MATH 4530 – Topology. HW 5

Please declare any collaborations with classmates; if you find solutions in books or online, acknowledge your sources in either case, write your answers in your own words. Please attempt all questions and justify your answers.

Write the proofs in complete sentences.

(1) Show that any infinite set with the finite complement topology is connected.

(2) Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on a set X. If $\mathcal{T}$ is finer than $\mathcal{T}'$, then the connectedness of which topology will implies the one of the other? Justify your answer.

(3) Let $p : X \to Y$ be a quotient map. Show that, if $p^{-1}(y)$ is connected for each $y \in Y$ and Y is connected, then X is connected.

(4) Let $f : X \to Y$ be a continuous map. Show that if X is path-connected, then $\operatorname{Im} f$ is path-connected.

(5) Show that there is no homeomorphism between $(0, 1)$ and $(0, 1]$ by using the connectedness. Hint: if we remove a point from each of the spaces, what happens?

(6) Show that if X is connected, then any continuous map $f : X \to Y$ where Y is a topological space with discrete topology is a constant map, i.e. $f(X) = \{y\}$ for some $y \in Y$.

(7) Consider the quotient space of $\mathbb{R}^2$ by the identification $(x, y) \sim (x + n, y + m)$ for all $(n, m) \in \mathbb{Z}^2$. Show that it is connected and compact.

(8) Recall that a square matrix M is orthogonal if $MM^t = I$. This condition is equivalent to "the set of row vectors form an orthonormal basis" and also to $\langle M\vec{v}, M\vec{w}\rangle = \langle \vec{v}, \vec{w}\rangle$. In particular, if M is orthogonal, then $\det M = \pm 1$. Let $\mathrm{O}(n, \mathbb{R})$ be the set of orthogonal matrices of size n. Show that it is not connected.

References

[M] Munkres, Topology.

[S] Basic Set Theory, http://www.math.cornell.edu/~matsumura/math4530/basic set theory.pdf

[L] Lecture notes, available at http://www.math.cornell.edu/~matsumura/math4530/math4530web.html