Math 304
Homework 8 Solutions

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Problem 1 Both directions are false:

To show that $f(A) = B$ does not imply that $f^{-1}(B) = A$: Let $f(x) = x^2$, $A = [0, 2]$, $B = [0, 4]$. Then $f(A) = B$, but $f^{-1}(B) = [-2, 2] \neq A$.

To show that $f^{-1}(B) = A$ does not imply $f(A) = B$: Let $A = X = [0, 1]$, $B = Y = [0, 2]$ and $f(x) = x$ for $x \in [0, 1]$. Then $f^{-1}(B) = A$, but $f(A) = [0, 1] \neq B$.

Problem 2 In order to calculate $f^{-1}(\mathbb{R} - \{0\})$ we must find all x for which $f(x) \neq 0$.

We know that $f(0) = 0$. If $x \neq 0$, $f(x) = x \sin(\pi/x)$. Then $f(x) = 0$ if $x = 0$ or $\sin(\pi/x) = 0$. Since $x \neq 0$, $\sin(\pi/x) = 0$. For any y, $\sin y = 0$ iff $y = \pi k$ for some integer k. So, if $x \neq 0$ and $f(x) = 0$, then $x = 1/k$ for some integer, and $f(1/k) = 0$ for all integers k.

Therefore, $f^{-1}(\mathbb{R} - \{0\}) = [0, 1] - (\{1/k \mid k \in \mathbb{N}\} \cup \{0\})$.

Problem 3 Given that the f from Problem 2 is continuous, $f^{-1}(\mathbb{R} - \{0\})$ is relatively open in $[0, 1]$, since $\mathbb{R} - \{0\}$ is an open set. By Problem 2, this set is $[0, 1] - (\{1/k \mid k \in \mathbb{N}\} \cup \{0\})$ and we are done.