MATH 4530 – Topology. Prelim II, Solutions

TAKE HOME

1. Degree

Let $S^1 := \{e^{2\pi i\theta}\} \subset \mathbb{C}$. We define the ***degree*** of a continuous map $S^1 \to S^1$ as follows. Let $x_0 \in S^1$ and let α be a path from $1 \in S^1$ to x_0.

(1) Show that if γ is a generator of $\pi_1(S^1, 1)$, then $\hat{\alpha}(\gamma)$ is a generator of $\pi_1(S^1, x_0)$. See Appendix $*1$.

> **Solution:** Let $[f] \in \pi_1(S^1, x_0)$. Then $\hat{\alpha}^{-1}([f]) = [\alpha] * [f] * [\bar{\alpha}] \in \pi_1(S^1, 1)$. Since γ is a generator for $\pi_1(S^1, 1)$, there is some n such that $[\alpha] * [f] * [\bar{\alpha}] = \gamma^n$ which implies
> $$[f] = [\bar{\alpha}] * \gamma^n * [\alpha] = ([\bar{\alpha}] * \gamma * [\alpha])^n = \hat{\alpha}(\gamma)^n.$$

(2) Show that $\hat{\alpha}(\gamma)$ depends only on x_0 but not on paths α.

> **Solution:** Take another path β from 1 to x_0. Then $\beta * \bar{\alpha}$ is a loop at 1. Now
> $$\hat{\beta}(\gamma) = [\bar{\beta}] * \gamma * [\beta] = [\bar{\beta}] * [\beta * \bar{\alpha}] * \gamma * [\beta * \bar{\alpha}]^{-1} * [\beta] = [\bar{\beta}] * [\beta] * [\bar{\alpha}] * \gamma * [\alpha] * [\bar{\beta}] * [\beta] = \hat{\alpha}(\gamma).$$

By (2), it is OK to write $\gamma_{x_0} := \alpha[\gamma]$ for a path α from 1 to x_0. Now let $h : S^1 \to S^1$ be a continuous map. Let $x_0 \in S^1$ and let $x_1 := h(x_0)$. Define ***degree*** of h to be an integer d such that

$$h_*(\gamma_{x_0}) = (\gamma_{x_1})^d.$$

It is well-defined because, by (1), γ_{x_1} is a generator of $\pi_1(S^1, x_1)$ respectively.

(3) Show that d is independent of the choice of x_0.

> **Solution:** Let $y_0 \in S^1$ and let β be a path from x_0 to y_0. Let $y_1 := h(y_0)$, then $\beta' := h \circ \beta$ is a path from x_1 to y_1.
> $$\gamma_{y_1} = [\bar{\beta}'] * \gamma_{x_1} * [\beta']$$
> $$h_*(\gamma_{y_0}) = h_*([\bar{\beta}] * \gamma_{x_0} * [\beta]) = [\bar{\beta}'] * h_*(\gamma_{x_0}) * [\beta'] = [\bar{\beta}'] * (\gamma_{x_1})^d * [\beta'] = ([\bar{\beta}'] * \gamma_{x_1} * [\beta'])^d = (\gamma_{y_1})^d.$$

(4) Show that d is independent of the choice of γ.

> **Solution:** By $*1$, a generator is either γ or γ^{-1}. We we use γ^{-1}, then
> $$(\gamma^{-1})_{x_0} = (\gamma_{x_0})^{-1}, \quad (\gamma^{-1})_{x_1} = (\gamma_{x_1})^{-1}.$$
> Thus $h_*((\gamma_{x_0})^{-1}) = ((\gamma_{x_1})^d)^{-1} = ((\gamma_{x_1})^{-1})^d$.

By (3) and (4), we have defined the degree of a map h, which is independent of all choices. Now we consider the properties of this degree:

(5) Show that if $h, k : S^1 \to S^1$ are homotopic, they have the same degree.

> **Solution:** It follows from Theorem 11.1 Lecture Notes.

(6) Show that $\deg h \circ k = \deg h \cdot \deg k$.

> **Solution:** It follows from (8) and (7).

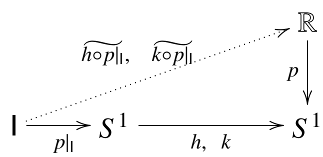
(7) Compute the degree of the map $h(z) = z^n$ where $n \in \mathbb{Z}$.

Solution: HW 9 (5).

(8) (Optional) Show that if $h, k : S^1 \to S^1$ have the same degree, then they are homotopic.

Solution:

- Consider $h_* : \pi_1(S^1, 1) \to \pi_1(S^1, h(1))$ and $k_* : \pi_1(S^1, 1) \to \pi_1(S^1, k(1))$. By the assumption, there is n such that $h_*(\gamma) = \gamma^n_{h(1)}$ and $k_*(\gamma) = \gamma^n_{k(1)}$.
- Let $\gamma := [p|_{\mathsf{I}}]$ where $p : \mathbb{R} \to S^1, t \mapsto e^{2\pi i t}$ is the standard map and $\mathsf{I} := [0, 1]$. Consider the following lifting diagram,



$\widetilde{h \circ p}|_{\mathsf{I}}$ and $\widetilde{k \circ p}|_{\mathsf{I}}$ are the lifts of $h \circ p|_{\mathsf{I}}$ and $k \circ p|_{\mathsf{I}}$ at $h_0 \in p^{-1}(h(1))$ and $k_0 \in p^{-1}(k(1))$ respectively. Let $h_1 := \widetilde{h \circ p}|_{\mathsf{I}}(1)$ and $k_1 := \widetilde{k \circ p}|_{\mathsf{I}}(1)$.
- Since $h_*(\gamma) = [h \circ p|_{\mathsf{I}}] = \gamma^n_{h(1)}$ and $k_*(\gamma) = [k \circ p|_{\mathsf{I}}] = \gamma^n_{k(1)}$, we have $h_1 - h_0 = k_1 - k_0 = n$.
- I wanted to find a path-homotopy between $\widetilde{h \circ p}|_{\mathsf{I}}$ and $\widetilde{k \circ p}|_{\mathsf{I}}$, but because the starting points and ending points are different, I can find it. So I will shift one of them. Define

$$\widetilde{\widetilde{h \circ p}}|_{\mathsf{I}}(s) := \widetilde{h \circ p}|_{\mathsf{I}}(s) - h_0 + k_0.$$

Now it is a path from k_0 to k_1. You can check it by evaluating at 0 and 1. Thus there is a path-homotopy $\tilde{F}$ from $\widetilde{\widetilde{h \circ p}}|_{\mathsf{I}}$ to $\widetilde{k \circ p}|_{\mathsf{I}}$, because $\mathbb{R}$ is a contractible space.
- Consider $\tilde{h}(x) := h(x) \cdot e^{2\pi i(k_0 - h_0)}$. Then $\widetilde{\widetilde{h \circ p}}|_{\mathsf{I}}$ is the lift of $\tilde{h} \circ p|_{\mathsf{I}}$ at k_0. Then we check that $p \circ \tilde{F}$ is a path homotopy from $\tilde{h} \circ p|_{\mathsf{I}}$ to $k \circ p|_{\mathsf{I}}$. Now since $p \circ \tilde{F}$ is a *path*-homotopy, it factors through (p, id), giving a homotopy F from $\tilde{h}$ to k.
- Now the homotopy from h to $\tilde{h}$ is easy to find:

$$G(x, t) := h(x) e^{2\pi i t(k_0 - h_0)}.$$

- By constructing F and G, we have shown that h is homotopic to k.

(5) says the degree is a homotopy invariant, i.e. if h, k have the different degrees, they can not be homotopic to each other. Together with (7) and (8), it classifies all homotopy equivalence classes of maps $S^1 \to S^1$. (6) says associating degrees have a certain algebraic structure.

Appendix

*1 An infinite cyclic group G is a group isomorphic to $(\mathbb{Z}, +)$. We say $g \in \mathsf{G}$ is a generator, if $\mathsf{G} = \{g^n \mid n \in \mathbb{Z}\}$. If g is a generator, then g^{-1} is also a generator and any generator is either g or g^{-1}.