MATH 4530 – Topology. Practice Problems For Final Part I

(1) Apply Theorem 12.6 [L] (See also Remark 12.7 [L], Theorem 72.1 [M]) and compute the fundamental group of $\mathbb{RP}^2\sharp$(Klein Bottle) where $\sharp$ means the connected sum defined in Section 12.1 [L].

(2) Let $h : S^1 \to S^1 \subset \mathbb{R}^2 - \{\vec{0}\}$ be a continuous map. Show that $\deg h = n(h, \vec{0})$. The $\deg h$ is defined in Prelim II and $n(h, \vec{0})$ is the winding number of h around $\vec{0}$ defined in Section 13 [L].

(3) Consider the group $\mathsf{G} = \langle x, y \mid x^2, y^2, xy = yx\rangle$.
   (a) Show that $\mathsf{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
   (b) Find a space B such that $\pi_1(B, b) \cong \mathsf{G}$ and its simply-connected covering space. Justify your answer.

(4) Find a space X with the fundamental group which is isomorphic to $\langle x, y, z \mid yz = zy\rangle$.

(5) Show that if a space X has the property that every continuous map $f : X \to X$ has a fixed point, then each retract Y of X has the property too.

(6) Find an example that Borsuk Lemma 13.15 (3) [L] doesn't hold if the map f is not injective.

(7) Let G be a topological group. Show that $\pi_1(\mathsf{G}, 1_{\mathsf{G}})$ is a commutative (abelian) group.

(8) Prove or disprove the following statement:
   (a) If X is connected, then $\pi_1(X, x_1)$ is isomorphic to $\pi_1(X, x_2)$ for every $x_1, x_2 \in X$.
   (b) If A and B are deformation retract of C, then A are B are homotopy equivalent (i.e. have the same homotopy type.)
   (c) If $p : E \to B$ is a covering map with $p(e) = b$ and $\pi_1(B, b)$ is abelian, then $\pi_1(E, e)$ is abelian, too.

(9) Let $\mathrm{GL}(2, \mathbb{R})$ denote the general linear group, the group of invertible 2×2 matrices with real coefficients. Let $\mathrm{O}(2)$ denote the orthogonal group, the 2×2 orthogonal matrices. Finally, $\mathrm{SO}(2)$ is the special orthogonal group, those matrices $Q \in \mathrm{O}(2)$ satisfying $\det(Q) = 1$. Let I denote the two-by-two identity matrix.
   (a) Determine whether or not $\mathrm{GL}(2, \mathbb{R})$ is connected. Justify your response.
   (b) The Gram-Schmidt orthogonalization process allows us to write a matrix $A \in \mathrm{GL}(2, \mathbb{R})$ uniquely as a product $A = QR$ where $Q \in O(2)$ is orthogonal and R is upper triangular with positive entries on the diagonal. Use this to produce a deformation retraction from $\mathrm{GL}(2, \mathbb{R})$ to $\mathrm{O}(2)$.
   (c) Show that $\mathrm{SO}(2)$, with matrix multiplication, is homeomorphic(as a topological space) to and isomorphic (as a group) to $S^1 = (\mathbb{R}, +)/\mathbb{Z}$.
   (d) Using the fact that $\mathrm{O}(2) = \mathrm{SO}(2) \times \{1, -1\}$, compute $\pi_1(\mathrm{GL}(2, \mathbb{R}), I)$.

(10) Let A be a compact contractible subspace of S^2. Show that A does not separate S^2.

References

[M] Munkres, Topology.
[S] Basic Set Theory, http://www.math.cornell.edu/~matsumura/math4530/basic set theory.pdf
[L] Lecture notes, available at http://www.math.cornell.edu/~matsumura/math4530/math4530web.html