1. A topological space X is **limit point compact** if every infinite subset of X has a limit point (in X). Prove that every compact space is limit point compact. Prove that $\mathbb{R}$ with the standard topology is not limit point compact.
2. Suppose that (X, ρ) is a metric space. Recall that for any subset $A \subseteq X$, we may define the distance from A to be $\rho_A : X \to \mathbb{R}$ given by $\rho_A(x) = \text{g.l.b.}\{\rho(x, a)\}$, where the greatest lower bound is taken over all elements of A. Suppose that A and B are disjoint closed subsets of X, with the metric topology. Show that the function
$$f(x) = \frac{\rho_A(x) - \rho_B(x)}{\rho_A(x) + \rho_B(x)}$$
is a continuous real-valued function $f : X \to [-1, 1]$, with $f^{-1}(\{1\}) = B$ and $f^{-1}(\{-1\}) = A$.
3. Suppose that X is a connected normal topological space. Show that if X contains more than one point, then X is uncountable.
4. Suppose that X is a compact Hausdorff space. Show that X is a normal space.
5. Suppose that X is a locally compact Hausdorff space. Show that if $A \subseteq X$ is an open or closed subset, then A with the subspace topology is also locally compact.
6. Suppose that $f : X \to Y$ is a homeomorphism of locally compact Hausdorff spaces. Show that this extends to a homeomorphism of the one point compactifications.
7. Suppose that X is locally compact, and $f : X \to Y$ is continuous. Is the image $f(X)$ locally compact? What if f is a continuous and open map? Justify your answer.
8. Let $\mathbb{Z}_+$ denote the positive integers, a subspace of $\mathbb{R}$. Show that the one point compactification of $\mathbb{Z}_+$ is homeomorphic to $\{0\} \cup \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{Z}_+ \right\} \subseteq \mathbb{R}$.
9. (EXTRA CREDIT) Let $O(n)$ denote the $n \times n$ orthogonal matrices with real entries. That is, $O(n)$ consists of those matrices whose columns form an orthonormal basis of $\mathbb{R}^n$. This is a subset of all $n \times n$ matrices, $\mathcal{M}_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$, and hence a topological space with the subspace topology.
   a. Show that $O(n)$ is a closed (topological) subspace of $\mathbb{R}^{n^2}$, by considering the dot products $v_i \cdot v_j$ of columns as continuous functions $\mathbb{R}^{n^2} \to \mathbb{R}$.
   b. Show that $O(n)$ is compact.