Math 304
Homework 2 Solutions

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February 9, 2008

Problem #1 The statement in 2.17 is false. A true statement is: "If x and y are real numbers such that $x^2 + 6y^2 = 25$ and $y^2 + x = 3$, then $y = \pm 2$."

We can prove that statement as follows: Suppose that $x^2 + 6y^2 = 25$ and $y^2 + x = 3$. Rearranging the second equation, we get $y^2 = 3 - x$. We can substitute that into the first equation to get that $x^2 + 6(3 - x) = 25$. Solving, we get that $x = -1$ or $x = 7$.

If $x = 7$, then $y^2 + 7 = 3$, so that $y^2 < 0$, which is not possible if y is real.

Thus, x must be -1 and we get $y^2 - 1 = 3$, which gives $y^2 = 4$. Therefore, $y = \pm 2$.

Some people incorrectly started this problem by assuming that $y = 2$, citing the "backwards method." The backwards method does not allow you to prove something by assuming its conclusion. It is instead a method of coming up with a proof of a statement which starts by contemplating the conclusion of the statement rather than the hypotheses.

Problem #2 "Being friendly" is not an equivalence relation since it is not transitive. We can show this by producing triangles A, B, and C such that A is friendly with B, B is friendly with C, but A is not friendly with C.

It is a fact from geometry that if x, y, and z are positive real numbers, then there is a triangle with side lengths x, y, and z if no one of the three numbers is bigger than or equal to the sum of the other two.

Therefore, there is a triangle with side lengths 2, 3, 4, which we will let be A, there is a triangle with side lengths 3, 4, 5, which we will let be B, and there is a triangle with side lengths 4, 5, 6, which we will let be C. It is then easy to verify that A and B are friendly, B and C are friendly, but B and C are not friendly.

Problem #3 a) Reflexivity: Let x be a real number. Then $x - x = 0$, which is rational.

Symmetry: Suppose x and y are rationally different, so that $x - y$ is rational. Then $y - x = -(x - y)$ is rational as well, so that y and x are rationally different.

Transitivity: Suppose that x and y are rationally different and that y and z are rationally different. Then $x - y$ is rational and $y - z$ is rational. Therefore $(x - y) + (y - z) = x - z$ is rational. Therefore x and z are rationally different.

b) Suppose that x and x' are rationally different (i.e., $x' \in [x]$) and that y and y' are rationally different (i.e., $y' \in [y]$). We must show that $x + y$ and $x' + y'$ are rationally different.

Since x and x' are rationally different, $x - x'$ is rational. Since y and y' are rationally different, $y - y'$ is rational. Therefore $(x - x') + (y - y') = (x+y) - (x'+y')$ is rational. Therefore $x+y$ and $x'+y'$ are rationally different.

c) This "definition" is not well-defined. For a counterexample: 0 is rationally different from 1, and π is rationally different from π, but $0 \cdot \pi = 0$ is not rationally different from $1 \cdot \pi = \pi$.

A couple of people did this part in the following way: They supposed that $x - x'$ and $y - y'$ was rational like in part b) and tried to show that $xy - x'y'$ was rational by showing that it was equal to $(x - x') \cdot (y - y')$. When that failed, they concluded that the definition was not well-defined.

This is not correct. Although showing that $(x - x') \cdot (y - y') = xy - x'y'$ would be one method of showing that $xy - x'y'$ is rational, it's not the only possible method, so if that fails, you cannot conclude that $xy - x'y'$ need not be rational.