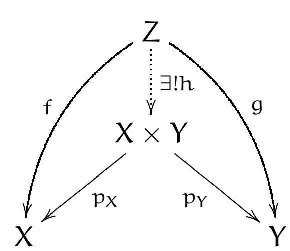
1. Suppose that $f : X \to Y$ is a continuous bijection. Show that if X is compact and Y is Hausdorff, then f is a homeomorphism.
2. Show that if Y is a compact space, then the projection $p_X : X \times Y \to X$ is a **closed map**: one that carries closed sets to closed sets.
3. Suppose that $A \subset X$. A **retraction** of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for every $a \in A$. Show that a retraction is a quotient map.
4. Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are both topologies on X.
   a. Suppose $\mathcal{T} \subseteq \mathcal{T}'$. What does compactness of X under one of these topologies imply about compactness under the other?
   b. If X is compact and Hausdorff under both $\mathcal{T}$ and $\mathcal{T}'$, show that either $\mathcal{T} = \mathcal{T}'$, or the topologies are incomparable.
5. If X is a set, recall that in the **finite complement topology**, a subset $A \subseteq X$ is open if its complement $X - A$ is either a finite set or all of X. Show that in this topology, every subspace is compact.
6. Let X be a topological space. Consider the relation $x \sim y$ if there exists a connected subspace $A \subseteq X$ with $x, y \in A$.
   a. Show that this is an equivalence relation on X. It's equivalence classes are called **components** or **connected components**.
   b. Show that each component is connected, any two components are equal or disjoint, and that the union of all the components is X.
   c. Show that a component is a closed subset of X.
7. Show that a finite set in a Hausdorff space is closed.
8. Let $\{X_\alpha\}$ be a collection of topological spaces, and let p_α denote the projection to the α coordinate. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be a sequence of points in the product $\prod X_\alpha$. Show that this sequence converges to a point $\mathbf{x}$ (in the product topology) if and only if the sequence $p_\alpha(\mathbf{x}_1), p_\alpha(\mathbf{x}_2), \ldots$ converges to $p_\alpha(\mathbf{x})$ for every α. Is this fact true if you use the box topology rather than the product topology?
9. (EXTRA CREDIT) If X and Y are topological spaces, recall that the product topology on $X \times Y$ is the coarsest topology on the set so that the projection maps $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are continuous. Show that $X \times Y$ satisfies that following property:

   For every topological space Z and pair of continuous maps $f : Z \to X$ and $g : Z \to Y$, there exists a unique continuous map $h : Z \to X \times Y$ so that $f = p_X \circ h$ and $g = p_Y \circ h$.

   This can be described by the following diagram:



   No matter which way you follow maps from Z down to X or to Y, whether or not you pass through $X \times Y$, you end up with the same result (because $f = p_X \circ h$ and $g = p_Y \circ h$). Such a diagram is called a **commutative diagram**.

   Finally, show that $X \times Y$ is the unique topological space that satisfies this property.

   A defining property like this is called a **universal property**.