Math 304
Homework 5 Solutions

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Problem #1 Let $A = \{2n - 1 \mid n \in \mathbb{N}_+\}$, and let F be as given in the problem statement using that definition of A.

We must show that F is a bijection from $\mathbb{N}_+$ to $\mathbb{N}_+$; i.e., that it is an injection and that it is a surjection. Observe first that if x is even then $F(x)$ is odd, and if x is odd then $F(x)$ is even.

To show that F is an injection: Suppose that $F(x) = F(y)$. First suppose that $F(x) = F(y)$ is even. Then x and y must be odd and hence $F(x) = x + 1$ and $F(y) = y + 1$. Thus, from $F(x) = F(y)$ we see that $x + 1 = y + 1$ and therefore $x = y$. Now suppose that $F(x) = F(y)$ is odd. Then x and y are even and $F(x) = x - 1$ and $F(y) = y - 1$. Thus, from $F(x) = F(y)$ we see that $x - 1 = y - 1$ and therefore $x = y$.

To show that F is a surjection: Let $y \in \mathbb{N}_+$. If y is even, then $y - 1$ is odd and therefore $F(y - 1) = (y - 1) + 1 = y$. If y is odd, then $y + 1$ is even and therefore $F(y + 1) = (y + 1) - 1 = y$.

Problem #2 Let $A = \{4^k \cdot m \mid m \text{ odd}, k \in \mathbb{N}\}$, and let F be as given in the problem statement using that definition of A. We will show that F is a bijection.

For any $x \in \mathbb{N}_+$, let $p(x)$ be the greatest k such that 2^k divides x. That is, for any x, $x = 2^{p(x)} \cdot m$, where m is odd. Observe that if $p(x)$ is even then $p(F(x))$ is odd, and if $p(x)$ is odd then $p(F(x))$ is even.

To show that F is an injection: Suppose that $F(x) = F(y)$. First suppose that $p(F(x))$ is even. Then $p(x)$ and $p(y)$ must be odd and hence $F(x) = 2x$ and $F(y) = 2y$. Thus, from $F(x) = F(y)$ we see that $2x = 2y$ and therefore $x = y$. Now suppose that $p(F(x))$ is odd. Then $p(x)$ and $p(y)$ are even and $F(x) = x/2$ and $F(y) = y/2$. Thus, from $F(x) = F(y)$ we see that $x/2 = y/2$ and therefore $x = y$.

To show that F is a surjection: Let $y \in \mathbb{N}_+$. If $p(y)$ is even, then $p(y/2)$ is odd and therefore $F(y/2) = 2(y/2) = y$. If $p(y)$ is odd, then $p(2y)$ is even and therefore $F(2y) = (2y)/2 = y$.

Problem #3 In the Halmos reading, we are told that if A is infinite, then $|A| = |A \times A|$. The problem follows immediately from that.

Alternatively, we can construct an explicit bijection between $\mathbb{R}$ and $\mathbb{R}^2$ or construct injections both ways and use the Schroeder-Bernstein theorem. I'll demonstrate how to do the latter.

To construct an injection from $\mathbb{R}$ to $\mathbb{R}^2$: take $f(x) = (x, 0)$ for example.

To construct an injection from $\mathbb{R}^2$ to $\mathbb{R}$:

A common attempt is as follows: First we construct an injection from $[0,1)^2$ to $\mathbb{R}$. This will suffice since we can cover $\mathbb{R}^2$ with countably infinitely many half-open squares $[0,1)^2$ (and $\mathbb{R}$ is infinite).

We do that as follows: Given a pair of real numbers in decimal form $(0.x_1x_3x_5\ldots, 0.x_2x_4x_6\ldots)$ send it to the real number $0.x_1x_2x_3x_4\ldots \in [0,1)$. This almost works, but it has the problem that decimal expansions are not unique. (For example, $0.09999\ldots = 0.1$.) This means that the map is not actually injective: the pairs $(0.09999\ldots, 0.999\cdots)$ and $(0.1, 0.0)$ are different but get sent to the same number. In fact that map is not even well-defined, since $(0.1, 0.0)$ could get mapped to 0.1 or to $0.00909090\ldots$ depending on the representation of 0.1 that you use.

However, we can observe the following: Let $\mathbb{I}$ be the set of irrationals in $[0,1)$. Let $\mathbb{Q}_{[0,1)} = \mathbb{Q} \cap [0,1)$ be the set of rationals in $[0,1)$. Let g be a bijection from $\mathbb{Q}$ to $\mathbb{N}$. The above *does* give an injection from $\mathbb{I}^2$ to $\mathbb{I}$ since the irrational number have a unique decimal expansion. Since $[0,1)^2 = \mathbb{I}^2 \cup (\mathbb{Q}_{[0,1)} \times \mathbb{I}) \cup (\mathbb{I} \times \mathbb{Q}_{[0,1)})$, we can complete the injection by having it send (x, y) to $2f(x) + y$ when x is rational and $x + 2f(y) + 1$ when y is rational.

Problem #4 Let $B = \{f \mid f \text{ is a bijection from } \mathbb{N}_+ \text{ to } \mathbb{N}_+\}$. We will show that $|B| = 2^{\aleph_0}$. Recall that $2^{\aleph_0} = |\mathcal{P}(\mathbb{N}_+)|$, the cardinality of the power set of $\mathbb{N}$. To do this we will show that there is an injection from $\mathcal{P}(\mathbb{N}_+)$ to B and an injection from B to $\mathcal{P}(\mathbb{N}_+)$.

To find an injection from $\mathcal{P}(\mathbb{N}_+)$ to B: Given $S \subset \mathbb{N}_+$, define $f_S \colon \mathbb{N}_+ \to \mathbb{N}_+$ as follows: If $x \in S$, let $f_S(2x) = 2x - 1$ and $f_S(2x - 1) = 2x$. If $x \notin S$, let $f_S(2x) = 2x$ and $f_S(2x - 1) = 2x - 1$.

We must show both that for any S, f_S is a bijection and that the map $S \mapsto f_S$ is an injection. I will omit the proof that f_S is always a bijection as it is similar to problem 1. To show that the map $S \mapsto f_S$ is an injection, suppose that $S_1 \neq S_2$ and we will show that $f_{S_1} \neq f_{S_2}$. If $S_1 \neq S_2$, then one of the two sets must contain an element not in the other. Without loss of generality, suppose that $x \in S_1$ and $x \notin S_2$. Then $f_{S_1}(2x) = 2x - 1$, but $f_{S_2}(2x) = 2x$. Thus, $f_{S_1} \neq f_{S_2}$.

To find an injection from B to $\mathcal{P}(\mathbb{N}_+)$: Given a bijection $f \in B$, let $S_f = \{f(1), f(1)+f(2), f(1)+f(2)+f(3), \ldots\}$. (Ryan Zhou came up with this idea.) We will show that the map $f \mapsto S_f$ is an injection. Let $S_f = S_g$. We will show by induction on x that $f(x) = g(x)$, thus showing that $f = g$.

Base case: It must be that $f(1) = g(1)$ since they must both be the smallest element of $S_f = S_g$.

Inductive case: Suppose that $f(x') = g(x')$ for all $x' \leq x$. Then $f(x+1)$ must be the $(x+1)$th smallest element of S_f minus $f(1) + f(2) + \cdots f(x)$. Since $S_f = S_g$ and $f(x') = g(x')$ for $x' \leq x$, $f(x+1) = g(x+1)$.

Problem #5 The only difficult part of this problem is that, in order to show that y is unique, you must show both that there *is* a y with the required property, and that there is *at most one* y with the required property.

Problem #6 Let $P(n,m)$ be the following statement: If there are $2n+1$ chocolates and $2m+1$ caramels in the machine, then the machine must eventually dispense a chocolate/caramel pair. We will prove $\forall m\, P(n,m)$ by induction on n.

The base case is when $n = 0$: In this case there is one chocolate and we must prove that no matter what odd number of caramels are in the machine, it must eventually dispense a chocolate/caramel pair. Since we want to prove $\forall m\, P(0,m)$, we can use a subinduction on m.

The base case of the subinduction is $m = 0$: Now we're in the case where there is one chocolate and one caramel. Obviously, the machine can do nothing but give a chocolate/caramel pair.

For the inductive case of the subinduction we assume that we've proven $P(0,m)$ and try to prove $P(0,m+1)$. Suppose that we have 1 chocolate and $2(m+1)+1$ caramels. Either the machine can dispense a chocolate/caramel pair, in which case we're done, or it can dispense two caramels. In that case we're left with one chocolate and $2m+1$ caramels, and we're done by the inductive hypothesis.

For the inductive case (of the main induction) we assume that we've proven $\forall m\, P(n,m)$ and try to show $\forall m\, P(n+1,m)$. We will show it by a subinduction on m.

The base case of this subinduction is $m = 0$. In this case we know that $\forall m\, P(n,m)$ holds and we want to show that $P(n+1,0)$ holds. So we have one caramel and $2(n+1)+1$ chocolates. Either the machine can dispense a chocolate/caramel pair in which case we're done, or it can dispense two chocolates. In that case we have one caramel and $2n+1$ chocolates left and we're done by induction.

For the inductive case of this subinduction we assume that we know $\forall m\, P(n,m)$ (the main inductive hypothesis) and $P(n+1,m)$ (the subinductive hypothesis) and try to show $P(n+1,m+1)$. So we assume we have $2(n+1)+1$ chocolates and $2(m+1)+1$ caramels in the machine. Either the machine can dispense a chocolate/caramel pair and be done, or it can dispense two chocolates and we're done by the main inductive hypothesis, or it can dispense two caramels and we're done by the subinductive hypothesis.