MATH 4530 – Topology. HW 9 solutions

Please declare any collaborations with classmates; if you find solutions in books or online, acknowledge your sources in either case, write your answers in your own words. Please attempt all questions and justify your answers.

Write the proofs in complete sentences.

(1) Let $p : E \to B$ be a covering map. Let f, g are *composable paths* in B, i.e. $f(1) = g(0)$. If $\tilde{f}, \tilde{g}$ are composable paths lifted from f, g, then show that $\tilde{f} * \tilde{g}$ is a lifting of $f * g$.

> **Solution (Ex 54.3** [M]**):** We need to show that $p \circ (\tilde{f} * \tilde{g}) = f * g$. If $s \in [0, 1/2]$, then $p \circ (\tilde{f} * \tilde{g})(s) = p \circ \tilde{f}(2s) = f(2s) = (f * g)(s)$. If $s \in [1/2, 1]$, then $p \circ (\tilde{f} * \tilde{g})(s) = p \circ \tilde{g}(2s - 1) = g(2s - 1) = (f * g)(s)$.

(2) Show that the fundamental group of a torus $S^1 \times S^1$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ as groups (additive on $\mathbb{Z} \times \mathbb{Z}$). (Hint; generalized the proof of $\pi_1(S^1, b_0) \cong \mathbb{Z}$.)

> **Solution (Ex 54.7** [M]**):** Since $\mathbb{R} \times \mathbb{R}$ is simply-connected, by Theorem 10.7 [L], $\phi_{e_0} : \pi_1(S^1 \times S^1, b_0) \to \mathbb{Z} \times \mathbb{Z}, [f] \mapsto [\tilde{f}(1)]$ is a bijective map where $b_0 = ((1, 0), (1, 0)) \in S^1 \times S^1$ and $e_0 = (0, 0) \in \mathbb{Z} \times \mathbb{Z}$. We need to show that this map is a group homomorphism.

(3) A group G acts on a set X ***from right*** if there is an action map $\mathsf{G} \times X \to X, (g, x) \mapsto xg$ which satisfies $x = x1_{\mathsf{G}}$ and $x(gh) = (xg)h$. Show that there is a natural right action of $\pi_1(B, b_0)$ on $p^{-1}(b_0)$ if $p : E \to B$ is a covering map. (Hint: use ϕ_{e_0} in Section 10.2 [L]).

> **Solution:** Define an action map
> $$\pi_1(B, b_0) \times p^{-1}(b_0) \to p^{-1}(b_0), \ \ ([f], e_0) \mapsto \phi_{e_0}([f]) = \tilde{f}(1).$$
> If f is a constant path at b_0, then $[f] = 1$ is the identity. The lift of f at e_0 is a constant path at e_0, thus $\phi_{e_0}([f]) = e_0$. This proves $x1_{\mathsf{G}} = x$ axiom.
> By Problem (1),
> $$e_0([f] * [g]) = e_0([f * g]) = \widetilde{f * g}(1) = (\tilde{f} * \tilde{g})(1) = \tilde{g}(1)$$
> where $\widetilde{f * g}$ is the lift of $f * g$ at e_0 and $\tilde{f}$ is the lift of f at e_0 and $\tilde{g}$ is the lift of g at $\tilde{f}(1)$.
> $$(e_0[f])[g] = (\tilde{f}(1))[g] = \tilde{g}(1).$$

(4) Let B be a simply-connected space.Then any covering map $p : E \to B$ with E path-connected, is a homeomorphism.

> **Solution (Ex 54.8** [M]**):** Since B is simply-connected, $\pi_1(B, b_0)$ is trivial. Since $\phi_{e_0} : \pi_1(B, b_0) \to p^{-1}(b_0)$ is surjective (E path-connected and Theorem 10.7 [L]), $p^{-1}(b_0) = \{e_0\}$. This is true for any b_0, therefore p must be bijective. Since p is an open map, the inverse is continuous too: p is a homeomorphism.

(5) Show that the map $p : S^1 \to S^1, z \mapsto z^n$ induces $p_* : \pi_1(S^1, b) \to \pi_1(S^1, b), [f] \mapsto [f]^n$. In other words, through the isomorphism in Section 10.4 [L], $\mathbb{Z} \to \mathbb{Z}, m \mapsto nm$.

References

[M] Munkres, Topology.
[S] Basic Set Theory, http://www.math.cornell.edu/~matsumura/math4530/basic set theory.pdf
[L] Lecture notes, available at http://www.math.cornell.edu/~matsumura/math4530/math4530web.html