MATH 304 **Midterm Prelim Solutions** **March 13, 2008**

Please do not use any books or notes. Please hand in your exam booklet, and don't forget to put your name on the outside.

Problem 1. (16 points) Which of the following statements are true, and provide a counterexample when the statement is false.

(a) For all real numbers $a < b$, there is a rational number q such that $a < q < b$.
Solution: True. Use the decimal expansion of real numbers.

(b) For all real numbers a, b, there is a real number c such that $ac = b$.
Solution: False. If $a = 0$ and $b \neq 0$, then there is no c such that $ac = b$, since $a0 = 0$.

(c) If the statement A implies the statement B, then when A is false B is false. (A truth table might help here.)
Solution: False: If A is false and B is true, A implies B, but B is true, i.e. not false.

Problem 2. (12 points) Negate the following statements.

(a) Either $1 + 1 = 2$ or $1 + 1 \neq 0$.
Solution: $1 + 1 \neq 2$ and $1 + 1 = 0$.

(b) $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that $\delta < \epsilon/20000$ and δ is rational.
Solution: $\exists\ \epsilon > 0$ such that $\forall\ \delta > 0,\ \ \delta \geq \epsilon/20000$ or δ is irrational.

(c) For all x a real number, the statement $A(x)$ implies the statement B.
Solution: There is a real number x such that $A(x)$ is true and the statement B is false. OR There is a real number x such that $A(x)$ does not imply the statement B.

Problem 3. (18 points)

(a) Define what it means for a function $f : A \to B$ to be injective and what it means for f to be surjective, where A and B are sets.
Solution: A function $f : A \to B$ is injective if there is a function $g : B \to A$ such that for all $x \in A$, $g(f(x)) = x$. OR you can use the definition of the function being one-to-one: A function $f : A \to B$ is one-to-one if for all x and y in A, $f(x) = f(y)$ implies $x = y$. OR A function $f : A \to B$ is one-to-one if for all $x \neq y$ in A, $f(x) \neq f(y)$. Some people said "If $f : A \to B$ is injective, every element in A maps to somewhere in B ..." or something similar. This is just the definition of what it means for f to be a function. It is NOT the definition of being injective.

A function $f : A \to B$ is surjective if there is a function $g : B \to A$ such that for all $y \in B$, $f(g(y)) = y$. OR you can use the definition of the function being onto: A function $f : A \to B$ is onto if for all y in B, there is an x in A such that $f(x) = y$.

(b) What is the definition of the statement that two sets A and B have the same cardinality.
Solution: Two sets A and B have the same cardinality if there is a function $f : A \to B$ that is both an injection and surjection. In other words, f is a bijection.

(c) State the Schröder-Bernstein Theorem.
Solution: If there is an injection from the set A to the set B, and there is an injection from the set B to the set A, then there is a bijection between A and B. In other words, A and B have the same cardinality.

(d) Use the Schröder-Bernstein Theorem to prove that if $f : A \to B$ is injective and $g : A \to B$ is surjective, then A and B have the same cardinality.
Solution: Since g is surjective, there is a function $f' : B \to A$, such that for all y in B, $f'(g(y)) = y$. This means that f' is injective. Thus the Schröder-Bernstein Theorem implies that there is a bijection between A and B.

Problem 4. (10 points) What is the cardinality of the following subset of the plane: $\{(x\sqrt{2}, y\pi) \mid x \in \mathbb{Q}, y \in \mathbb{N}_+\}$? The set $\mathbb{Q}$ is the set of rational numbers. The set $\mathbb{N}_+ = \{1, 2, \dots\}$ the positive natural numbers.
Solution: The cardinality is $\aleph_0$. This is because the cardinality of both $\mathbb{Q}$ and $\mathbb{N}_+$ is $\aleph_0$, and so each set $\{(x\sqrt{2}, y\pi) \mid x \in \mathbb{Q}\}$ for y in $\mathbb{N}_+$ is countably infinite, which implies that the set $\{(x\sqrt{2}, y\pi) \mid x \in \mathbb{Q}, y \in \mathbb{N}_+\}$ is the countable union of countably infinite sets, and thus it is countably infinite as well.

Problem 5. (12 points)

(a) What is the definition of the statement that a real valued function $f(x)$ is differentiable at $x = 0$?
Solution: There is a number $f'(0)$ such that for every $\epsilon > 0$, there is a $\delta > 0$ such that for all $0 < |h| < \delta$, $|(f(x + h) - f(x))/h - f'(0)| < \epsilon$. (Many people forgot to require $|h| > 0$, committing the cardinal sin of calculus, division by 0.)

(b) Using the definition in Problem 5a to prove that $f(x) = x^2$ is differentiable at $x = 0$.
Solution: Choose $f'(0) = 0$ and $\delta = \epsilon$. When $0 < |h| < \delta = \epsilon$, then $|(f(x+h)-f(x))/h - f'(0)| = |(h^2 - 0)/h - 0| = |h| < \epsilon$.

Problem 6. (16 points)

(a) Define what an equivalence relation is, and what an equivalence class is.
Solution: An equivalence relation $\sim$ is a binary relation on a set X that is reflexive ($x \sim x$ for all x in X), symmetric ($x \sim y$ implies $y \sim x$ for all x and y in X), and transitive ($x \sim y$, $y \sim z$ implies $x \sim z$ for all x, y, z in X).

(b) Say that two real numbers x and y are equivalent if $|x - y|$ is an integer. Prove that this is an equivalence relation.
Solution: if $|x - y|$ is an integer it is equivalent to saying that $y - x = n$, where n is an integer. Since $x - x = 0$, the relation is reflexive. Since if $y - x = n$ an integer, $x - y = -n$ is also an integer. Thus the relation is symmetric. If $y - x = n$ and $z - y = m$ are integers, so are $z - y + y - x = z - x = m + n$. Thus the relation is transitive.

(c) For the equivalence relation in Problem 6b, define $[x] + [y] = [x + y]$, where $[x]$, $[y]$ and $[x + y]$ are the equivalence classes of x, y and $x + y$ respectively. Prove that this definition is well-defined. (For the original definition that defined x equivalent to y if $|x - y| = 1$: This is NOT an equivalence relation. For example, it is not reflexive. $|x - x| = 0 \neq 1$.)
Solution: If $[x] = [x']$, and $[y] = [y']$ it means that $x' - x = n$ and $y' - y = m$ are integers. Then $x' + y' - (x + y) = x' - x + y' - y = n + m$ is an integer. Thus $[x + y] = [x' + y']$ and addition of equivalence classes is well-defined.

Problem 7. (16 points)

(a) Prove that the polynomial $(x - r)$ divides the polynomial $x^n - r^n$, for $n = 1, 2, \dots$.
Solution: $x^n - r^n = (x - r)(x^{n-1} + x^{n-2}r + \cdots + xr^{n-2} + r^{n-1})$ since all the terms cancel except the first and last. (The right hand term is a finite geometric series.)

(b) Use the result of Problem 7a to prove that if a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with real coefficients $a_n, a_{n-1}, \ldots a_0$ has a real root r, then $(x - r)$ divides $p(x)$. A real number r is a root of $p(x)$ if $p(r) = 0$. (Hint: Subtract $p(r)$ from $p(x)$ term-by-term.)
Solution: $p(x) = p(x) - p(r) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 - a_n r^n - a_{n-1}r^{n-1} - \cdots - a_0$. So $p(x) = a_n(x^n - r^n) + a_{n-1}(x^{n-1} - r^{n-1}) + \cdots + a_1(x - r)$, and $(x - r)$ divides each term by Problem 7a. Hence $(x - r)$ divides $p(x)$.

(c) Use induction to prove that a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with real coefficients $a_n, a_{n-1}, \ldots a_0$ has at most n distinct real roots. Please do not use the fundamental theorem of algebra.
Solution: We should exclude the 0 polynomial. We prove that any non-zero polynomial $p(x)$ has at most n distinct real roots by induction on n the degree of $p(x)$. When $n = 1$, $p(x) = a_1 x + a_0$, and it has one real root $x = -a_0/a_1$ if $a_1 \neq 0$ or no roots if $a_1 = 0$ since then $a_0 \neq 0$. We assume that $p(x)$ is a polynomial of degree $n + 1$, $n \geq 1$, and that all non-zero polynomials of degree n have at most n real roots. We will show that $p(x)$ has at most $n + 1$ real roots. If there are no real roots for $p(x)$, we are done. If r is a root, then by Problem 7b, $(x - r)$ divides $p(x)$. So $p(x) = (x - r)q(x)$, where $q(x)$ is a non-zero polynomial of degree n. By the induction hypothesis $q(x)$ has at most n real roots. So there are at most $n + 1$ roots in all.