MATH 4530 – Topology. HW 9

Please declare any collaborations with classmates; if you find solutions in books or online, acknowledge your sources in either case, write your answers in your own words. Please attempt all questions and justify your answers.

Write the proofs in complete sentences.

(1) Let $p : E \to B$ be a covering map. Let f, g are *composable paths* in B, i.e. $f(1) = g(0)$. If $\tilde{f}, \tilde{g}$ are composable paths lifted from f, g, then show that $\tilde{f} * \tilde{g}$ is a lifting of $f * g$.

(2) Show that the fundamental group of a torus $S^1 \times S^1$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ as groups (additive on $\mathbb{Z} \times \mathbb{Z}$). (Hint; generalized the proof of $\pi_1(S^1, b_0) \cong \mathbb{Z}$.)

Solution
Let $S^1 := \{e^{i\theta}, \theta \in \mathbb{R}\}$ and $p : \mathbb{R} \to S^1, \theta \mapsto e^{2\pi i\theta}$. Let $\mathbf{p} := (\mathbf{p}, \mathbf{p}) : \mathbb{R} \times \mathbb{R} \to \mathbf{S^1} \times \mathbf{S^1}$.
Let $[f], [g] \in \pi_1(S^1 \times S^1, (1, 1))$. Let $\tilde{f}, \tilde{g}$ be the lifts of f and g at $(0, 0)$ along $\mathbf{p}$ and $\tilde{f}(1) =: (n_1, m_1)$ and $\tilde{g}(1) =: (n_2, m_2)$. We need to show that if $f \tilde{*} g$ is the lift of $f * g$ at $(0, 0)$ along $\mathbf{p}$, then $\widetilde{f * g}(1) = (n_1 + n_2, m_1 + m_2)$. Consider $\tilde{g}_1(s) := (n_1, m_1) + \tilde{g}(s)$. Then $\tilde{g}$ is the lift of g at (n_1, m_1) along $\mathbf{p}$ by the invariance of p under the shifting $(p(\vec{x} + (n + m)) = p(\vec{x}))$. Since $\tilde{g}_1(0) = (n_1, m_1)$, by (1), $\widetilde{f * g} = \tilde{f} * \tilde{g}_1$. Thus $\widetilde{f * g}(1) = \tilde{g}_1(1) = (n_1 + n_2, m_1 + m_2)$.

(3) A group G acts on a set X ***from right*** if there is an action map $\mathsf{G} \times X \to X, (g, x) \mapsto xg$ which satisfies $x = x1_{\mathsf{G}}$ and $x(gh) = (xg)h$. Show that there is a natural right action of $\pi_1(B, b_0)$ on $p^{-1}(b_0)$ if $p : E \to B$ is a covering map. (Hint: use ϕ_{e_0} in Section 10.2 [L]).

(4) Let B be a simply-connected space.Then any covering map $p : E \to B$ with E path-connected, is a homeomorphism.

(5) Show that the map $p : S^1 \to S^1, z \mapsto z^n$ induces $p_* : \pi_1(S^1, b) \to \pi_1(S^1, b), [f] \mapsto [f]^n$. In other words, through the isomorphism in Section 10.4 [L], $\mathbb{Z} \to \mathbb{Z}, m \mapsto nm$.

Solution Under the isomorphism in Theorem 10.8 [L], it is enough to prove $p_*(1) = n$, since $\mathbb{Z}$ is generated by 1. More concretely, it is enough because $f(m) = f(1 + \cdots + 1) = f(1) + \cdots + f(1) = mf(1) = mn$. Let $S^1 = \{e^{2\pi i\theta}\}$ and $\bar{p} : \mathbb{R} \to S^1, \theta \mapsto e^{2\pi i\theta}$. Then it is clear that $\bar{p}|_{\mathsf{I}} : \mathsf{I} \to S^1$ is a loop at 1 and its lift at $0 \in \mathbb{R}$ is the inclusion $j : \mathsf{I} \hookrightarrow \mathbb{R}$. Thus $\phi_0([\bar{p}_{\mathsf{I}}]) = j(1) = 1$. $p_*(1) = p_*([\bar{p}|_{\mathsf{I}}]) = [p \circ \bar{p}|_{\mathsf{I}}]$. Then the lift of $p \circ \bar{p}_{\mathsf{I}}$ at 0 is $\widetilde{p \circ p}|_{\mathsf{I}} : \mathsf{I} \to \mathbb{R}, s \mapsto ns$ (Check this!). Now $\widetilde{p \circ p}|_{\mathsf{I}}(1) = n$.

References

[M] Munkres, Topology.
[S] Basic Set Theory, http://www.math.cornell.edu/~matsumura/math4530/basic set theory.pdf
[L] Lecture notes, available at http://www.math.cornell.edu/~matsumura/math4530/math4530web.html