Math 304
Homework 6 Solutions

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C.2 Let a and b be such that $a|b$ and $b|a$. Then there is a k_1 and a k_2 such that $a = k_1 b$ and $b = k_2 a$. Therefore, $a = k_1 k_2 a$, and $(1 - k1k_2)a = 0$.

Therefore, either $a = 0$ or $1 - k_1 k_2 = 0$. If $a = 0$, then b must be 0 as well, and $a = b$.

If $1 - k_1 k_2 = 0$, then $k_1 k_2 = 1$. Since k_1 and k_2 are integers, either $k_1 = k_2 = 1$ or $k_1 = k_2 = -1$. In either case, $a = \pm b$.

A note on C.5 The proof actually has a mistake in it. When Solow says "$r = a(1 - mq) + b(-nq) \in M$," actually r is not in M. We would have that $r \in M$ if $r > 0$, but since we know that r cannot be in M (since $d \in M$ is the least element of M and $r < d$), we know that $r = 0$.

Garbage Tag Problem By dividing by 5 we may reduce the problem to that of determining which natural numbers are nonnegative combinations of 4 and 7.

Suppose that x is such that each of x, $x+1$, $x+2$, and $x+3$ are all nonnegative combinations of 4 and 7. Say

$$x = 4a_0 + 7b_0$$

$$x + 1 = 4a_1 + 7b_1$$

$$x + 2 = 4a_2 + 7b_2$$

$$x + 3 = 4a_3 + 7b_3$$

where each a_i and each b_i is nonnegative.

Then each $y \geq x$ is also a nonnegative combination of 4 and 7: Let $y = x + 4d + r$ where $d \geq 0$ and $0 \leq r \leq 3$. Then $y = 4(a_r + d) + 7b_r$.

Similarly, if x is a nonnegative combination of 4 and 7, then so is $x + 4k$ for any nonnegative k. Since 0 is congruent to 0 mod 4, 7 is congruent to 1 mod 4, 14 is congruent to 2 mod 4, and 21 is congruent to 3 mod 4, it follows that 18, 19, 20 and 21 are all nonnegative combinations of 4 and 7. Therefore, all greater numbers are as well. So, to complete the problem, we just have to determine which numbers below 18 are positive combinations of 4 and 7.

You could do this by hand, or notice that by our argument above, any number which is a nonnegative combination of 4 and 7 can be written as a nonnegative combination $4a + 7b$ where $0 \leq b \leq 3$. Therefore, just find the numbers below 18 of the form $4a$, $4a + 7$, $4a + 14$, and $4a + 21$.