1. Let $A$ be a subset of a topological space $X$. Prove that every closed set $V$ that contains $A$ also contains the closure $\overline{A}$.
2. Let $X$ be a topological space, and $Y$ a subspace. Show that if $A$ is closed in $Y$, and $Y$ is closed in $X$, then $A$ is closed in $X$. Prove the same statement for open subsets.
3. Let $X$ be a topological space, $U$ an open subset, and $A$ a closed subset. Show that
$$U - A = \{u \in U \mid u \notin A\}$$
is open in $X$, and
$$A - U = \{a \in A \mid a \notin U\}$$
is closed in $X$.
4. Let $A$ and $B$ be subsets of a topological space $X$. Determine which of the following equalities hold. If the equality does not hold, determine if one containment or the other holds. Justify your answer with a proof or counterexample.
   (a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
   (b) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
   (c) $\overline{(\bigcup_\alpha A_\alpha)} = \bigcup_\alpha \overline{A}_\alpha$.
5. Let $Y$ be a subspace of $X$, and $Z$ a subspace of $Y$. Prove that $Z$ is a subspace of $X$.
6. Let $C_i$ denote the circle of radius $\frac{1}{i}$ in $\mathbb{R}^2$, and let
$$X = \bigcup_{i=1}^{\infty} C_i.$$
Determine whether or not $X$ is a closed subset of $\mathbb{R}^2$ (with the usual topology). Why or why not?
7. Consider the set
$$\left\{ [a, b) = \{x \mid a \leq x < b\} \;\middle|\; a, b \in \mathbb{R} \right\}.$$
Show that this is a basis for a topology on $\mathbb{R}$. This topology is called the **lower limit topology**.
8. Declare a subset of $\mathbb{R}$ to be **open** if its complement is finite or all of $\mathbb{R}$. Show that this defines a topology on $\mathbb{R}$. This is known as the **finite complement** topology. What is the closure of a subset $A$ in $\mathbb{R}$?
9. Let $I_a$ denote the subset $(a, \infty)$ of $\mathbb{R}$. Notice that $I_\infty = \emptyset$ and $I_{-\infty} = \mathbb{R}$. Show that the collection $\mathcal{O} = \{I_a \mid a \in \mathbb{R}\}$, is a topology on $\mathbb{R}$. What is the closure of a subset $A$ in $\mathbb{R}$?
10. Compare the topologies on $\mathbb{R}$ that we have seen so far:
    (a) the discrete topology $\mathcal{D}$;
    (b) the finite complement topology $\mathcal{F}$;
    (c) the topology $\mathcal{O}$;
    (d) the standard topology $\mathcal{S}$; and
    (e) the topology $\mathcal{T} = \{\emptyset, \mathbb{R}\}$.
    What is the partial ordering on these? That is, which are finer than which? Which are incomparable?

*11. (Kuratowski) Consider the power set of a topological space $X$: the set of all subsets of $X$. The operations **closure** $A \mapsto \overline{A}$ and **complement** $A \mapsto A'$ are functions on the power set.
    (a) Show that for any set $A \subseteq X$, one can form no more than 14 distinct sets by successively applying these two operations.
    (b) Find a subset $A$ of $\mathbb{R}$ with the standard topology that achieves this maximum.