

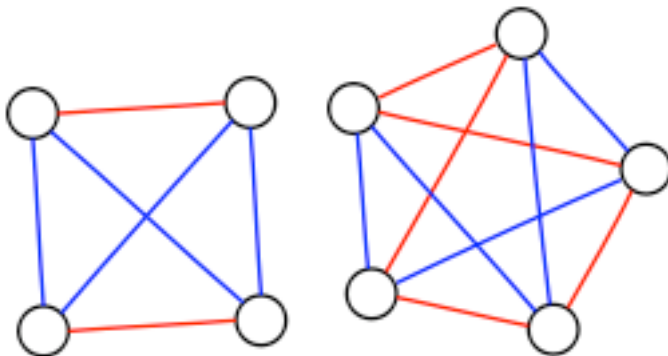
GRAPH THEORY SOLUTIONS

1. Friends and Strangers

1. K_3 has 3 edges. K_4 has 6 edges. K_5 has 10 edges. The most straight forward way to do this is to just start adding edges in a systematic way. If you are making K_n you can draw n edges from the first vertex, $n - 1$ for the second, and so on until you draw 1 edge for the n th vertex. Thus there are $\sum_{i=1}^n i = n(n - 1)/2$ edges. This formula might be familiar or it might not. In case of the latter, the identity can be seen by thinking about taking an $n - 1$ by n rectangle and putting stacks of height i inside this rectangle.

However this is a clever way to do this. Each edge in K_n represents a choice of 2 out of n vertices, and since we want all possible edges to be present there must be $C(n, 2) = n(n - 1)/2$ edges.

2. Here are examples of such colorings:



2. Influence Model

1. Since $p(v, u)$ is non-zero only if v has the same color as u so we have that $p_{net}(v, R) = \sum_u p(u, v) = d(R)t$ and similarly $p_{net}(v, B) = d(B)(1 - t)$. The vertex v should choose to be red when $p_{net}(v, R) \geq p_{net}(v, B)$, and this happens when $d(R)t \geq (1 - t)d(B)$. Solving for t we get that $t \geq d(B)/(d(R) + d(B) = d(B)/deg(v)$. Thus if t is at least as large as the proportion of blue neighbors of v then v should become red.

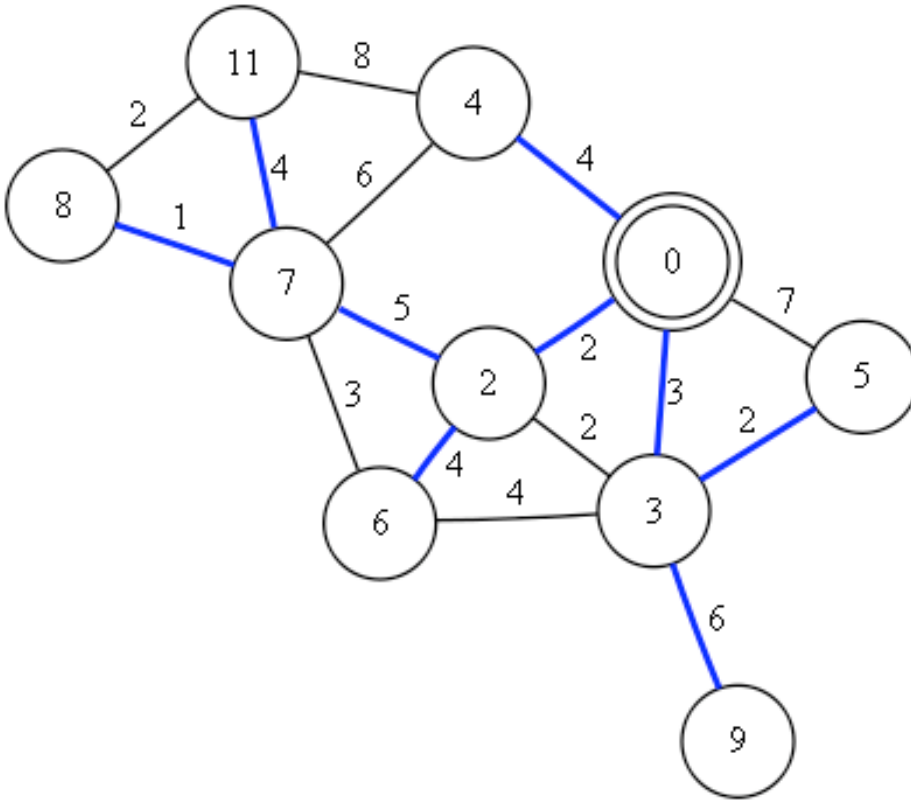
2. For $t < 0.5$ nothing happens. Otherwise the red nodes spread. If we label the vertices as the integers with the early adopter at zero we see the red vertices spread outwards, alternating between even and odd vertices. If we start with 0 and 1 as early adopters then the trend spreads without alternating between evens and odds.

3. A weighted graph which will always be taken over by red is a ray. Put down an early adopter and send an edge from it to a new vertex v_1 . Each vertex v_i then sends an edge to v_{i+1} .

For red to spread at the first step we need $t \geq (1 - t)w$, so $w \leq t/(1 - t)$. After that the same behavior as seen in the first part of 1. occurs.

3. Finding Shortest Paths

1. Here is a version of the graph with the minimal spanning tree in blue and the distances from the source labeled.

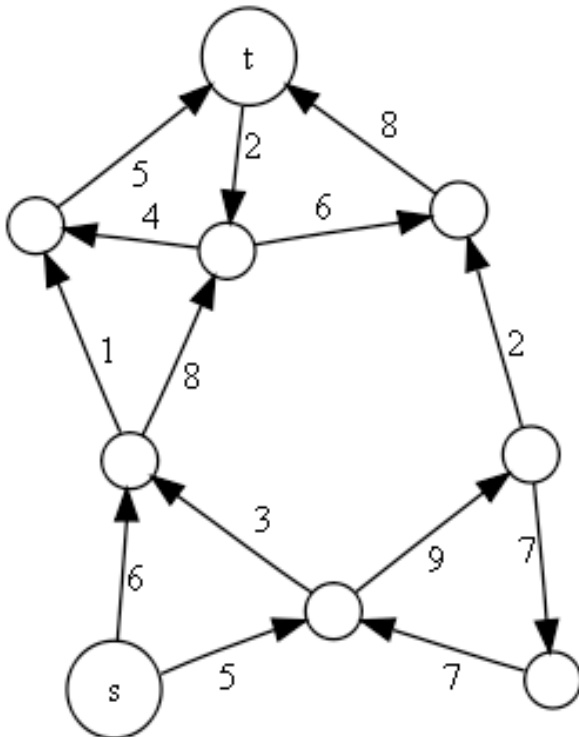


2. The only way it would be worse to take C would be if you knew it took longer than B on average. Thus $12p + 20(1 - p) > 17$. Solving for p shows that C should not be taken in $p < 3/8$.

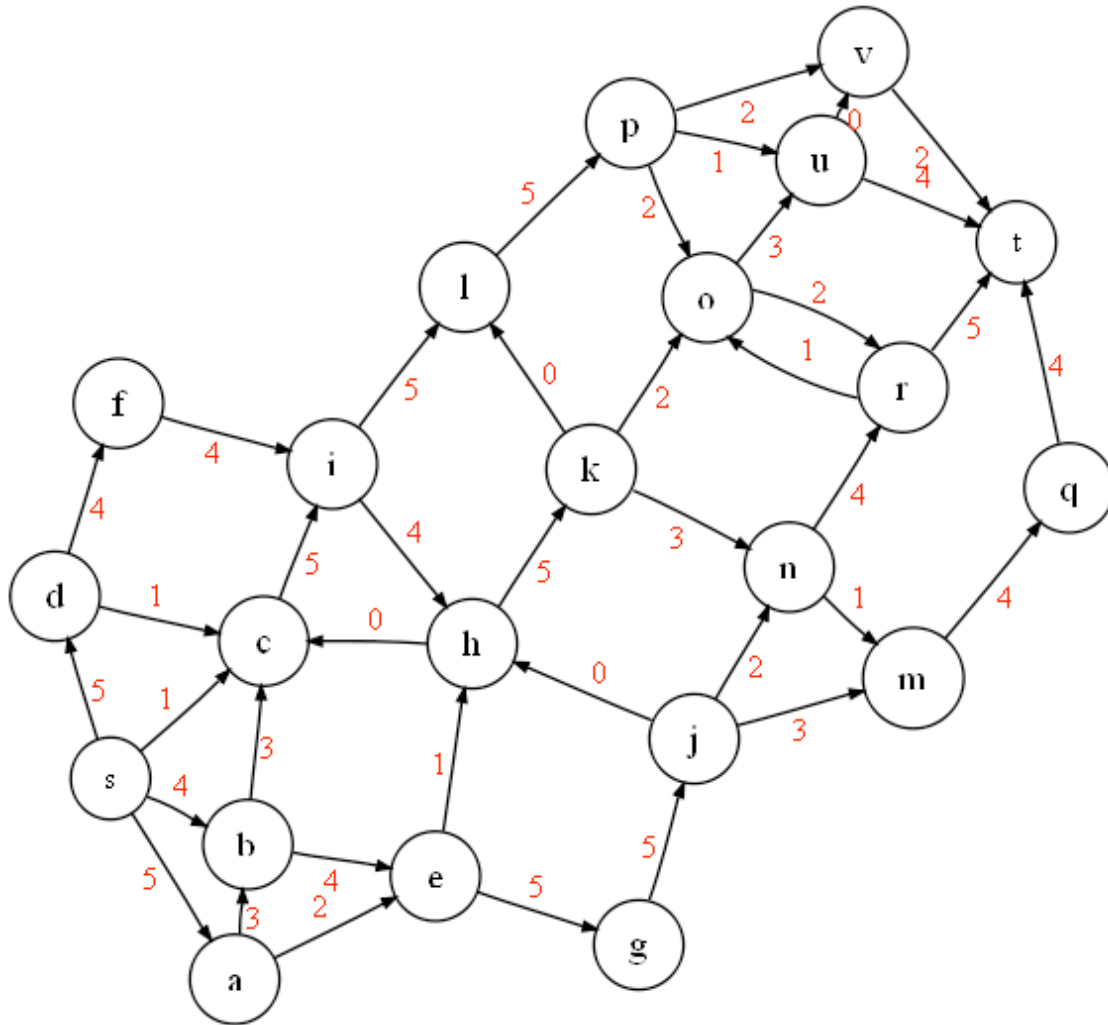
4. Flows

1. The first graph needs 2 cuts. The edges leaving s or t suffice. The 2nd graph needs 3, and the edges around s or t again suffice. The paths in this one will need to take advantage of one of the double edges. The 3rd graph needs 3 cuts. This time the cuts need to be made in the middle of the right hand side of the graph. The particular edges can be found by drawing several paths from s to t and seeing where they all tend to intersect.

2. One possible labeling is shown below. The flow out of s is equal to the flow into t , which is 11.



3. Below is a labeled version of the graph: $S = \{s, a, b, c, d, e, f, g\}$. The capacity of the cut and the value of the flow is 15.



5. Random Walks

1. An easy example on which $q(2n + 1) = 0$ is the graph of \mathbb{Z} used in the article. Such graphs are generally called bipartite graphs. This means that you can split the vertices into 2 sets L and R and the edges in the graph only go between these two sets. If the walker

starts in L then he will always be in R for odd times and in L for even times. If we want $q(2n + 1) > 0$ for large enough n we can just take a bipartite graph and add some edges between vertices in R . If we let the walker go long enough eventually he will be able to cross this inter- R edge, and will then be able to be in R at even times with positive probability. Similarly if the walker can be in R for even times then he will also be able to be in L at odd times. A simple example of such a graph is a triangle.

2. At each step the walker moves left with probability $1/2$ and right with probability $1/2$. We can view the walker's current position as an integer i . Steps to the left are like adding -1 and steps to the right are like adding 1 . Thus $p(i, i - 1) = 1/2$, $p(i, i + 1) = 1/2$, and $p(i, j) = 0$ if $|i - j| \neq 1$.

3. To get back to 0 in $2n$ steps n of those steps must be to the right and n to the left. There are $C(2n, n)$ ways to arrange these left and right steps (You've got $2n$ steps, and you chose which n of them should be to the left. The rest are rights). There are 2^{2n} total walks with $2n$ steps, so $q(2n) = C(2n, n)/2^{2n}$. We then use Stirling's Approximation (after plugging in factorials for $C(2n, n)$) to get

$$(1) \quad q(2n) \approx 2^{-2n} \frac{\sqrt{4\pi n} (2n)^{2n} e^{-2n}}{(\sqrt{2\pi n} (n)^n e^{-n})^2}$$

$$(2) \quad = \frac{1}{\sqrt{\pi n}}.$$

4. Lets examine the possible values of $(Y(n) + Z(n))/2$. These are $(1, 0)$, $(-1, 0)$, $(0, 1)$ and $(0, -1)$. Each of these occurs with probability $1/4$. To see this we calculate one example and note that an analogous calculation holds for the other values. With probability $1/2$, $Y(n) = (1, 1)$ and with probability $1/2$, $Z(n) = (1, -1)$. As these walks are independent these events happen simultaneously with probability $(1/2)(1/2) = 1/4$, so $(Y(n) + Z(n))/2 = (1, 0)$ with probability $1/4$.

5. We have n steps to assign our 3 directions into. First we place the Us. There are $C(n, i)$ ways to do this. For each placement of the Us we still need to place j Ns in the remaining $n - i$ places. There are $C(n - i, j)$ to do this. The Es are put into the remaining $n - i - j$ spaces. Thus there are $C(n, i)C(n - i, j)$ ways to place the Us, Ns, and Es.