MATH 304 **Notes and Problems** **Review for the Final**

Final is on Tuesday, May 13, 2008, 7:00-9:30 PM, in Rockefeller 104

The following are some sample questions of the sort that might be asked on our Final. This covers roughly what we covered in the course. If you have questions, please come to see me. I will be in most days mostly between 10 AM and 4 PM.

Problem 1. **Logic:** Negate the following statements

(a) All men are dishonest.

(b) For all real numbers $a < b$ there is an integer n such that $a \leq n \leq b$.

(c) All continuous functions are differentiable.

Problem 2. **Truth tables:** Construct the truth table for the statement A and B implies C or D.

Problem 3. **Definitions:** Define a rational number p/q to be even if p and q are even integers. Is this well-defined?

Problem 4. **Functions:** Prove or find a counterexample to the following statement: If a function $f : \{1,2,3\} \to \{1,2,3,4\}$ is injective, and $g : \{1,2,3,4\} \to \{1,2,3\}$ is surjective, then the composition $g(f(x))$ is bijective.

Problem 5. **Cardinality:** Prove that cardinality of the set of roots of non-zero polynomials with integer coefficients is countable. (These numbers are called *algebraic numbers*.)

Problem 6. **Schröder-Bernstein Theorem:** State the definition of two sets having the same cardinality and the Schröder-Bernstein Theorem. Use use these definitions to prove that if $f : X \to Y$ is one-to-one and $g : X \to Y$ is onto then X and Y have the same cardinality.

Problem 7. **Calculus:** State the definition of continuity and use it to prove that the composition of two continuous functions is continuous.

Problem 8. **Induction:** Consider the following sequence $a_n, n = 1, 2, \ldots$ defined by $a_1 = 1$, and $a_{n+1} = a_n + 6n$ for $n \geq 1$. Use induction to prove that $a_n = 3n^2 - 3n + 1$.

Problem 9. **Number Theory:** Prove that if an integer d divides the integer a, d divides the integer b, and there are integers m and n such that $am + nb = 3$, then $d = \pm 1$ or $d = \pm 3$.

Problem 10. **Infima:** Define the concept of the infimum of a set and use it to prove what the infimum of the set $X = \{x \mid x = m^2 n^2/(m^2 + n^2), m \in \mathbb{Z}, n \in \mathbb{Z}, m \neq 0, n \neq 0\}$ is.

Problem 11. **Topology:** Define what an open subset of $\mathbb{R}$ is and use it to prove that the set $X = \{x \mid \sin(1/x) > 0, x \neq 0\}$ is open.

Problem 12. **Set equations:** Define what $f^{-1}(A)$ means, where $f : X \to Y$ is a function. Let $g : Y \to Z$ be another function, and let h be the composition of f and g. I.e. $h(x) = g(f(x))$ for $x \in X$. Prove that for any $A \subset Z$, $h^{-1}(A) = f^{-1}(g^{-1}(A))$.

Problem 13. **Monotone functions:** Define what it means for a function $f : [0,1] \to \mathbb{R}$ to be monotone increasing (but not necessarily strictly monotone increasing). Use it to show that if f is continuous, then the function $g(x) =$ maximum of $\{f(t) \mid t \in [0, x]\}$ is monotone increasing and continuous.

Problem 14. **Intermediate Value theorem:** Prove that there is a unique real number x such that $\cos x = x$.

Problem 15. **Inverse Function theorem:** Prove or disprove: If $f : [0, 1] \to \mathbb{R}$ is differentiable and strictly monotone increasing, then the inverse function exists and is differentiable.

Problem 16. **Self-similar sets:** Calculate the Hausdorf dimension of the Cantor set obtained by removing the middle fourth of the interval $[0, 1]$ and then removing the middle fourth of each remaining interval successively.

Problem 17. **Geometry:** Prove that for a parallelogram P in the plane, there is a point p in the plane such that 180 degree rotation about p takes P into itself. If a quadrilateral Q in the plane has opposite sides the same length, but is self-intersecting, prove there is a line of symmetry L in the plane, such that reflection about L takes Q into itself.