Math 304
Homework 10 Solutions

Michael O'Connor

May 4, 2008

Problem 1 Since $f(-10) < 0$, $f(-1) > 0$, $f(1) < 0$ and $f(10) > 0$, it follows by the intermediate value theorem that f has a root in each of $(-10, -1)$, $(-1, 1)$, $(1, 10)$. Therefore f has at least three real roots. By a problem on the first prelim, we know that f can have at most three real roots.

Problem 2 False. Let $f(x) = x^3$. Then $f'(0) = 0$, but $g(x) = \sqrt[3]{x}$ is such that $f(g(x)) = g(f(x)) = x$ for all x in $\mathbb{R}$.

Problem 3 Assume that f is increasing. Let U be an open set. We would like to show that $f(U)$ is open. So let $y \in f(U)$ and we must find an ϵ so that $N_\epsilon(y) \subset f(U)$.

Since $y \in f(U)$ there is some $x \in U$ such that $f(x) = y$. Since U is open there is some ϵ' such that $N_{\epsilon'}(x) \subset U$. Since $N_{\epsilon'}(x) = [x - \epsilon', x + \epsilon']$ by the inverse function theorem, $f(N_{\epsilon'}(x)) = [f(x - \epsilon'), f(x + \epsilon')] \subset f(U)$.

Since $y = f(x) \in [f(x - \epsilon'), f(x + \epsilon')]$, we may take $\epsilon = \min\{y - f(x - \epsilon'), f(x + \epsilon') - y\}$ and we have $N_\epsilon(y) \subset f(U)$.

For the case where f is decreasing, we may simply take $g(x) = -f(x)$. Then g is strictly monotonically increasing, and we may use the same proof.

Problem 4 Since 3 copies of T_∞ scaled down by 2 in each direction cover T_∞, its fractal dimension is $\log_2 3$.