)i}\right) \leq C_4 \left(\frac{1}{i} + R^{-i}\right) \#F(i).$$

Since $\log_*[\#F(i)] \simeq i$, it follows that there exists a constant C such that

$$\#\partial F(i) \leq \frac{C\#F(i)}{\log_*[\#F(i)]}.$$

This ends the proof of Lemma 7.3.8.

9.4 Lower bound on ϕ

In this section we prove the following result.

Theorem 9.4.1 *Let (G, S) be a Cayley graph of a polycyclic group G. Then the probability of return after $2n$ steps satisfies*

$$\phi(n) \succeq \exp(-n^{1/3}).$$

By invariance under quai-isometry, this implies that the same result holds for groups containing a polycyclic subgroup of finite index. As in Section 7.3, it suffices to prove this result for groups G admitting an exact sequence

$$0 \to N \to G \to \mathbb{Z}^r$$

where N is a finitely generated nilpotent group. Let ψ denote the canonical projection $\psi : G \mapsto G/N = \mathbb{Z}^r$. Fix a symmetric generating set B in N and

$a_1, \dots, a_r \in G$ such that $\psi(a_1) = e_1, \dots, \psi(a_r) = e_r$ where $(e_1, \dots, e_r)$ is the canonical basis of $\mathbb{Z}^r$. As in Lemma 7.3.3, set

$$S = A \cup B$$

where $A = \{a_i^{\pm 1}, i = 1, \dots, r\}$. Recall that, if $k = (k_1, \dots, k_r) \in \mathbb{Z}^r$, then

$$|k| = \sum_1^r |k_i| \text{ and } \mathbf{k} = a_1^{k_1} \cdots a_r^{k_r}.$$

As in 7.3.1, we set $K = \{\mathbf{k} : k \in \mathbb{Z}^r\}$. Thus any $x \in G$ can be written uniquely as

$$x = y\,\mathbf{k} \quad \text{with} \quad y \in N \text{ and } \mathbf{k} \in K.$$

We set $\mu(g) = \frac{1}{\#S}\mathbf{1}_S(g)$ and consider the left-invariant random walk $(X_n)_{n\geq 0}$ on G with independent increments $X_n^{-1}X_{n+1}$ of law μ. In other words, $X_n = \xi_0\xi_1 \dots \xi_n$ where each ξ_i, $i \geq 1$, is picked independently and uniformly (i.e., according to μ) in S.

Recall that this induces a family of probability measure $\mathbf{P}_x$, indexed by the starting point $x \in G$, on the path space $G^{\mathbb{N}}$ so that

$$\mathbf{P}_x(X_n = y) = \mu^{(n)}(x^{-1}y).$$

Let $\alpha = \#A$, $\beta = \#B$ so that $\#S = \alpha + \beta$, and consider the random variable $Z_n = \psi(X_n) \in \mathbb{Z}^r$. Then

$$Z_n = Z_0 + \sum_1^n \zeta_i$$

where $\zeta_i = \psi(X_{i-1}^{-1}X_i) = \psi(\xi_i)$ with ξ_i is distributed according to μ. Thus $(Z_n)_{n\geq 0}$ can be interpreted as a random walk on $\mathbb{Z}^r$ with independent identically distributed increments ζ_i of law ν given by

$$\nu(\zeta = 0) = \frac{\beta}{\alpha + \beta}, \quad \nu(\zeta = \pm e_j) = \frac{\alpha}{2r(\alpha + \beta)}, \; j = 1, \dots, r. \tag{9.4.1}$$

The law ν induces a family of probability measures $\mathbb{P}_z$, indexed by the starting point $z \in \mathbf{Z}^r$, on the path space $[\mathbb{Z}^r]^{\mathbb{N}}$ so that

$$\mathbb{P}_z(Z_n = z') = \nu^{(n)}(z' - z).$$

Let

$$M_n = \max_{0\leq j\leq n} |Z_j|. \tag{9.4.2}$$

Let q be the integer given by Lemma 7.3.6 and consider the sets

$$D_m^\ell = \{x = y\,\mathbf{k} \in G : |y|_B \leq \ell\, q^m, \;\; |k| \leq m\}.$$

Lemma 7.3.6 has the following consequence.

Lemma 9.4.2 *With the notation introduced above, for all integers ℓ and m,*

$$\mathbf{P}_{\mathrm{id}}(X_\ell \in D_m^\ell) \geq \mathbb{P}_0(M_\ell \leq m).$$

Proof: For any sequence $\sigma = (s_1, \ldots, s_\ell)$, let

$$k(i) = \psi(s_1 \cdots s_i), \quad t(\sigma) = \max_{1 \leq i \leq \ell} |k(i)|.$$

Then Lemma 7.3.6 says that

$$\begin{aligned} &\{x = s_1 \cdots s_\ell : \sigma = (s_i)_1^\ell \in S^\ell,\ \ t(\sigma) \leq m\} \\ &\qquad \subset\ \{x = y\,\mathbf{k} : y \in N,\ \mathbf{k} \in K,\ |y|_B \leq \ell q^m,\ |k| \leq m\}. \end{aligned}$$

Thus

$$\mathbb{P}_0(M_\ell \leq m) = \mathbf{P}_{\mathrm{id}}\left(\sup_{1 \leq i \leq \ell} |Z_i| \leq m\right) \leq \mathbf{P}_{\mathrm{id}}(X_\ell \in D_m^\ell).$$

Proof of Theorem 7.4.1: Recall from Lemma 2.1.1 that

$$\mu^{(2\ell)}(\mathrm{id}) = \sup_{g \in G} \mu^{(2\ell)}(g).$$

Thus, for any finite set $D \subset G$,

$$\mathbf{P}_{\mathrm{id}}(X_{2\ell} \in D) = \sum_{g \in D} \mu^{(2\ell)}(g) \leq [\#D]\, \mu^{(2\ell)}(\mathrm{id}).$$

Applying this to $D = D_m^{2\ell}$, and using Lemma 7.4.2, we get

$$\phi_S(\ell) = \mu^{(2\ell)}(\mathrm{id}) \geq \left[\#D_m^{2\ell}\right]^{-1} \mathbb{P}_0(M_{2\ell} \leq m). \tag{9.4.3}$$

Since the group N is nilpotent, $V_B(n) \simeq (1+n)^d$ for some integer $d \geq 0$. Thus there exists a constant C_0 such that

$$\#D_m^{2\ell} \leq C_0(1+m)^r(1+\ell)^d q^{dm} \leq C_0(1+\ell+m)^{r+d} q^{dm}. \tag{9.4.4}$$

Suppose that there exists a constant C_1 such that $\mathbb{P}_0(M_{2\ell} \leq m) \geq e^{-C_1\ell/m^2}$. Then (7.4.3), (7.4.4) yield

$$\phi_S(\ell) \geq \exp\left(-C_1 \frac{\ell}{m^2} - C_2\, m - C_3\, \log \ell\right)$$

for all $\ell, m \geq 2$ with $C_2 = d \log q$, $C_3 = \log(r+d)$. Taking $m = \ell^{1/3}$ yields the result stated in Theorem 7.4.1. Thus the proof of Theorem 7.4.1 reduces to the following result in random walk on $\mathbb{Z}^r$.

Lemma 9.4.3 *Let ν be the probability measure on $\mathbb{Z}^r$ given by* (7.4.1) *with $\alpha, \beta > 0$. Consider the random walk $(Z_n)_{n \geq 0}$ generated by ν and let $(M_n)_{n \geq 0}$ be as in* (7.4.2)*. Then there exists a constant C such that*

$$\mathbb{P}_0(M_\ell \leq m) \geq e^{-C\ell/m^2}.$$

This lemma is essentially well known although it seems hard to find a reference in text books. We give a complete proof for the convenience of the reader.

Lemma 9.4.4 *Referring to the setting of Lemma* 7.4.3*,*

$$\mathbb{P}_0(M_\ell \geq m) \leq 2\,\mathbb{P}_0(|Z_\ell| \geq m).$$

Proof: This follows from what is known as the reflection principle. Let τ the first time that $(Z_n)_{n\geq 0}$ hits the set of $S(m) = \{k \in \mathbb{Z}^r : |k| = m\}$. Because the walk takes only nearest neighbor steps, it has to hit $S(m)$ to exit $B = B(0,m) = \{k \in \mathbb{Z}^r : |k| \leq m\}$. Hence

$$\mathbb{P}_0(M_\ell \geq m) = \mathbb{P}_0(\tau \leq m).$$

Write

$$\mathbb{P}_0(\tau \leq m) = \mathbb{P}_0(\tau \leq m; |Z_\ell| \leq m) + \mathbb{P}_0(\tau \leq m; |Z_\ell| \geq m).$$

Then, using the Markov property,

$$\mathbb{P}_0(\tau \leq m; |Z_\ell| \geq m) = \sum_{1\leq j\leq m} \sum_{y:|y|=\ell} \mathbb{P}_0(\tau = j; Z_\tau = y)\mathbb{P}_y(|Z_{\ell-j}| \geq m) \quad (9.4.5)$$

and, similarily,

$$\mathbb{P}_0(\tau \leq m; |Z_\ell| \leq m) = \sum_{1\leq j\leq m} \sum_{y:|y|=\ell} \mathbb{P}_0(\tau = j; Z_\tau = y)\mathbb{P}_y(|Z_{\ell-j}| \leq m). \quad (9.4.6)$$

Fix $y \in S(m)$. The point y belongs to one of the faces of dimension $r-1$ of the convexe polytope B. These faces are the intersections of B with the hyperplanes

$$H^m_\epsilon = \left\{ y : \sum_{i=1}^r \epsilon_i y_i = m \right\}$$

where $\epsilon = (\epsilon_i)_1^r$ with $\epsilon_i = \pm 1$. Let R^m_ϵ be the affine reflection through the hyperplane H^m_ϵ. Observe that the random walk $(Z_n)_{n\geq 0}$ has the symmetry property

$$\mathbb{P}_x(Z_n = z) = \mathbb{P}_{R^m_\epsilon(x)}(Z_n = R^m_\epsilon(z)).$$

with respect to each R^m_ϵ. This is because the reflections R^m_ϵ are automorphisms of the usual graph structure of $\mathbb{Z}^r$ and, moreover,

$$\nu(z - x) = \nu(R^m_\epsilon(z) - R^m_\epsilon(z)).$$

Let $\epsilon = \epsilon(y)$ be such that $y \in B \cap H^m_\epsilon$. Let $B' = B(z_0, m) = R^m_\epsilon(B)$. By construction, B' lies in $\{y \in \mathbb{Z}^r : |y| \geq m\}$. Moreover, since $R^m_\epsilon(y) = y$, $R^m_\epsilon(B) = B'$, we have

$$\mathbb{P}_y(|Z_n| \leq m) = \mathbb{P}_y(|Z_n| \in B) = \mathbb{P}_y(|Z_n| \in B') \leq \mathbb{P}_y(|Z_n| \geq m). \quad (9.4.7)$$

By (7.4.5), (7.4.6) and (7.4.7), we have

$$\mathbb{P}_0(\tau \leq m; |Z_\ell| \leq m) \leq \mathbb{P}_0(\tau \leq m; |Z_\ell| \geq m).$$

Hence

$$\mathbb{P}_0(\tau \leq m) \leq 2\,\mathbb{P}_0(\tau \leq m; |Z_\ell| \geq m) = 2\,\mathbb{P}_0(|Z_\ell| \geq m).$$

Lemma 9.4.5 *Referring to the setting of Lemma* 7.4.3, *we have*

$$\mathbb{P}_0(|Z_\ell| \geq m) \leq 2r\, e^{-m^2/[2r^2\,\ell]}.$$

Proof: First let us note that the constants in this estimate are not optimal. Still this bound will suffices for our purpose and it has a simple proof. For $k = (k_1, \ldots, k_r)$, let $\|k\| = \max\{|k_i| : 1 \leq i \leq r\}$ be the sup-norm of k. Write $Z_\ell = (Z_\ell^i)_1^r$ in coordinates. Of course, $|Z_\ell| \leq r\|Z_\ell\|$. Thus

$$\mathbb{P}_0(|Z_\ell| \geq m) \leq \mathbb{P}_0(\|Z_\ell\| \geq m/r) \leq r\,\mathbb{P}_0(|Z_\ell^1| \geq m/r).$$

Now, Z_ℓ^1 is distributed as a one dimensional random walk with independent increments taking values $0, \pm 1$ with probability

$$1 - \frac{\alpha}{r(\alpha+\beta)} \text{ and } \frac{\alpha}{2r(\alpha+\beta)} \text{ respectively.}$$

Denote by T a random variable with this law. Hence, for any $s, t > 0$,

$$\begin{aligned}
\mathbb{P}_0(|Z_\ell^1| \geq t) &= 2\,\mathbb{P}_0\left(Z_\ell^1 \geq t\right) = 2\,\mathbb{P}_0\left(e^{s\,Z_\ell^1} \geq e^{st}\right) \\
&\leq 2e^{-st} E_0\left(e^{s\,Z_\ell^1}\right) = 2e^{-st}\left[E_0\left(e^{s\,T}\right)\right]^\ell \\
&\leq 2e^{-st}\left[1 - \frac{\alpha}{r(\alpha+\beta)} + \frac{\alpha}{r(\alpha+\beta)}\cosh(s)\right]^\ell \\
&\leq 2e^{-st}\left[\cosh(s)\right]^\ell \leq 2e^{-st+\frac{1}{2}s^2\ell}.
\end{aligned}$$

Here, we have used the elemetary inequality $\cosh(s) \leq e^{\frac{1}{2}s^2}$ which can be proved by comparing the power series of both side. Setting $s = t/\ell$ yields

$$\mathbb{P}_0(|Z_\ell^1| \geq t) \leq 2e^{-\frac{t^2}{2\ell}}.$$

For $t = m/r$, this proves Lemma 7.4.5.

Corollary 9.4.6 *Referring to the setting of Lemma* 7.4.3, *we have*

$$\mathbb{P}_x(M_\ell \geq m) \leq 4r\, e^{-m^2/[8r^2\,\ell]}$$

for all $x \in \mathbb{Z}^r$ *such that* $|x| \leq m/2$.

Proof: This follows from Lemmas 7.4.4, 7.4.5 and

$$\begin{aligned}
\mathbb{P}_x(M_\ell \geq m) &= \mathbb{P}_0\left(\sup_{0\leq i\leq \ell} |x + Z_i| \geq m\right) \\
&\leq \mathbb{P}_0(M_\ell \geq m - |x|) \leq \mathbb{P}_0(M_\ell \geq m/2) \\
&\leq 2\mathbb{P}_0(|Z_\ell| \geq m/2) \leq 4r\, e^{-m^2/[8r^2\,\ell]}.
\end{aligned}$$

Lemma 9.4.7 *Referring to the setting of Lemma* 7.4.3*, there exists* $\epsilon \in (0,1)$*such that*

$$\mathbb{P}_x(|Z_\ell| \geq m/2) \leq 1-\epsilon$$

for all m, ℓ *and all* $x \in \mathbb{Z}^r$ *such that* $\ell \leq m^2$*,* $|x| \leq m/2$*.*

Proof: This can be proved using the central limit theorem. An alternative (more elementary) is as follows. By Theorem 4.2.6,

$$\mathbb{P}_x(|Z_\ell| \leq m/2) \geq c_1 \sum_{|y|\leq m/2} (1+\ell)^{-r/2} \exp(-c_2|y-x|^2/\ell).$$

Let x_ℓ be a point at distance $[\sqrt{\ell}]$ from x and such that $|x\ell| = |x| - [\sqrt{\ell}] \leq m/2 - [\sqrt{\ell}]$. Such a point can be found by moving along a geodesic path from 0 to x. Then, $B(x_\ell, [\sqrt{\ell}]) \subset B(0, m/2)$, $\#B(x_\ell, [\sqrt{\ell}]) \simeq (1+\ell)^{r/2}$ and $d(x,y) \leq 2\sqrt{\ell}$ for all $y \in B(x_\ell, [\sqrt{\ell}])$. Hence,

$$\mathbb{P}_x(|Z_\ell| \leq m/2) \geq c_3 \sum_{|y-x_\ell|\leq[\sqrt{\ell}]} (1+\ell)^{-r/2} \geq \epsilon.$$

Lemma 9.4.8 *Referring to the setting of Lemma* 7.4.3*, there exist* $\eta, \epsilon \in (0,1)$ *such that*

$$\mathbb{P}_x(M_\ell \leq m;\ |Z_\ell| \leq m/2) \geq \epsilon$$

for all m, ℓ *and all* $x \in \mathbb{Z}^r$ *such that* $\ell \leq \eta m^2$*,* $|x| \leq m/2$*.*

Proof: Fix x, ℓ, m with $|x| \leq m/2$, $\ell \leq m^2$, and write

$$\mathbb{P}_x(M_\ell \leq m;\ |Z_\ell| \leq m/2) \geq 1 - \mathbb{P}_x(M_\ell| \leq m) - \mathbb{P}_x(|Z_\ell| \leq m/2).$$

Then, by lemma 7.4.7 and Corollary 7.4.6, there exists $\epsilon > 0$ such that

$$\mathbb{P}_x(M_\ell \leq m;\ |Z_\ell| \leq m/2) \geq \epsilon - 4re^{-m^2/[8r^2\ell]}.$$

It follows that

$$\mathbb{P}_x(M_\ell \leq m;\ |Z_\ell| \leq m/2) \geq \epsilon/2$$

for all x, ℓ, m with $|x| \leq m/2$, $\ell \leq \eta m^2$ if $\eta^{-1} = 8r^2 \log(8r/\epsilon)$.

Proof of Lemma 7.4.3: Let η, ϵ be as given by Lemma 7.4.8. Fix m such that $m^2 \geq 1/\eta$. We are going to prove by induction on n that

$$\mathbb{P}_x(M_\ell \leq m;\ |Z_\ell| \leq m/2) \geq \epsilon^{(1+n)} \tag{9.4.8}$$

if $|x| \leq m/2$ and $n[\eta m^2] < \ell \leq (n+1)[\eta m^2]$. Lemma 7.4.8 gives the desired result if $n = 0$. Assume that (7.4.8) holds for n and let ℓ be such that $(n+1)[\eta m^2] < \ell \leq (n+2)[\eta m^2]$. Set $\ell' = (n+1)[\eta m^2]$. Then

$$\begin{aligned}
&\mathbb{P}_x(M_\ell \leq m;\ |Z_\ell| \leq m/2) \\
&\quad \geq \ \mathbb{P}_x(M_\ell \leq m;\ |Z_{\ell'}| \leq m/2;\ |Z_\ell| \leq m/2)
\end{aligned}$$

$$
\begin{aligned}
&\geq \mathbb{P}_x\left(M_{\ell'} \leq m;\ \sup_{\ell' \leq i \leq \ell} |Z_{\ell'} + (Z_\ell - Z_{\ell'})| \leq m;\ |Z_{\ell'}| \leq m/2;\ |Z_\ell| \leq m/2\right)\\
&= E_x\left(\mathbf{1}_{\{M_{\ell'} \leq m;\ |Z_{\ell'}| \leq m/2\}} \mathbb{P}_{Z'_\ell}\left(M_{\ell-\ell'} \leq m;\ |Z_{\ell-\ell'}| \leq m/2\right)\right)\\
&\geq \epsilon E_x\left(\mathbf{1}_{\{M_{\ell'} \leq m;\ |Z_{\ell'}| \leq m/2\}}\right) = \epsilon \mathbb{P}_x\left(M_{\ell'} \leq m;\ |Z_{\ell'}| \leq m/2\right)\\
&\geq \epsilon^{n+2}.
\end{aligned}
$$

This proves (7.4.8). Since n is of order ℓ/m^2 we get that there exists $C > 0$ such that

$$\mathbb{P}_x(M_\ell \leq m;\ |Z_\ell| \leq m/2) \geq e^{-C\ell/m^2}$$

for all $m \geq 1/\sqrt{\eta}$, $|x| \leq m/2$ and all ℓ. The case where $m \leq 1/\sqrt{\eta}$ is easily treated by inspection. In particular,

$$\mathbb{P}_0(M_\ell \leq m) \geq e^{-C\ell/m^2}$$

as stated in Lemma 7.4.3.

9.5 Discrete subgroups of Lie groups

The aim of this section is to provide a proof of Theorem 7.2.3 which states the following: *Let Γ be a discrete subgroup of a Lie group with finitely many connected components. Then either Γ contains a free subgroup on two generators or Γ contains a finite index subgroup which is polycyclic.*

Tits alternative [36] applied to a Lie group G with a finite number of connected components shows that any subgroup of G either contains a free subgroup on two generators or a finite index subgroup which is solvable. If we add the hypothesis that the subgroup is discrete we get the following result.

th:main **Theorem 9.5.1** *Let G be a Lie group with a finite number of connected components and let Γ be a discrete subgroup of G. The following are equivalent.*

1. *The group Γ contains a finite index subgroup which is polycyclic*
2. *Any subgroup of Γ is finitely generated*
3. *The group Γ contains no free subgroup on two generators.*

For example this shows that a solvable subgroup of $Gl(n, \mathbb{Z})$ is polycyclic (a result due to Mal'cev). Note that the hypothesis that the subgroup is discrete is necessary. For exemple, consider the subgroup of the affine group of the real line generated by $x \mapsto x + 1$ and $x \mapsto \lambda x$. If $\lambda \in \mathbb{N}, \lambda > 1$ it has derived group isomorphic to $\mathbb{Z}[1/p_1, 1/p_2, ..., 1/p_k]$ where $p_1, p_2, ..., p_k$ are the primes appearing in the decomposition of λ. If λ is a positive transcendental real, the group we get is not finitely presented. It is isomorphic to the wreath product of two infinite cyclic groups [34] 15.1 Ex.. In this section we explain how 3) implies 1), using classical results. The facts that 1) implies 2) and that 2) implies 3) are easy.

th:abelian **Theorem 9.5.2** *Let G be a Lie group with a finite number of connected components and let A be a discrete abelian subgroup of G. Then A is finitely generated.*

Theorem 7.5.2 clearly follows from Theorem 7.5.1 but actually Theorem 7.5.2 combined with Tits alternative and the following caracterisation of polycyclic groups among solvable one (see [34] 15.2.1) implies Theorem 9.5.1.

th: mal **Theorem 9.5.3 (Mal'cev)** *Let Γ be a solvable group such that any abelian subgroup is finitely generated. Then Γ is polycyclic.*

Let us know recall some facts needed for the proof of 9.5.2. If the group G is solvable, the following proposition ([33] 3.8) implies Theorem 9.5.2.

prop:Rag **Proposition 9.5.4** *Let G be a connected solvable Lie group. Let H be a closed subgroup of G. Let H^0 be the connected component of the identity of H. Then H/H^0 is finitely generated.*

We will need the following well known result.

lem:center **Lemma 9.5.5** *The center $Z(G)$ of a semi-simple connected Lie group G is a discrete finitely generated abelian group.*

Proof: The center $Z(G)$ of a Lie group is a closed subgroup. By the Von Neumann Cartan theorem it is a Lie subgroup [38] 2.12.6. As G is semi-simple, the connected component $Z(G)^0$ is the identity. We conclude that $Z(G)$ is discrete. Hence the projection

$$G \to G/Z(G)$$

is a Galois covering. We have the associated exact sequence

$$\pi_1(G) \to \pi_1(G/Z(G)) \to Z(G).$$

But a connected Lie group has the same homotopy type of a compact subgroup [21] XV 3.1. This implies that $\pi_1(G/Z(G))$ is finitely generated.

lem:abel **Lemma 9.5.6** *Let G be an algebraic group. Let A be an abelian subgroup. Then the Zariski closure $\overline{A}$ of A is an abelian group.*

Proof: If $g \in G$ we denote by ϕ_g the algebraic map from G to itself given by

$$x \mapsto xgx^{-1}g^{-1}.$$

Let e be the identity element of G and let

$$I_A = \bigcap_{a \in A} \phi_a^{-1}(e).$$

As A is abelian we have $A \subset I_A$. Let

$$J_A = \bigcap_{x \in I_A} \phi_x^{-1}(e).$$

As $A \subset I_A$ we have that $J_A \subset I_A$. Elements of J_A commutes because if $x, y \in J_A$ then $x \in I_A$, hence $y \in \phi_x^{-1}(e)$. Notice that J_A still contains A. This implies that

$$A \subset \overline{A} \subset J_A.$$

This shows that $\overline{A}$ is abelian.

Proof of Theorem 9.5.2: Let G^0 be the connected component of G. The quotient $A/A \cap G^0$ injects into G/G^0 which is finite by hypothesis. Hence we can assume that G is connected. Let R be its radical. Let p_1 be the projection on G/R which is semi-simple. We take the quotient of G/R by its center $Z(G/R)$ and denote by p_2 the projection. This quotient is a connected semi-simple Lie group with trivial center hence it is the connected component $H^0_{\mathbb{R}}$ of the real points of an algebraic group H defined over $\mathbb{R}$ (in fact over $\mathbb{Q}$ but here we don't care)[Zim] 3.1.6. We denote by $p = p_2 p_1$ the projection from G onto $H^0_{\mathbb{R}}$. Let A be our discrete abelian subgroup of G. Let $\overline{p(A)}$ be the Zariski closure of the abelian subgroup $p(A)$ of H. This is an algebraic group defined over $\mathbb{R}$ [4] 1.3 b) and by Lemma 9.5.6 it is an abelian group. Hence the real points of this Zariski closure form an abelian Lie subgroup of $H_{\mathbb{R}}$ with a finite number of connected (in the sense of the Euclidean topology) components [25]. Let Q be the connected component of the identity of this abelian Lie group. Let $B = A \cap p^{-1}(Q)$. As Q is of finite index in $\overline{p(A)}_{\mathbb{R}}$, the group B is of finite index in A. We have the exact sequence

$$Z(G/R) \to p_2^{-1}(Q) \to Q.$$

According to Lemma 9.5.5, as G/R is semi-simple its center $Z(G/R)$ is abelian of finite type, say $\mathbb{Z}^k \times T$ with T finite and as $Z(G/R)$ is a discrete subgroup, the projection in the above exact sequence is a covering map. As Q is connected it implies that the connected component of the Lie group $p_2^{-1}(Q)$ maps isomorphicaly on Q. Hence the sequence splits and as the kernel is central we get a direct product

$$p_2^{-1}(Q) = Q \times \mathbb{Z}^k \times T.$$

We also have the exact sequence

$$R \to p_1^{-1}(Q \times \mathbb{Z}^k \times T) \to Q \times \mathbb{Z}^k \times T.$$

We denote the Lie group $p_1^{-1}(Q \times \mathbb{Z}^k \times T)$ by X. Let X^0 be the connected component of X. It is a solvable Lie group because it is an extension of Q by R and we have the exact sequence

$$X^0 \to X \to \mathbb{Z}^k \times T.$$

The discrete subgroup B sits in X, its intersection with X^0 is finitely generated according to Proposition 9.5.4 and its projection in $\mathbb{Z}^k \times T$ is of course finitely generated. Hence B is finitely generated and so is A.

Remarks: 1. Following the same lines as above, we can prove Theorem 9.5.1 without using Theorem 9.5.3. Instead of considering an abelian discrete subgroup A, we consider a solvable discrete subgroup S. Instead of Lemma 9.5.6 we use the fact that in an algebraic group, the Zariski closure of a solvable subgroup is again a solvable subgroup (this easily follows from [4] AG. 6.6 applied to a morphism of algebraic varieties from a product $G \times G \times ... \times G$ (with the Zariski topology of the product) to G given by a multi-commutator of rank big enough so that the image of this multi-commutator is trivial). To conclude, instead of Theorem 9.5.3, we only need the fact that a solvable group with all its subgroups finitely generated is polycyclic (this is obtained by refining the derived series).

2. Notice that Proposition 9.5.4 is easily proved using Theorem 9.5.3. First it is enough to prove the proposition for simply connected solvable Lie groups and for discrete subgoups. See [33], 3.8. Those Lie groups are closed subgroups of complex linear groups [33] 1.4. So if A is abelian discrete in a complex linear group G, the Zariski closure $\overline{A}$ of A in G is an abelian Lie group with a finite number of connected components (see [35] Book 3 ch.7,2). Hence A is finitely generated.

Chapter 10

Baumslag-Solitar groups

10.1

In this chapter we consider the group defined by

$$G = \langle a, b : aba^{-1} = b^2 \rangle.$$

More generally, one can consider the family of Baumslag-Solitar groups $G_q = \langle a, b : aba^{-1} = b^q \rangle$. The groups G_q and G_r are not necessary quasi-isometric when $q \neq r$. Farb and Mosher [?] prove that these groups are quasi-isometric if and only if they are commensurable (i.e. contain isomorphic subgroups of finite index). It is easy to check that the argument developped below for $G = G_2$ also work for $G = G_q$. These groups are solvable but not polycyclic. This chapter priovides a detailled proof of the following result.

Theorem 10.1.1 *Let $G = \langle a, b : aba^{-1} = b^2 \rangle$. Then*

(1) G has exponential growth.

(2) $\phi(n) \simeq \exp\left(-n^{1/3}\right)$.

(3) $J(n) \simeq \log_*(n)$.

10.2 The group $\langle a, b : aba^{-1} = b^2 \rangle$

The aim of this section is to give several description of $G = \langle a, b : aba^{-1} = b^2 \rangle$ and of the Cayley graph (G, S) with $S = \{a^{\pm}, b^{\pm}\}$).

Proposition 10.2.1 *Let $G = \langle a, b : aba^{-1} = b^2 \rangle$. Each element $g \in G$ can be written uniquely as*

$$a^k (ba^{k_1})(ba^{k_2}) \cdots (ba^{k_n}) b^{\ell}$$

with $k, \ell \in \mathbb{Z}$, n an integer, and $k_1, \ldots, k_n$ positive integers if $n \neq 0$.

Proof: Let $g \in G$ be a word in the alphabet $\{a, a^{-1}, b, b^{-1}\}$:

$$g = a^{\alpha_1} b^{\beta_1} \cdots a^{\alpha_j} b^{\beta_j}$$

with $\alpha_i, \beta_i \in \mathbb{Z}$, α_1 or $\beta_1 \neq 0$ $\alpha_i, \beta_i \neq 0$ for $1 < i < j-1$, $\alpha_j \neq 0$. (we assume that words are automatically simplified using the rules $x^n x^m = x^{n+m}$, $x^n x^{-n} = \mathrm{id}$). We say that this is a normal form if the condition of the proposition are satisfied. Given a word

$$a^{\alpha_1} b^{\beta_1} \cdots a^{\alpha_j} b^{\beta_j},$$

using the rules $ba^{-1} = a^{-1}b^2$ and $b^{-1}a^{-1} = a^{-1}b^{-2}$ and working from right to left, we can move all the negative power of a to the front. Thus we can assume that

$$g = a^{\alpha_1} b^{\beta_1} \cdots a^{\alpha_j} b^{\beta_j}$$

with $\alpha_i, \beta_i \in \mathbb{Z}$, α_1 or $\beta_1 \neq 0$ $\alpha_i > 0$, $\beta_i \neq 0$ for $1 < i < j-1$, $\alpha_j \neq 0$. Now, let $t = j+1$ if this is a normal form. If not, there exists $1 < i < j$ such that $\beta_i \neq 1$, let t be the smallest such integer. Reading from left to right, t is the first time that one of the normal form conditions is violated. Let $N = t - j + 1$. N is the number of powers of a that appear after t (including t).

The existence of a normal form as asserted in the proposition will be proved by induction on N using the rules $b^2 a = ab$, $b^{-1}a = bab^{-1}$. Observe that these rules do not affect the positivity of the power of a.

If $N = 0$ the present writing of g is a normal form. $N = 1$ never occurs.

If $N = 2$, we have $g = a^{\alpha_1}(ba^{\alpha_2}) \cdots (ba^{\alpha_{j-2}}) b^{\beta_{j-1}} a^{\alpha_j} b^{\beta_j}$ with $\alpha_2, \dots, \alpha_j > 0$ and $\beta_{j-1} \neq 1$. Using $b^2 a = ab$, $b^{-1}a = bab^{-1}$, we move $b^{\beta_{j-1}}$ forward and get

$$g = \begin{cases} a^{\alpha_1}(ba^{\alpha_2}) \cdots (ba^{\alpha_j}) b^{\ell} & \text{if } \beta_{j-1} \text{ is odd} \\ a^{\alpha_1}(ba^{\alpha_2}) \cdots (ba^{\alpha_{j-2}})(ba^{\alpha_{j-1}+\alpha_j}) b^{\ell} & \text{if } \beta_{j-1} \text{ is even.} \end{cases}.$$

Both words are normal forms.

Let $n \geq 1$ be an integer. Suppose that any g which can be written as a word with $N \leq n$ has a normal form. Let h be an element which can be written as a word with $N = n+1$. Thus,

$$h = a^{\alpha_1}(ba^{\alpha_2}) \cdots (ba^{\alpha_t}) b^{\beta_t} a^{\alpha_{t+1}} \cdots b^{\beta_{j-1}} a^{\alpha_j} b^{\beta_j}$$

with $\alpha_2, \dots, \alpha_j$ positive integers and $\beta_t \neq 1$ By using the allowed rules as above, we can write

$$h = \begin{cases} a^{\alpha_1}(ba^{\alpha_2}) \cdots (ba^{\alpha_t})(ba^{\alpha'}) b^{\beta'} a^{\alpha_{t+2}} \cdots a^{\alpha_j} b^{\beta_j} & \text{if } \beta_{j-1} \text{ is odd} \\ a^{\alpha_1}(ba^{\alpha_2}) \cdots (ba^{\alpha_t + \alpha_{t+1}}) b^{\beta'} a^{\alpha_{t+2}} \cdots a^{\alpha_j} b^{\beta_j} & \text{if } \beta_{j-1} \text{ is odd} \end{cases}$$

In both cases, the new writing of h has $N = n$. This proves the existence of a normal form. Unicity will be proved as part of the next proposition.

Proposition 10.2.2 *The group $G = \langle a, b : aba^{-1} = b^2 \rangle$ is isomorphic to the group of affine transformations of the line generated by $A(z) = 2z$, $B(z) = z+1$, that is, the group of 2 by 2 matrices generated by*

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Proof: Since $ABA^{-1} = B^2$, there exists a homorphism from G to the matrix group generated by A, B that send a to A and b to B. To prove injectivity, observe that

$$h = a^k (ba^{k_1})(ba^{k_2}) \cdots (ba^{k_n}) b^\ell$$

with $k, \ell \in \mathbb{Z}$, n an integer, and $k_1, \ldots, k_n$ positive integers if $n \neq 0$, is send to

$$H = A^k (BA^{k_1})(BA^{k_2}) \cdots (BA^{k_n}) B^\ell = \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix}$$

with

$$u = 2^{k + \sum_1^n k_i}, \quad v = \ell 2^{k + \sum_1^n k_i} + 2^k \left(\sum_{i=1}^{n} 2^{\sum_{m=1}^{i-1} k_m} \right).$$

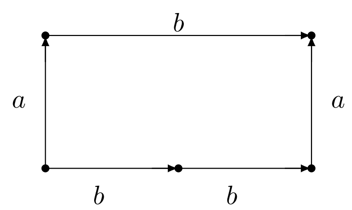
If H is the identity matrix and $n = 0$, we must have $k = \ell = 0$. If H is the identity matrix and $n > 0$, then $k + \sum_1^n k_i = 0$, thus $k \leq 0$. Furthermore,

$$-\ell = 2^k \left[2^{k_1 + \cdots + k_{n-1}} + \cdots + 2^{k_1} + 1 \right].$$

The term in brackets is not divisible by 2 and $\ell \in \mathbb{Z}$ implies that $k \geq 0$. Thus $k = 0$. It follows that $\ell = 0$. This proves that our homorphism is an isomorphism and the missing unicity in Proposition 8.2.1.

We now descibe and draw the Cayley graph of $G = \langle a, b : aba^{-1} = b^2 \rangle$. The basic building block of this Cayley graph is the cell shown in Figure 8.1.

Figure 10.1: The basic cell



One way to construct the desired Cayley graph is to first consider the planar graph shown in Figure 8.2 which represents those elements of G that can be written as $a^k b^\ell$ with $k, \ell \in \mathbb{Z}$. This is only a small part of the Cayley graph of G. Consider what happens when one translate the set $P = \{a^k b^\ell : k, \ell \in \mathbb{Z}\}$ by b on the left. We obtain a new copy of Figure 8.2 since left translation are automorphisms of the Cayley graph. However, if $k \leq 0$, then $ba^k b^\ell = a^k b^{\ell+2} \in P$ whereas if $k > 0$, $ba^k b^\ell \notin P$. Thus the part P_- of P that lies below the horizontal line containing the identity (including this line) is globally preserved under $x \to bx$ whereas the part P_+ of P that lies above this line is send to an independent copy of P_+, call it P'_+, that is attach to P as shown in Figure 8.3. This procedure must be carried over infinitely many times to obtain the full Cayley graph of G.

If one identifies all the points on a same horizontal line, that is the points

$$a^k(ba^{k_1})(ba^{k_2})\cdots(ba^{k_n})b^\ell,\ \ \ell\in\mathbb{Z}$$

for fixed integers $k, k_1, \ldots k_n$ with $k_1 > 0, \ldots, k_n > 0$, the Cayley graph of G collaps to a binary tree shown in Figure 8.5.

Remark: The normal form of an element g given by Proposition 8.2.1 does not always have minimal length as a word in $a^{\pm 1}, b^{\pm 1}$. For instance $g = a^{-2}b^{32}$ in normal form is also equal to b^8a^{-2}. Similarly, $g = a^{-2}bab^4$ can be written $b^2a^{-2}ba$.

10.3 Følner sets for $\langle a, b : aba^{-1} = b^2\rangle$

From Proposition 8.2.1 it clearly follows that $G = \langle a, b : aba^{-1} = b^2\rangle$ has exponential volume growth. Indeed, the ball of radius $3n$ around the identity contains all the points of the form

$$a^{k_1}(ba)^{\ell_1}a^{k_2}(ba)^{\ell_2}\cdots a^{k_j}(ba)^{\ell_j}$$

with k_i, ℓ_i integers such that $k_1 \geq 0, \ell_j \geq 0,\ k_2, \ldots, k_j > 0,\ \ell_1, \ldots, \ell_{j-1} > 0$, $\sum_1^j k_i + \ell_i \leq n$. Moreover all these points are distincts. Thus the ball of radius $3n$ in G contains the ball of radius n of a free semigroup generated by two letters a and (ba). It follows that $V(3n) \geq 2^{n+1} - 1$.

Figure 10.2: The elements $a^k b^\ell$

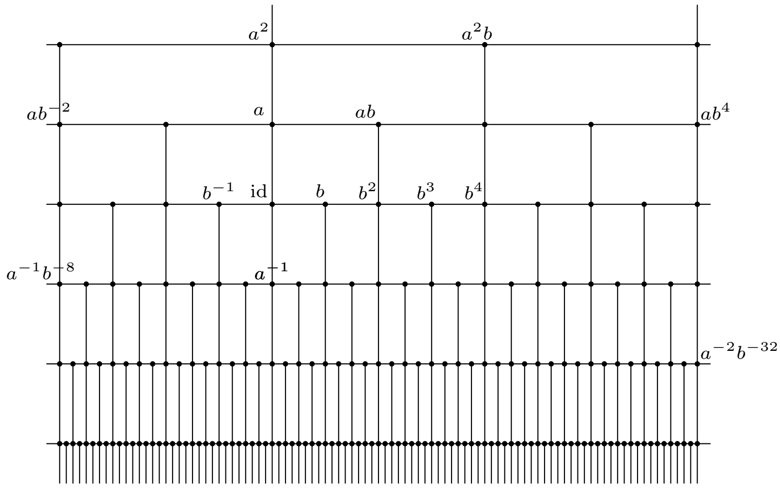


Figure 10.3: P and bP

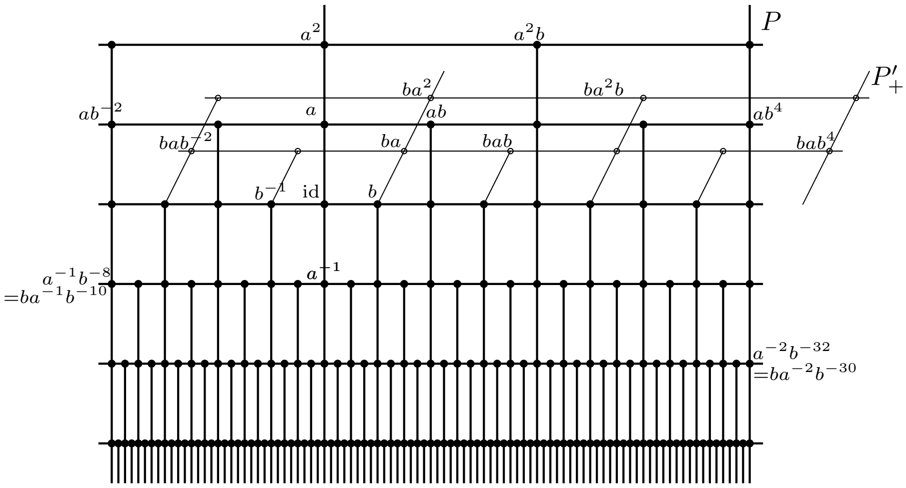


Figure 10.4: A piece of the Cayley graph of $G = \langle a, b : aba^{-1} = b^2 \rangle$

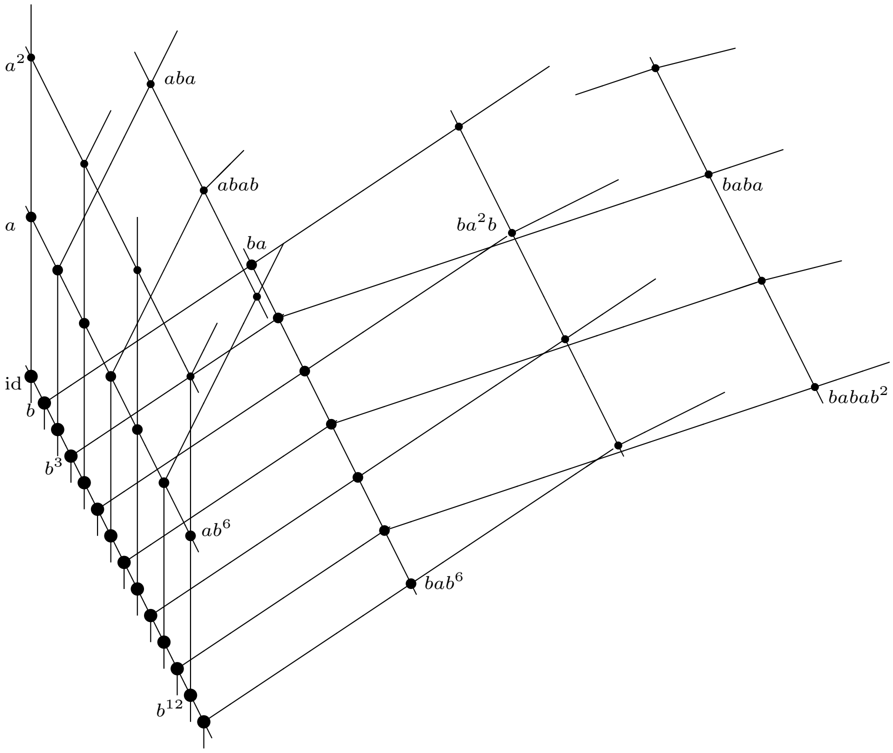
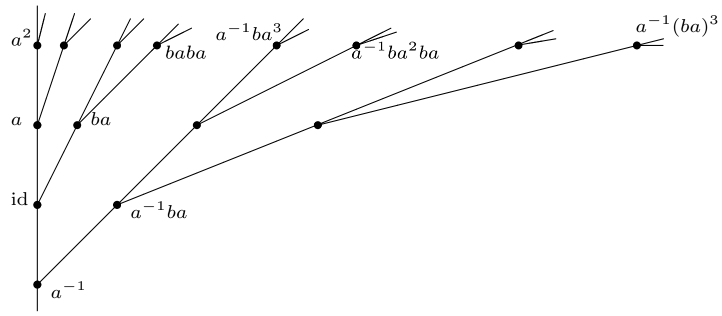


Figure 10.5: The tree profile of G



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