MATH 4530 – Topology. Practice Problems For Final solutions

Write the proofs in complete sentences.

(1) Apply Theorem 12.6 [L] (See also Remark 12.7 [L], Theorem 72.1 [M]) and compute the fundamental group of $\mathbb{RP}^2 \sharp$(Klein Bottle) where $\sharp$ means the connected sum defined in Section 12.1 [L].

Solution

(2) Let $h : S^1 \to S^1 \subset \mathbb{R}^2 - \{\vec{0}\}$ be a continuous map. Show that $\deg h = n(h, \vec{0})$. The $\deg h$ is defined in Prelim II and $n(h, \vec{0})$ is the winding number of h around $\vec{0}$ defined in Section 13 [L].

Solution Let $S^1 = \{e^{2\pi i\theta}\} \subset \mathbb{C} - \{0\}$. It is natural to define $n(h, \vec{0}) := n(h \circ p|_{\mathsf{I}}, \vec{0})$ (Definition 13.3 [L]) where $p : \mathbb{R} \to S^1$ is the standard covering. Let $n := n(h, \vec{0})$, i.e. if $\tilde{g}$ is a lift of $g := h \circ p|_{\mathsf{I}}$, then $\tilde{g}(1) - \tilde{g}(0) = n$. On the other hand, for a generator $\gamma = [p|_{\mathsf{I}}]$ of $\pi_1(S^1, 1)$ and if $m := \deg h$, we have $h_*\gamma = [h \circ p|_{\mathsf{I}}] = [(\bar{\alpha} * p|_{\mathsf{I}} * \alpha)^m]$ for a path from 1 to $h(1)$ in S^1. Since $h \circ p|_{\mathsf{I}}$ is path homotopic to $(\bar{\alpha} * p|_{\mathsf{I}} * \alpha)^m$, their lifts to $\mathbb{R}$ at the same point are path homotopic too. Therefore the ending points are the same. Thus if $\tilde{\beta}$ is a lift of $(\bar{\alpha} * p|_{\mathsf{I}} * \alpha)^m = \bar{\alpha} * p|_{\mathsf{I}}^m * \alpha$, then $\tilde{\beta}(1) - \tilde{\beta}(0) = n$. A lift of $p|_{\mathsf{I}}^m$ is given by $\tilde{q} : \mathsf{I} \to \mathbb{R}, t \mapsto mt$. If $\tilde{\alpha}$ is a lift of α, then $\tilde{\beta} := \bar{\tilde{\alpha}} * (\tilde{q} + \tilde{\alpha}(0)) * (\tilde{\alpha} + m)$ is a lift of $\bar{\alpha} * p|_{\mathsf{I}}^m * \alpha$. $\tilde{\beta}(1) = \tilde{\alpha}(1) + m$ and $\tilde{\beta}(0) = \tilde{\alpha}(1)$, thus $\tilde{\beta}(1) - \tilde{\beta}(0) = m = n$.

(3) Consider the group $\mathsf{G} = \langle x, y \mid x^2, y^2, xy = yx\rangle$.
  (a) Show that $\mathsf{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
  (b) Find a space B such that $\pi_1(B, b) \cong \mathsf{G}$ and its simply-connected covering space. Justify your answer.

Solution
  (a) Consider $\varphi : G \to \mathbb{Z}_2 \times \mathbb{Z}_2$, defined by extending $x \mapsto (1, 0)$, $y \mapsto (0, 1)$ to a homomorphism.
  (b) We know that $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$ and that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$. So if you think of $\mathbb{R}P^2$ as a square with sides appropriately identified, you can think of B as four squares glued together accordingly. Since the map identifying antipodal points $S^2 \to \mathbb{RP}^2$ is a covering map and S^2 is simply-connected, $S^2 \times S^2$ is a simply-connected covering space of $\mathbb{RP}^2 \times \mathbb{RP}^2$.

(4) Find a space X with the fundamental group which is isomorphic to $\langle x, y, z \mid yz = zy\rangle$.

Solution First think about just y and z. We want them to be the generators that commute, so that gives us a torus. Now we want another generator x that does not commute with either x or y. We can achieve this by attaching a loop to some point of the torus. So we get the space that is a torus and a circle sharing a point.

(5) Show that if a space X has the property that every continuous map $f : X \to X$ has a fixed point, then each retract Y of X has the property too.

Solution Let $r : X \to Y$ be the retraction map. Let $g : Y \to Y$ be any continuous map. Define $g' : X \to X$ to be the extension of g, say $g' = g \circ r$. Clearly g' is continuous, so there is a fixed point, say $g'(x_0) = x_0$. Let $y_0 = r(x_0)$, then $y_0 = r(x_0) = r(g'(x_0)) = g'(x_0) = g(r(x_0)) = g(y_0)$, where the middle equality is true since $g'(x_0) \in Y$. Hence, y_0 is the fixed point in Y under g.

(6) Find an example that Borsuk Lemma 13.15 (3) [L] doesn't hold if the map f is not injective.

Solution WLOG assume a and b are the north and the south pole of S^2 and the center of the sphere is at the origin. Consider a map $g : [0, 2] \to S^2$ given by $g(0) = g(1) = g(2) = 1, 0, 0,$ from $g([0, 1])$ is a clockwise loop around the equator and $g([1, 2])$ is a counter-clockwise loop around the equator. Clearly, g is nulhomotopic, but a and b lie in different components of $S^2 - g([0, 2])$.

(7) Let G be a topological group. Show that $\pi_1(\mathsf{G}, 1_{\mathsf{G}})$ is a commutative (abelian) group.

Solution

(8) Prove or disprove the following statement:
- (a) If X is connected, then $\pi_1(X, x_1)$ is isomorphic to $\pi_1(X, x_2)$ for every $x_1, x_2 \in X$.
- (b) If A and B are deformation retract of C, then A are B are homotopy equivalent (i.e. have the same homotopy type.)
- (c) If $p : E \to B$ is a covering map with $p(e) = b$ and $\pi_1(B, b)$ is abelian, then $\pi_1(E, e)$ is abelian, too.

Solution
- (a) False. Need path-connectedness to make the sentence true. The counter example would be as follows. The toplogist's sine curve is the space which is connected but not path-connected. Each path connected component is simply connected so add half circle going from $(0, 1)$ to $(0, -1)$ counterclockwise. Then the fundamental group of one component is $\mathbb{Z}$ but the other one is trivial.
- (b) True.
- (c) True. By Thm 54.6, we know that $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is a monomorphism.

(9) Let $\mathrm{GL}(2, \mathbb{R})$ denote the general linear group, the group of invertible 2×2 matrices with real coefficients. Let $\mathrm{O}(2)$ denote the orthogonal group, the 2×2 orthogonal matrices. Finally, $\mathrm{SO}(2)$ is the special orthogonal group, those matrices $Q \in \mathrm{O}(2)$ satisfying $\det(Q) = 1$. Let I denote the two-by-two identity matrix.
- (a) Determine whether or not $\mathrm{GL}(2, \mathbb{R})$ is connected. Justify your response.
- (b) The Gram-Schmidt orthogonalization process allows us to write a matrix $A \in \mathrm{GL}(2, \mathbb{R})$ uniquely as a product $A = QR$ where $Q \in O(2)$ is orthogonal and R is upper triangular with positive entries on the diagonal. Use this to produce a deformation retraction from $\mathrm{GL}(2, \mathbb{R})$ to $\mathrm{O}(2)$.
- (c) Show that $\mathrm{SO}(2)$, with matrix multiplication, is homeomorphic(as a topological space) to and isomorphic (as a group) to $S^1 = (\mathbb{R}, +)/\mathbb{Z}$.
- (d) Using the fact that $\mathrm{O}(2) = \mathrm{SO}(2) \times \{1, -1\}$, compute $\pi_1(\mathrm{GL}(2, \mathbb{R}), I)$.

Solution:
- (a) Consider the continuous map $\det : \mathrm{GL}(2, \mathbb{R}) \to \mathbb{R} - \{0\}, A \mapsto \det A$. This map is surjective. If $\mathrm{GL}(2, \mathbb{R})$ is connected, then $\mathbb{R} - \{0\}$ must be connected since the image of a continuous map from a connected space is connected. But $\mathbb{R} - \{0\}$ is not connected. Thus $\mathrm{GL}(2, \mathbb{R})$ is not connected.

Solution

(b) (i) Let T_+ be the set of all upper triangular matrices in $\mathrm{GL}(2,\mathbb{R})$ with positive diagonal entries. Consider the deformation retract $H : \mathbb{R} \times [0,1] \to \mathbb{R}, (x,t) \mapsto tx$ of $\mathbb{R}$ to a point $\{0\}$. Consider the homeomorphism $\ln : \mathbb{R}_{>0} \to \mathbb{R}, x \mapsto \ln x$. Use ln to identify $\mathbb{R}_{>0}$ and $\mathbb{R}$, and get a deformation retract $H' : \mathbb{R}_{>0} \times [0,1] \to \mathbb{R}_{>0}$ of $\mathbb{R}_{>0}$ to a point $\{1\}$. Use H and H' to give the deformation retract $F : T_+ \times [0,1] \to T_+$ of T_+ to $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$.

(ii) Consider the map $m : O(2) \times T_+ \to \mathrm{GL}(2,\mathbb{R}), (Q,R) \mapsto QR$, which is a continuous bijective map. It is an open map, since if $U \times V \subset O(2) \times T_+$, then $m(U \times V) = UV = \cup_{v \in V} Uv$ which is a union of open sets. Thus it is a homeomorphism.

(iii) Now the map $(\mathrm{id}_{O(2)}, F) : O(2) \times T_+ \times [0,1] \to O(2) \times T_+$ is a deformation retract of $O(2) \times T_+$ to $O(2)$. By the homeomorphism m, we have found the desired deformation retract.

(c)

$$\mathrm{SO}(2) = \left\{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, 0 \le t < 2\pi\right\}.$$

Define a map f from $S^1 = \{e^{i\theta}, 0 \le \theta < 2\pi\}$ to SO(2) by

$$e^{i\theta} \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

It is a continuous map since it can be obtained from the continuous map $g : [0,2\pi] \to \mathrm{SO}(2), t \mapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ by factoring through the standard quotient map $p : [0,2\pi] \to S^1$. The bijectivity can be checked directly. Since it is a map from compact to Hausdorff, it is a homeomorphism. It is a group homomorphism because

$$f(e^{it}) \cdot f(e^{is}) = \begin{pmatrix} \cos t \cos s - \sin t \sin s & \cos t \sin s + \sin t \cos s \\ -\cos t \sin s + \sin t \cos s & \cos t \cos s - \sin t \sin s \end{pmatrix} = \begin{pmatrix} \cos(t+s) & \sin(t+s) \\ -\sin(t+s) & \cos(t+s) \end{pmatrix}$$

(d) $\pi(\mathrm{GL}(2,\mathbb{R}), I) = \pi(O(2), I) = \pi(\mathrm{SO}(2), I) = \pi(S^1, 1) = \mathbb{Z}$.

(10) Let A be a compact contractible subspace of S^2. Show that A does not separate S^2.

Solution Consider any two points, $a, b \in S^2 \setminus A$. Define $f : A \to S^2 - a - b$ to be the identity map. Then by Borsuk lemma, a and b lie in the same component of $S^2 \setminus f(A) = S^2 \setminus A$. Since a, b were chosen randomly, all points of $S^2 \setminus A$ lie in the same component and so A does not separate S^2.

References

[M] Munkres, Topology.

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[L] Lecture notes, available at http://www.math.cornell.edu/~matsumura/math4530/math4530web.html