MATH 4530 – Topology. HW 3 Solution

(1) Show that X is a Hausdorff space if and only if the ***diagonal*** $\Delta := \{(x, x) \mid x \in X\} \subset X \times X$ is closed with respect to the product topology.

Solution:

$\Leftarrow$: Assume that the diagonal is closed. Consider $a, b \in X$ with $a \neq b$, note that $(a, b) \in X \times X \setminus \Delta$. Now since Δ is closed with respect to the product topology, $\exists$ open set $U = U_1 \times U_2 \in \mathcal{T}_{X\times X}$ such that $U \cap \Delta = \varnothing$. This implies that we found $U_1, U_2 \in \mathcal{T}_X$ satisfying $a \in U_1$, $b \in U_2$ and $U_1 \cap U_2 = \varnothing$. So X is Hausdorff.

$\Rightarrow$: Assume that X is Hausdorff. Given a point $(a, b) \in X \times X$, with $a \neq b$, we will show that (a, b) is not a limit point of Δ. Consider the open sets $U_1, U_2 \in \mathcal{T}_X$, with $a \in U_1$, $b \in U_2$ and $U_1 \cap U_2 = \varnothing$ (such sets exit since X is Hausdorff). Then $U_1 \times U_2 \subset X \times X \setminus \Delta$ is an open set in the product topology containing the point (a, b). Since Δ contains all its limit points, it is closed.

(2) Find all points that the sequence $\{x_n = 1/n \mid \mathbb{Z}_{>0}\}$ converges to with respect to the following topology of $\mathbb{R}$. Justify your answer.

(a) Standard Topology
(b) Finite Complement Topology
(c) Discrete Topology
(d) Lower Limit Topology

Solution:

(a) $\{0\}$. For any open set (a, b) around 0, we have $1/n \in (a, b)$ for all $n > 1/b$.
(b) $\mathbb{R}$. Since the complement of any non-empty open set, U, in the finite complement topology is finite, we must have U contains infinitely many points of the sequence $\{x_n\}$.
(c) $\varnothing$. For any point in discrete topology, we can consider an open set containing just that point.
(d) $\{0\}$. For any open set $[a, b)$ around 0, we have $1/n \in [a, b)$ for all $n > 1/b$.

(3) Let X and Y be topological spaces. Prove that $f : X \to Y$ is continuous if and only if for every subset A of X, we have $f(\overline{A}) \subset \overline{f(A)}$.

Solution:

($\Rightarrow$) f is continuous, so by Ex 3.14 [**?**], for every closed subset C_Y in Y, $f^{-1}(C_Y)$ is closed in X.

$$\Rightarrow \quad f^{-1}(\overline{f(A)}) \text{ is closed in } X \text{ and } A \subset f^{-1}(\overline{f(A)})$$

$$\Rightarrow \overline{A} \subset f^{-1}(\overline{f(A)}) \quad \text{by the definition of closure.}$$

$$\Rightarrow f(\overline{A}) \subset \overline{f(A)} \quad \text{by the definition of preimages}$$

($\Leftarrow$) Let C_Y be a closed set in Y. Let $A := f^{-1}(C_Y)$. To show that $f^{-1}(C_Y)$ is closed, we just need to show $\bar{A} = A$. By the assumption, $f(\bar{A}) \subset \overline{f(A)} = \overline{C_Y} = C_Y$. Thus $\bar{A} \subset f^{-1}(C_Y) = A$. Since $\bar{A} \supset A$ by the definition of closure, we have $\bar{A} = A$.

(4) Define a map $f : \mathbb{R} \to \mathbb{R}$ by

$$x \mapsto \begin{cases} |x| & \text{if } x \text{ is rational} \\ -|x| & \text{if } x \text{ is irrational.} \end{cases}$$

Show that f is continuous at $x = 0$ but not continuous at other points.

Solution: First let's prove continuity at 0. Consider any basis element of $\mathbb{R}$ containing $f(0) = 0$, say $(-a, b)$. Note that the pre-image of $(-a, b)$ contains the open interval $(-\min\{a, b\}, \min\{a, b\})$, and hence f is continuous at 0. Now for any positive $p \in \mathbb{R}$, consider an open set $(p/2, p + 1)$. Note that the pre-image of this set consists only of rational points and thus does not contain any open subsets around p. So, f is not continuous for p positive. Similarly, for any negative $p \in \mathbb{R}$, consider an open set $(p - 1, p/2)$. Note that the pre-image of this set consists only of irrational points and thus does not contain any open subsets around p. So, f is not continuous for p negative.

(5) Let X and Y be sets. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projections to the first and the second factors.
   (a) (**Set Theory**) For a given set Z with maps $f_1 : Z \to X$ and $f_2 : Z \to Y$, find a map $g : Z \to X \times Y$ such that $f_1 = \pi_1 \circ g$ and $f_2 = \pi_2 \circ g$. Show that such g is unique .
   (b) Suppose X and Y are topological spaces. Show that π_1 and π_2 are continuous maps with respect to the product topology $\mathcal{T}_{prod}$ on $X \times Y$. Show that any topology $\mathcal{T}$ on $X \times Y$ such that π_1 and π_2 are continuous, must be finer than the product topology.
   (c) Suppose X and Y are topological spaces. For a given topological space Z with continuous maps $f_1 : Z \to X$ and $f_2 : Z \to Y$, show that the map g you found in (a) is continuous with respect to the product topology on $X \times Y$.
   (d) Explain why there is no finer topology on $X \times Y$ than the product topology such that (c) holds.

Solution:
   (a) $g : Z \to X \times Y$ is given by $g(z) = (f_1(z), f_2(z))$, which clearly satisfies the required conditions. Let $g' : Z \to X \times Y$ be a map satisfying the conditions. Let $g'(z) = (g'_1(z), g'_2(z))$. Then the conditions make sure that $g'_1 = f_1$ and $g'_2 = f_2$.
   (b) For any open set U_1 in X, $\pi_1^{-1}(U_1) = U_1 \times Y$ which is an open set in the product topology. The same works for π_2. Say $\mathcal{T}$ is some topology on $X \times Y$ such that π_1 and π_2 are continuous. Then $U_1 \times Y$ and $X \times U_2$ must be open sets in $(X \times Y, \mathcal{T})$, thus $U_1 \times U_2 = (U_1 \times Y) \cap (X \times U_2) \in \mathcal{T}$. Thus $\mathcal{T}_{prod} \subset \mathcal{T}$.
   (c) For any open set $U_1 \times U_2$ in $X \times Y$, we need to show that $g^{-1}(U_1 \times U_2) \subset Z$ is open.

$$z \in g^{-1}(U_1 \times U_2) \Leftrightarrow g(z) \in U_1 \times U_2 \Leftrightarrow f_1(z) \in U_1 \text{ and } f_2(z) \in U_2$$

$$\Leftrightarrow z \in f_1^{-1}(U_1) \text{ and } z \in f_2^{-1}(U_2) \Leftrightarrow z \in f^{-1}(U_1) \cap f^{-1}(U_2).$$

   Thus $g^{-1}(U_1 \times U_2) = f_1^{-1}(U_1) \cap f_2^{-1}(U_2)$ which is open since f_1 and f_2 are continuous.
   (d) Apply the construction in (a) to $Z = X \times Y$ and π_1 and π_2. Then $g = \mathrm{id}_{X \times Y}$. Say we put some topology $\mathcal{T}$ in $X \times Y$. Since $\mathrm{id}_X : (X \times Y, \mathcal{T}_{prod}) \to (X \times Y, \mathcal{T})$ must be continuous, $\mathcal{T}$ cannot be finer than the product topology.

(6) Show that the open interval $(-\pi/2, \pi/2)$ of $\mathbb{R}$ with the subspace topology is homeomorphic to $\mathbb{R}$. Show that any open interval (a, b) is homeomorphic to $\mathbb{R}$.

Solution: Define $f : (-\pi/2, \pi/2) \to \mathbb{R}$ to be $f(\theta) := \tan\theta$ which is one to one and continuous. The function $\tan^{-1}$ is also continuous. Thus f is a homeomorphism. Now any two interval (a, b) and (c, d) are homeomorphic, by

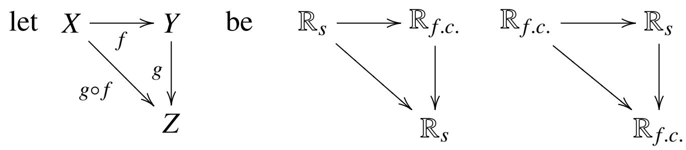
$$(a, b) \xrightarrow[x-a]{} (0, b-a) \xrightarrow[x/(b-a)]{} (0, 1) \xrightarrow[(d-c)x]{} (0, d-c) \xrightarrow[x+c]{} (c, d).$$

So (a, b) is homeomorphic to $(-\pi/2, \pi/2)$ and $(-\pi/2, \pi/2)$ is homeomorphic to $\mathbb{R}$, so (a, b) is homeomorphic to $\mathbb{R}$.

(7) Suppose that X, Y, Z are topological spaces. Let $f : X \to Y$ and $g : Y \to Z$ be maps of sets. Prove or disprove the following statement:

(a) If $f : X \to Y$ is continuous and the composition map $g \circ f : X \to Z$ is continuous, then $g : Y \to Z$ is continuous.
(b) If $g : Y \to Z$ is continuous and the composition map $g \circ f : X \to Z$ is continuous, then $f : X \to Y$ is continuous.

Solution: Let $\mathbb{R}_s$ be $\mathbb{R}$ with standard topology and let $\mathbb{R}_{f.c.}$ be $\mathbb{R}$ with finite complement topology. The following diagrams will give the counterexamples for (a) and (b), respectively :



where all maps are identity maps. You can actually replace $\mathbb{R}_s$ and $\mathbb{R}_{f.c.}$ with any $(X, \mathcal{T})$ and $(X, \mathcal{T}')$ such that $\mathcal{T}$ is finer than $\mathcal{T}'$.

References

[M] Munkres, Topology.
[S] Basic Set Theory, http://www.math.cornell.edu/~matsumura/math4530/basic set theory.pdf
[L] Lecture notes, available at http://www.math.cornell.edu/~matsumura/math4530/math4530web.html