MATH 4530 – Topology. HW 10

Please declare any collaborations with classmates; if you find solutions in books or online, acknowledge your sources in either case, write your answers in your own words. Please attempt all questions and justify your answers.

Write the proofs in complete sentences.

(1) (How to show two spaces can't be homeomorphic to each other)
  (a) Show that $\mathbb{R}^1$ is not homeomorphic to $\mathbb{R}^n, n > 1$.
  (b) Show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n, n > 2$.

  Hint: recall how you showed that $(0, 1]$ and $(0, 1)$ can't be homeomorphic to each other. That might help.

  Note: once we compute higher homotopy groups for S^n, we can show that $\mathbb{R}^n$ and $\mathbb{R}^m$ are note homeomorphic when $n \neq m$.

(2) Let X be the union of the two copies of $S^n, n \geq 2$ having a single point in common. What is the fundamental group of X?

(3) Assume that the following statement holds: *There is no retraction $r : B^{n+1} \to S^n$ for each n.*
  (a) Prove that the identity map $\mathrm{id}_{S^n} : S^n \to S^n$ is not *nullhomotopic*, i.e. is not homotopic to a constant map.
  (b) Prove that the inclusion map $j : S^n \hookrightarrow \mathbb{R}^{n+1} - \{\vec{0}\}$ is not nullhomotopic.
  (c) Explain why every continuous map $f : B^{n+1} \to B^{n+1}$ must have a fixed point, i.e. $f(x) = x$ for some $x \in B^{n+1}$.

  Again, once we compute the higher homotopic groups of S^n, we can prove the assumed statement similarly to $n = 1$ case.

References

[M] Munkres, Topology.
[S] Basic Set Theory, http://www.math.cornell.edu/~matsumura/math4530/basic set theory.pdf
[L] Lecture notes, available at http://www.math.cornell.edu/~matsumura/math4530/math4530web.html