Math 304
Homework 4 Solutions

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Problem #1 By a Cantor diagonalization argument which you saw in class, the set of positive rationals $\mathbb{Q}^+$ is countable. The set of all rationals is equal to $\mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$, where $\mathbb{Q}^-$ is the set of negative rationals.

By precisely the same argument, $\mathbb{Q}^-$ is countable, and $\{0\}$ is countable since it is finite. Therefore, $\mathbb{Q}$ is countable as it is the union of countably many (in fact, 3) countable sets.

Problem #2 By the Intermediate Value Theorem, f is surjective. Therefore, there is an injective function $g\colon [0,1] \to [0,1]$ such that $f(g(y)) = y$ for all $y \in [0,1]$. Assume that f maps no irrational to an irrational. Then g must map each irrational to a rational. But, since g is injective, this implies that $|\mathbb{R} - \mathbb{Q}| \leq |\mathbb{Q}|$ which is false, since $|\mathbb{Q}| = \aleph_0$ but $|\mathbb{R} - \mathbb{Q}| = 2^{\aleph_0}$.

Problem #3 Let A be a countable set. For each $n \in \mathbb{N}$, let $A_n = \{S \subset A \mid \text{card}(S) = n\}$.

Each A_n is countable: For all $n \in \mathbb{N}$, $A^n = \overbrace{A \times \cdots \times A}^{n \text{ times}}$ is countable and there is a surjection f_n from A^n to A_n defined by $f_n(\langle x_1, \ldots, x_n \rangle) = \{x_1, \ldots, x_n\}$. Since any finite set has cardinality some $n \in \mathbb{N}$, the set of finite subsets of A is equal to $\bigcup_{n\in\mathbb{N}} A_n$. Since this is a countable union of countable sets, there are only countably many finite subsets of A.