Chapter 1
The Emergence of Rigorous Calculus

1.1 What Is Mathematical Analysis?

Mathematical analysis[8] is the critical and careful study of calculus with an emphasis on understanding of its basic principles. As opposed to *discrete* mathematics or *finite* mathematics, mathematical analysis can be thought of as being a form of *infinite* mathematics. As such, it must rank as one of the greatest, most powerful, and most profound creations of the human mind.

> The infinite! No other question has ever moved so profoundly the spirit of man — David Hilbert (1921).

Now, as you may expect, great, profound, and powerful thoughts do not often appear overnight. In fact, it took the best part of 2500 years from the time the first calculus-like problems tormented Pythagoras, until the first really solid foundations of mathematical analysis were laid in the nineteenth century. During the seventeenth and eighteenth centuries calculus blossomed, becoming an important branch of mathematics and, at the same time, a powerful tool, able to describe such physical phenomena as the motion of the planets, the stability of a spinning top, the behavior of a wave, and the laws of electrodynamics. This period saw the emergence of almost all of the concepts that one might expect to see in an elementary calculus course today.

But if the blossoms of calculus were formed during the seventeenth and eighteenth centuries, then its roots were formed during the nineteenth. Calculus underwent a revolution during the nineteenth century, a revolution in which its fundamental ideas were revealed and in which its underlying theory was properly understood for the first time. In this revolution, calculus was rewritten from its foundations by a small band of pioneers, among whom were Bernhard Bolzano, Augustin Cauchy, Karl Weierstrass, Richard Dedekind, and Georg Cantor. You will see their names repeatedly in this book, for it was largely as a result of their efforts that the subject that we know today as *mathematical analysis* was born. Their work enabled us to appreciate the nature of our number system and gave us our first solid understanding of the concepts of limit, continuity, derivative, and

[8] Note to instructors: This chapter is not designed for in-class teaching. It is intended to be a reading assignment, possibly in conjunction with other material that the student can find in the library.

integral. This is the great and profound theory to which you, the reader of this book, are heir.

In this chapter we shall focus on three earth-shaking events that have taken place during the past 2500 years and which helped to pave the way for the emergence of rigorous mathematics as we know it today. These events are sometimes known as the **Pythagorean crisis**, the **Zeno crisis**, and the **set theory crisis.**

1.2 The Pythagorean Crisis

In about 500 B.C.E. an individual in the Pythagorean school noticed that, according to the Greek concepts of number and length, it is impossible to compare the length of a side of a square with the length of its diagonal. The Greek concept of length required that, in order to compare two line segments AB and CD, we need to be able to find a measuring rod that fits exactly a whole number of times into each of them. If, for example, the measuring rod fits 6 times into AB and 10 times into CD, as shown in Figure 1.1, then we have

$$\frac{AB}{CD} = \frac{6}{10}.$$

More generally, if the measuring rod fits exactly m times into AB and exactly n

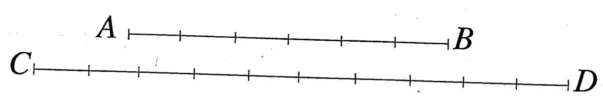


Figure 1.1

times into CD, then we have

$$\frac{AB}{CD} = \frac{m}{n}.$$

Note that this kind of comparison requires that the ratio of any two lengths must be a rational number.

The crisis came when the young Pythagorean drew a square with a side of one unit as shown in Figure 1.2 and applied the theorem of Pythagoras to find the length of the diagonal. As we know, the length of this diagonal is $\sqrt{2}$ units. From the fact that the number $\sqrt{2}$ is irrational he concluded that the equation

$$\frac{\sqrt{2}}{1} = \frac{m}{n}$$

is impossible if m and n are integers and that, consequently, it is impossible to compare the side of this square with its diagonal.

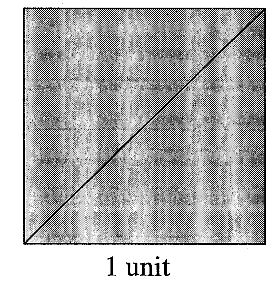


Figure 1.2

From our standpoint today, we can see that this discovery reveals the inadequacy of the rational number system and of the Greek concept of length; but to them, the discovery was a real shocker. Just how much of a shock it was can be gauged from the writings of the Greek philosopher Proclus, who tells us that the Pythagorean who made this terrible discovery suffered death by shipwreck as a punishment for it.

1.3 The Zeno Crisis

1.3.1 The Paradoxes of Zeno

In the fifth century B.C.E., Zeno of Elea came up with four innocent-sounding statements that plagued the philosophers all the way up to the time of Bolzano and Cauchy early in the nineteenth century. These four statements are known as the **paradoxes of Zeno**, and the first three of these appear in Bell [4] as follows:

1. *Motion is impossible, because whatever moves must reach the middle of its course before it reaches the end; but before it has reached the middle, it must have reached the quarter mark, and so on, indefinitely. Hence the motion can never start.*
2. *Achilles running to overtake a crawling tortoise ahead of him can never overtake it, because he must first reach the place from which the tortoise started; when Achilles reaches that place, the tortoise has departed and so is still ahead. Repeating the argument, we easily see that the tortoise will always be ahead.*
3. *A moving arrow at any instant is either at rest or not at rest, that is, moving. If the instant is indivisible, the arrow cannot move, for if it did, the instant would immediately be divided. But time is made up of instants. As the arrow cannot move in any one instant, it cannot move in any time. Hence it always remains at rest.*

Much has been said about these paradoxes, and, quite obviously, we are not going to do them justice here. But let's talk about the third paradox for a moment. At any one instant of time, the arrow does not move. Does that really mean that the arrow will not find its target? Would Zeno have been prepared to stand in front of the arrow? We think not. Then what was Zeno trying to tell us? Zeno's statement warns us that velocity can be meaningful in any physical sense only as an *average velocity over a period of time.* If an arrow covers a distance of 60 feet during the course of a second, we can say that the arrow has an average velocity of 60 feet per second. But Zeno's statement warns us that our senses can make nothing out of a notion of *velocity of the arrow at any one instant.*

1.3.2 Stating Zeno's Third Paradox in Terms of Slope

To state Zeno's third paradox in terms of slope, we shall suppose that A is the point $(x_1, f(x_1))$ on the graph of a function f, and that B is some other point $(x_1 + \Delta x, f(x_1 + \Delta x))$, as shown in Figure 1.3. As usual, the slope of the line

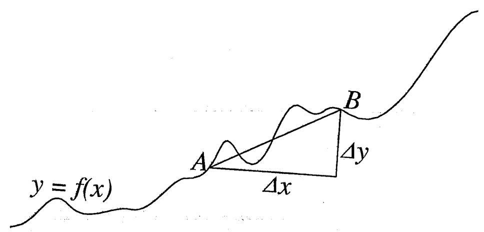


Figure 1.3

segment AB is defined to be to be the ratio $\Delta y/\Delta x$, where

$$\Delta y = f(x_1 + \Delta x) - f(x_1).$$

This ratio $\Delta y/\Delta x$ is the average slope of the graph of f between the points A and B. However, Zeno's third paradox serves as a warning that there is no obvious physical meaning to the notion of *slope of the graph at the point A*.

"But" you may ask, "isn't this what calculus is all about? Are the paradoxes of Zeno trying to tell us to abandon the idea of a derivative?" They are not. But what we should learn from these paradoxes is that if we want to *define* the derivative of the function f at the point A to be

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

then that's just fine with Zeno. Only we can't blame Zeno if this derivative that

we have *defined* doesn't measure how the function f increases at A, because, as Zeno quite rightly tells us, the function f can't change its value at any one point. We may therefore think of Zeno's paradoxes as telling us that (referring to Figure 1.3) even though we may speak of the slope $\frac{\Delta y}{\Delta x}$ of the line segment AB, and even though we may *define* the derivative of f at A and call it $\frac{dy}{dx}$ and have

$$\frac{\Delta y}{\Delta x} \to \frac{dy}{dx} \quad \text{as} \quad \Delta x \to 0,$$

we may not think of $\frac{dy}{dx}$ as the ratio of two quantities dy and dx, the amounts by which y and x increase at the point A, because, as Zeno quite rightly tells us, there are *no* increases in y and x at the point A.

1.3.3 The Rigorous Reformulation

Mathematics prior to the dawn of the nineteenth century was much less precise than mathematics as we know it today. The core of pre-nineteenth century mathematics was the calculus that had been developed by Newton, Leibniz, and others during the seventeenth century. That calculus represented a magnificent contribution. It gave us the notation for derivatives and integrals that we still use today and provided a mathematical basis for the understanding of such physical phenomena as the motion of the planets, the motion of a spinning top and the vibration of a violin string. But the calculus of Newton and Leibniz did not rest on a solid foundation.

The problem with Newtonian calculus is that it was not based on an adequate theory of limits. In fact, prior to the nineteenth century, there was not much understanding that calculus needs to be based on a theory of limits at all. Nor was there much understanding of the nature of the number system $\mathbf{R}$ and the role of what we call today the *completeness* of the number system $\mathbf{R}$. In a sense, the calculus of Newton and Leibniz did not pay sufficient heed to the paradoxes of Zeno. Although Newton and Leibniz themselves may have had some appreciation of the fundamental ideas upon which the concepts of derivative and integral depend, many of those who followed them did not. Until the end of the eighteenth century the majority of mathematicians based their work upon an impossible mythology. During this time, proofs of theorems in calculus commonly depended on a notion of "infinitely small" numbers, numbers that were zero for some purposes yet not for others. These were known as *evanescent numbers, differentials,* or *infinitesimals*, and, undeniably, their use provided a beautiful, revealing, and elegant way of looking at many of the important theorems of calculus. Even today we like to use the notion of an infinitesimal to motivate some of the theorems in calculus, and scientists use them even more frequently than mathematicians. But it is one thing to use the idea of an infinitesimal to *motivate* a theory, and it is quite another matter to base virtually the entire theory upon them. Today, the concept of an infinitesimal can actually be made precise in a modern mathematical theory that is known as *nonstandard analysis*, but there was no precision in the way infinitesimals were used in the eighteenth century.

During the eighteenth century, the voices of critics began to be heard. In 1733, Voltaire [32] described calculus as

> *The art of numbering and measuring exactly a thing whose existence cannot be conceived.*

Then, in 1734, Bishop George Berkeley, the philosopher, wrote an essay, Berkeley [5], in which he rebuked the mathematicians for the weak foundations upon which their calculus had been based, and he no doubt took great pleasure in asking

> *Whether the object, principles, and inferences of the modern analysis are more distinctly conceived, or more evidently deduced than religious mysteries and points of faith.*

Some mathematicians composed weak answers to Berkeley's criticism, and others tried vainly to make sense of the idea of infinitely small numbers, but it was not until the early nineteenth century that any real progress was made. The turning point came with the work of Bernhard Bolzano, who gave us the first coherent definition of limits and continuity and the first understanding of the need for a complete number system. Then came the work of Cauchy, Weierstrass, Dedekind, and Cantor that placed calculus on a rigorous foundation and settled many important questions about the nature of the number system $\mathbf{R}$. The work of these pioneers has made possible the understanding that we have promised you.

1.4 The Set Theory Crisis

Following the work of the nineteenth-century pioneers, the mathematical community began to believe that true understanding was at last within its grasp. All of the fundamental concepts seemed to be rooted solidly in Cantor's theory of sets. But the collective sigh of relief had hardly died away when a new kind of paradox burst upon the scene. In 1897, the Italian mathematician Burali-Forti discovered what is known today as the **Burali-Forti paradox**, which shows that there are serious flaws in Cantor's theory of sets, upon which our understanding of the real number system had been based. Then, a few years later, Bertrand Russell discovered his famous paradox. Like Burali-Forti's paradox, Russell's paradox demonstrates the presence of flaws in Cantor's set theory.

To see just how much these paradoxes stunned the mathematical community, one might want to look at Frege [9], *Grundgesetze der Arithmetik* (*The Fundamental Laws of Arithmetic*), which was written by the German philosopher Gottlob Frege and published in two volumes, the first in 1893 and the second in 1903. This book was Frege's life work, and it was his pride and joy. He had bestowed upon the mathematical community the first sound analysis of the meaning of number and the laws of arithmetic and, although the book is quite technical in places, it is worth skimming through, if only to see the sarcastic way in which Frege speaks of the "stupidity" of those who had come before him. An example of this sarcasm is Frege's description of his attempt to induce other mathematicians to tell him what the number *one* means. "One object," would be the reply. "Very well," answered Frege, "I choose the *moon*! Now I ask you please to tell me: *Is one plus one still equal to two*?" As things turned out, the second volume of Frege's book came out just after Russell had sent Frege his famous paradox. There was just enough space at the end of Frege's book for the following acknowledgment:

> *A scientist can hardly encounter anything more undesirable than to have the foundation collapse just as the work is finished. I was put in this position by a letter from Mr. Bertrand Russell when the work was almost through the press.*

As Frege said, the foundation collapsed. It would not be stretching the truth too much to say that all of mathematics perished in the fire storm that was ignited by the paradoxes of Russell and Burali-Forti. The mathematics that we know today is what emerged from that storm like a phoenix from the ashes, and it depends upon a new theory of sets that is known as **Zermelo-Fraenkel set theory** which was developed in the first few decades of the twentieth century. Within the framework of Zermelo-Fraenkel set theory, we can once again make use of Frege's important work.

One question that remains is whether we are now safe from new paradoxes that might ignite a new fire storm, and the answer is that we don't know. A theorem of Gödel guarantees that, unless someone actually discovers a new paradox that destroys Zermelo-Fraenkel set theory, we shall never know whether such a paradox exists. Thus it is entirely possible that you, the reader of this book, may stumble upon a snag that shows that mathematics as we know it does not work. But don't hold your breath. The chances of your encountering a new paradox are very remote.

Chapter 2
Mathematical Grammar

In this chapter you will learn how to read and understand the language in which mathematical proofs are written. Just like any other language, the language of mathematics contains its own special words and rules of grammar that govern the way in which these words may be combined into sentences. This chapter will acquaint you with many of these words, and it will teach you how the rules of grammar apply to them.

2.1 The Quantifiers *For Every* and *There Exists*

2.1.1 Unknowns in a Mathematical Statement

Some mathematical statements refer only to predefined mathematical objects. For example, the statement $1 + 1 = 2$ refers only to the numbers 1 and 2 and the operation $+$. However, mathematical statements can also refer to objects that have not already been introduced. We shall call such objects **unknowns** in the statement. To understand how unknowns can appear, look at the following simple question about elementary algebra:

Is it true that

$$(x+y)^2 = x^2 + y^2? \tag{2.1}$$

Perhaps the answer "no" is hovering on your lips. If so, you are being a little hasty, for is it not true that

$$(3+0)^2 = 3^2 + 0^2?$$

As you can see, the truth or falsity of Equation (2.1) depends on the values of x and y that we are talking about. Until we receive more information about x and y, we cannot say whether or not Equation (2.1) is true. Here are some analogs of the above question that can be answered specifically:

- Is it true that, if $x = 2$ and $y = 3$, then Equation (2.1) is true? No!
- Is it true that there exist numbers x and y such that Equation (2.1) holds? Yes!
- Is it true that for every number x there exists a number y such that Equation (2.1) holds? Yes!

- Is it true that there exists a number x such that for every number y Equation (2.1) holds? Yes!
- Is it true that for all numbers x and y Equation (2.1) holds? No!

If you look at any one of these five questions, you will see that each of the symbols x and y was introduced either by saying "for every" or by saying "there exists". The symbols in these questions have been what we call **quantified**. The statements that follow exhibit some more examples of quantified symbols.

1. *Every tall man in this theater is wearing a hat.* In this statement, "tall man" is introduced with *for every.*
2. *No tall man in this theater is wearing a hat.* This statement can be interpreted as saying that every man wearing a hat in this theater fails to be tall.
3. *Some tall men in this theater are wearing hats.* This statement introduces "tall man" with *there exists.* It says that there exists at least one tall man in this theater who is wearing a hat.
4. *Not all of the tall men in this theater are wearing hats.* This statement can be interpreted as saying that there exists at least one tall man in this theater who is not wearing a hat. Interpreted this way, the statement introduces "tall man" with *there exists.*
5. *For every positive integer n there is a prime number p such that $p > n$.* This statement contains two unknowns, n and p. The unknown n is introduced with *for every* and then, after n has been introduced, the unknown p is introduced with *there exists.*

2.1.2 The Quantifiers

The phrase *for every* is called the **universal quantifier** and, depending on the context, it appears sometimes simply as *every*, sometimes as *all*, and sometimes as the symbol $\forall$. The phrase *there exists* is called the **existential quantifier**, and, depending on the context, it sometimes appears as *there is, we can find, it is possible to find, there must be, there is at least one, some*, or as the symbol $\exists$.

2.1.3 Exercises on the Use of Quantifiers

Except in Exercise 2, decide whether the sentence that appears in the exercise is meaningful or meaningless. If the sentence is meaningful, say whether what it says is true or false.

1. (a) $\sqrt{x^2} = x$.
   (b) For every real number x we have $\sqrt{x^2} = x$.
   (c) For every positive number x we have $\sqrt{x^2} = x$.
2. (a) Point at the expression $\sqrt{x^2}$ and click on the Evaluate button.

(b) Point at the expression $\sqrt{x^2}$ and click on the Simplify button.

(c) Point at the equation $\sqrt{x^2} = x$, open the Compute menu, and click on Check Equality.

(d) Point at the equation $x = -2$ and click on the button to supply the definition $x = -2$ to *Scientific Notebook*. Then try a Check Equality on the equation $\sqrt{x^2} = x$.

3. For every number x and every number y there is a number z such that $z = x + y$.
4. For every number x there is a number z such that for every number y we have $z = x + y$.
5. For every number x and every number z there is a number y such that $z = x + y$.
6. $\sin^2 x + \cos^2 x = 1$.
7. For every number x we have $\sin^2 x + \cos^2 x = 1$.
8. For every integer $n > 1$, if $n^2 \leq 3$, then the number 57 is prime.

2.1.4 Order of Appearance of Unknowns in a Statement

As we have seen, if a statement contains some unknowns, then its truth or falsity depends on the way in which these unknowns have been quantified. If a statement contains more than one unknown, then the order in which these unknowns appear can also be very important. The following examples illustrate how we can change the meaning of a statement by changing the order in which two unknowns are introduced.

1. If two unknowns are introduced with $\forall$, one directly after the other, then the order in which they are introduced is not important. For example, the following two statements say exactly the same thing.

   (a) *For every positive number x and every negative number y, the number xy is negative.*

   (b) *For every negative number y and every positive number x, the number xy is negative.*

2. If two unknowns are introduced with $\exists$, one directly after the other, then the order in which they are introduced is not important. For example, the following two statements say exactly the same thing.

   (a) *There exists a positive integer m and a positive integer n such that $\sqrt{m^2 + n^3}$ is an integer.*

   (b) *There exists a positive integer n and a positive integer m such that $\sqrt{m^2 + n^3}$ is an integer.*

3. If one unknown in a sentence is introduced with $\forall$ and another is introduced with $\exists$, then the order in which they appear is very important. For example, compare the following two sentences:

   (a) *For every positive number x there exists a positive number y such that $y < x$.*

   (b) *There exists a positive number y such that for every positive number x we have $y < x$.*

   Although these statements may look similar, they do not say the same thing. As a matter of fact, the first one is true and the second one is false.

4. In this example we look at some sentences that contain three unknowns.

   (a) *For every number x and every number z, there exists a number y such that $x + y = z$.*

   (b) *For every number x there exists a number y such that for every number z we have $x + y = z$.*

   (c) *There exists a number y such that for every number x and every number z we have $x + y = z$.*

   If you look at these statements carefully, you will see that the first one is true and the other two are false.

5. In this example we look at two more statements with three unknowns. Try to decide whether they are true or false. We shall provide the answers in Subsection 3.8.1.

   (a) *For every number x there exists a positive number δ such that for every number t satisfying the inequality $|t - x| < \delta$ we have $|t^2 - x^2| < 1$.*

   (b) *There exists a positive number δ such that for every number x and for every number t satisfying the inequality $|t - x| < \delta$ we have $|t^2 - x^2| < 1$.*

6. In this example we look at some statements that contain four unknowns. Try to decide whether they are true or false. We shall provide some of the answers in Section 3.8.

   (a) *For every number $\varepsilon > 0$ and for every number $x \in [0, 1]$ there exists a positive integer N such that for every integer $n \geq N$ we have*

   $$\frac{nx}{1 + n^2x^2} < \varepsilon.$$

   (b) *For every number $x \in [0, 1]$ and for every number $\varepsilon > 0$ there exists a* *positive integer N such that for every integer $n \geq N$ we have*

$$\frac{nx}{1+n^2x^2} < \varepsilon.$$

(c) *For every number $\varepsilon > 0$ there exists a positive integer N such that for every number $x \in [0,1]$ and for every integer $n \geq N$ we have*

$$\frac{nx}{1+n^2x^2} < \varepsilon.$$

2.1.5 Exercises on Order of Appearance of Unknowns

For each of the following pairs of statements, decide whether or not the statements are saying the same thing. Except in the first two exercises, say whether or not the given statements are true.

1. (a) Every person in this room has seen a good movie that has started playing this week.
   (b) A good movie that has started playing this week has been seen by every person in this room.
2. (a) Only men wearing top hats may enter this hall.
   (b) Only men may enter this hall wearing top hats.
   (c) Men wearing top hats only may enter this hall.
   (d) Men wearing only top hats may enter this hall.
   (e) Men wearing top hats may enter this hall only.
3. (a) For every nonzero number x there is a number y such that $xy = 1$.
   (b) There is a number y for which the equation $xy = 1$ is true for every nonzero number x.
4. (a) For every number $x \in [0,1)$ there exists a number $y \in [0,1)$ such that $x < y$.
   (b) There is a number $y \in [0,1)$ satisfying $x < y$ for every number $x \in [0,1)$.
5. (a) For every number $x \in [0,1]$ there exists a number $y \in [0,1]$ such that $x < y$.
   (b) There is a number $y \in [0,1]$ satisfying $x < y$ for every number $x \in [0,1]$.
6. (a) For every number $x \in [0,1)$ there exists a number $y \in [0,1)$ such that $x \leq y$.
   (b) There is a number $y \in [0,1)$ satisfying $x \leq y$ for every number $x \in [0,1)$.

7. (a) For every number $x \in [0,1]$ there exists a number $y \in [0,1]$ such that $x \leq y$.
   (b) There is a number $y \in [0,1]$ satisfying $x \leq y$ for every number $x \in [0,1]$.
8. (a) For every odd integer m it is possible to find an integer n such that mn is even.
   (b) It is possible to find an integer n such that for every odd integer m the number mn is even.
9. (a) For every number x it is possible to find a number y such that $xy = 0$.
   (b) It is possible to find a number y such that for every number x we have $xy = 0$.
10. (a) For every number x it is possible to find a number y such that $xy \neq 0$.
    (b) It is possible to find a number y such that for every number x we have $xy \neq 0$.
11. (a) For every number a and every number b there exists a number c such that $ab = c$.
    (b) For every number a there exists a number c such that for every number b we have $ab = c$.
12. (a) For every number a and every number c there exists a number b such that $ab = c$.
    (b) For every number a there exists a number b such that for every number c we have $ab = c$.
13. (a) For every nonzero number a and every number c there exists a number b such that $ab = c$.
    (b) For every nonzero number a there exists a number b such that for every number c we have $ab = c$.

2.2 Negating a Mathematical Sentence

When you are trying to read and understand a mathematical statement P, you will often find it useful to ask yourself what it would mean to say that the statement P is false. The assertion that the given statement P is false is called the **negation** or **denial** of the statement P and is written as $\neg P$.

Thus, for example, if P is the assertion that the number 4037177 is prime, then the statement $\neg P$ is the assertion that the number 4037177 is composite. As a matter of fact, since

$$4037177 = 17 \times 19 \times 29 \times 431,$$

we know that the statement P is false and that the statement $\neg P$ is true. In general, if P is any mathematical statement, then one of the statements P and $\neg P$ will be true and the other will be false.

2.2.1 Negations and the Quantifiers

Suppose that a given statement P asserts that *Everyone in this room can speak French.* The denial of P does not claim that no one in the room can speak French. To deny the assertion that everyone in the room can speak French, all we have to do is find one person in the room who cannot speak French.

More generally, if a statement P contains the quantifier "for every", saying that every object x of a certain type has a certain property, then the denial of P says that there exists at least one object of this type that fails to have the required property.

On the other hand, if a statement P says that at least one person in this room has a dirty face, then the denial of P says that every person in the room has a clean face. Note how the quantifiers "for every" and "there exists" change places as we move from a statement to its denial.

2.2.2 Some Exercises on Negations and the Quantifiers

Write a negation for each of the following statements or say that the statement is meaningless.

1. All roses are red.
2. In Sam's flower shop there is at least one rose that is not red.
3. In every flower shop there is at least one rose that is not red.
4. I believe that all roses are red.
5. There is at least one person in this room who thinks that all roses are red.
6. Every person in this room believes that all roses are red.
7. At least half of the people in this room believe that all roses are red.
8. Every man believes that all women believe that all roses are red.
9. You were at least an hour late for work every day last week.
10. It has never rained on a day on which you have remembered to take your umbrella.
11. You told me that it has never rained on a day on which you have remembered to take your umbrella.
12. You lied when you told me that it has never rained on a day on which you have remembered to take your umbrella.

13. I was joking when I said that you lied when you told me that it has never rained on a day on which you have remembered to take your umbrella.
14. This, Watson, if I mistake not, is our client now.[9]
15. (a) For every real number x there exists a real number y such that

$$\frac{2x^2+xy-y^2}{x^3-y^3} = \frac{2}{3(x-y)} + \frac{5y+4x}{3(x^2+xy+y^2)}.$$

Is this statement true?

(b) There exists a real number x such that for every real number y we have

$$\frac{2x^2+xy-y^2}{x^3-y^3} = \frac{2}{3(x-y)} + \frac{5y+4x}{3(x^2+xy+y^2)}.$$

Is this statement true?

(c) For every real number x and every real number $y \neq x$ we have

$$\frac{2x^2+xy-y^2}{x^3-y^3} = \frac{2}{3(x-y)} + \frac{5y+4x}{3(x^2+xy+y^2)}.$$

Is this statement true?

2.3 Combining Two or More Statements

Two or more given statements can be combined into a single statement using one or more of the conjunctions and conditionals

and	or	if	only if	$\Rightarrow$	$\Leftarrow$	$\Longleftrightarrow$

In this section we study the way in which these combined statements may be formed and some of the relationships between statements of this type.

2.3.1 The Conjunction *and*

If P and Q are given statements, then the assertion

P and Q

says that both of the statements P and Q are true. This assertion is sometimes written as $P \wedge Q$.

[9] The purpose of this exercise is to invite you to discuss a rather strange statement that was made by Sherlock Holmes in one of the stories by Sir Arthur Conan Doyle. Is Holmes' statement meaningful?

For example, if P is the statement

Every tall man in this theater is wearing a hat.

and Q is the statement

Every man wearing a hat in this theater is inconsiderate.

then the statement $P \wedge Q$ says that

Every tall man in this theater is wearing a hat and every man wearing a hat in this theater is inconsiderate.

From the statement $P \wedge Q$ we can deduce that every tall man in this theater is inconsiderate.

2.3.2 The Conjunction *or*

If P and Q are given statements, then the assertion

$$P \text{ or } Q$$

says that at least one of the statements P and Q is true. This assertion is often written as $P \vee Q$.

For example, if P is the statement

You have a cracked radiator.

and Q is the statement

Your water pump needs replacing.

then the sentence P and Q says

Either you have a cracked radiator or your water pump needs replacing.

Note that the condition $P \vee Q$ includes the possibility that both P and Q are true. Thus, if you hear the words

Either you have a cracked radiator or your water pump needs replacing.

then you are going to have at least one job performed on your car but there is also the possibility that both your radiator and your water pump need to be replaced.

2.3.3 Some Examples on the Use of *and* and *or*

1. The equation $x^2 - 3x - 4 = 0$ is equivalent to the condition

$$x = -1 \quad \text{or} \quad x = 4.$$

It would be quite wrong to write the solution of the equation as $x = -1$ *and* $x = 4$ because the equation does not require x to be equal to both of the numbers -1 and 4. If x is equal to either one of the numbers -1 and 4, the equation will hold.

2. The inequality $x^2 - 3x - 4 \leq 0$ is equivalent to the condition

$$-1 \leq x \quad \text{and} \quad x \leq 4.$$

In order for this quadratic inequality to hold we have to have *both* of the conditions $-1 \leq x$ and $x \leq 4$.

3. The inequality $x^2 - 3x - 4 \geq 0$ is equivalent to the condition

$$x \leq -1 \quad \text{or} \quad x \geq 4.$$

4. In this example we assume that m and n are integers. The condition for the number mn to be even is equivalent to the condition

$$m \text{ is even} \quad \text{or} \quad n \text{ is even.}$$

Note that mn will certainly be even in the event that both of the integers m and n are even, but the condition that mn is even does not *require* that both of m and n are even. All it requires is that at least one of the integers m and n is even.

5. Again in this example we assume that m and n are integers. The condition for the number mn to be odd is equivalent to the condition

$$m \text{ is odd} \quad \text{and} \quad n \text{ is odd.}$$

6. In this example we assume that x and y are nonzero numbers. The condition for the equation $|x + y| = |x| + |y|$ to hold is that

$$\text{either } x < 0 \text{ and } y < 0, \text{ or } x > 0 \text{ and } y > 0.$$

2.3.4 The Conditional *if*

If P and Q are given statements, then the sentence

$$\text{If } P, \text{ then } Q$$

says that, in the event that P is true, the sentence Q must also be true. One of the most important facts about the sentence "If P, then Q" is that it places no demands on Q when P is false. The sentence "If P, then Q" can be thought of as saying that

I don't know whether or not P is true and I don't care. However, if the *statement P does happen to be true, then the statement Q must also be true.*

The condition "If P, then Q" can be written in several equivalent ways. One of these is

$$Q \text{ if } P,$$

because the latter statement also says that Q must hold true in the event that P is true. We can also write the condition "If P, then Q" in the form

$$P \text{ only if } Q,$$

because the latter condition says that the only way that P can be true is that Q is also true. The condition "If P, then Q" can also be written without using the word "if" because it says that

either P is false or Q is true.

The following table lists some of the many equivalent ways in which we can write the condition "If P, then Q":

If P, then Q	P implies Q
Q if P	$P \Rightarrow Q$
P only if Q	Q is implied by P
P is a sufficient condition for Q	$Q \Leftarrow P$
Q is a necessary condition for P	Either $\neg P$ or Q
$(\neg Q) \Rightarrow (\neg P)$	$(\neg P) \vee Q$

Finally, the sentence P if and only if Q can be written in any of the following equivalent forms:

P is necessary and sufficient for Q	P iff Q
$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$	$P \Leftrightarrow Q$

The assertion $P \Leftrightarrow Q$ tells us that either the statements P and Q are both true or they are both false.

2.3.5 Negations and the Conjunctions and Conditionals

Suppose that P and Q are given mathematical statements.

1. Since the condition $P \wedge Q$ asserts that both of the statements P and Q must be true, the denial of this condition says that one (or both) of the statements P and Q is false. Thus the denial of the condition $P \wedge Q$ says that either P is false or Q is false.
2. Since the condition $P \vee Q$ says that at least one of the statements P and Q must be true, the denial of this condition says that neither of the statements P and Q is true. In other words, the denial of the condition $P \vee Q$ says that P is false and Q is false.
3. Since the condition $P \Rightarrow Q$ says that either P is false or Q is true, the denial of the condition $P \Rightarrow Q$ says that P is true and Q is false. Thus, for example, the denial of the assertion

   If you eat that grape, you will die.

   says that

   You will eat that grape and you will not die.

2.3.6 Contrapositives and Converses

As we know, if P and Q are mathematical statements, then the assertion $P \Rightarrow Q$ says that either P is false or Q is true. We shall now make the observation that the assertion $(\neg Q) \Rightarrow (\neg P)$ says exactly the same thing. In fact, the assertion $(\neg Q) \Rightarrow (\neg P)$ says that either $\neg Q$ is false or $\neg P$ is true; in other words, it says that either Q is true or P is false.

Thus, if P and Q are mathematical statements, then the assertions $P \Rightarrow Q$ and $(\neg Q) \Rightarrow (\neg P)$ are logically equivalent to each other. We are therefore at liberty to look at whichever of these two assertions looks easier to understand. The assertion $(\neg Q) \Rightarrow (\neg P)$ is called the **contrapositive** form of the assertion $P \Rightarrow Q$.

Of course, the statements $P \Rightarrow Q$ and $Q \Rightarrow P$ do not say the same thing. We call the statement $Q \Rightarrow P$ the **converse** of the statement $P \Rightarrow Q$.

Look, for example, at the statement: *All roses are red.* This statement can be thought of as saying that if an object is a rose, then it must be red. The contrapositive form of this statement says that all things that are not red must fail to be roses. The converse of this statement says that all red things are roses.

2.3.7 A Word of Warning

The statement "If P, then Q" is often confused with the slightly more complex sentence "P, and therefore Q". The meaning of the latter sentence is as follows:

The statement P is known to be true and we also know that P implies Q.

Therefore we can assert that the statement Q must be true.

To help us understand the distinction between "If P, then Q" and "P, and therefore Q" we shall take P to be the statement: *It is raining outside* and Q to be the statement: *You will get wet.* The sentence "If P, then Q" says:

If it is raining outside, then you will get wet.

This sentence does not assert that it is raining. It says only that in the event that it is raining, you will get wet. This sentence is always true on a dry day. On the other hand, the sentence "P, and therefore Q" says:

It is raining outside and therefore you will get wet.

This sentence *does* assert that it is raining and also that you will get wet. The latter sentence is always false on a dry day.

2.3.8 Some Exercises on the Use of Conditionals

In the exercises that follow you should assume that P, Q, R, and S are given statements that may be either true or false.

1. **H** Write down the denial, the converse, and the contrapositive form of each of the following statements:
   (a) *All cats scratch.*
   (b) *If what you said yesterday is correct, then Jim has red hair.*
   (c) *If a triangle $\triangle ABC$ has a right angle at C, then*
   $$(AB)^2 = (AC)^2 + (BC)^2 .$$
   (d) *If some cats scratch, then all dogs bite.*
   (e) *It is with regret that I inform you that someone in this room is smoking.*
   (f) *If a function is differentiable at a given number, then it must be continuous at that number.*
   (g) *Every boy or girl alive is either a little liberal or else a conservative.*
2. **H** Write down a denial of each of the following statements.
   (a) *All cats scratch and some dogs bite.*
   (b) *Either some cats scratch or, if all dogs bite, then some birds sing.*
   (c) *He walked into my office this morning, told me a pack of lies and punched me on the nose.*
   (d) *No one has ever seen an Englishman who is not carrying an umbrella.*
   (e) *For every number x there exists a number y such that $y > x$.*

3. **H** In each of the following exercises we assume that f and g are given functions. Write down a denial of each of the following statements:
   (a) *Whenever $x > 50$, we have $f(x) = g(x)$.*
   (b) *There exists a number w such that $f(x) = g(x)$ for all numbers $x > w$.*
   (c) *For every number x there exists a number $\delta > 0$ such that for every number t satisfying the condition $|x - t| < \delta$ we have $|f(x) - f(t)| < 1$.*
   (d) *There exists a number $\delta > 0$ such that for every pair of numbers x and t satisfying the condition $|x - t| < \delta$ we have $|f(x) - f(t)| < 1$.*
   (e) *For every number $\varepsilon > 0$ and for every number x, there exists a number $\delta > 0$ such that for every number t satisfying $|x - t| < \delta$, we have $|f(t) - f(x)| < \varepsilon$.*
   (f) *For every positive number ε there exists a positive number δ such that for every pair of numbers x and t satisfying the condition $|x - t| < \delta$ we have $|f(x) - f(t)| < \varepsilon$.*
4. Explain why the statement $\neg (P \Rightarrow Q)$ is equivalent to the statement $P \wedge (\neg Q)$.
5. Explain why the statement $\neg (P \Leftrightarrow Q)$ is equivalent to the assertion that either (P is true and Q is false) or (P is false and Q is true).
6. Explain why the statement $\neg (P \vee Q)$ is equivalent to the statement $(\neg P) \wedge (\neg Q)$.
7. Explain why the statement $\neg (P \Rightarrow (Q \vee R))$ is equivalent to the assertion that P is true and that both of the statements Q and R are false.
8. Explain why the converse of the statement $P \Rightarrow (Q \vee R)$ is equivalent to the condition $(R \Rightarrow P) \wedge (Q \Rightarrow P)$.
9. Write the assertion $P \Rightarrow (Q \vee R)$ as simply as you can in its contrapositive form.
10. Write the assertion $(P \wedge Q) \Rightarrow (R \vee S)$ as simply as you can in its contrapositive form.

Chapter 3
Strategies for Writing Proofs

3.1 Introduction

The purpose of this chapter is to teach you how to study mathematical proofs that have been provided for you and how to think up proofs of your own. As you will see, these two tasks are almost the same because, in the process of reading a proof, you are constantly filling in details in order to justify the assertions that the author has made. Thus the key to learning a proof is *understanding*. A time honored practice that students have employed in order to "learn" a proof without having to go to the trouble of actually understanding it is to commit it to memory. Don't make that mistake! Memorizing a proof that you do not understand makes about as much sense as studying a great musical composition by memorizing the notes and musical symbols such as ♭, ♮, and ♯ that appear on a piece of paper, without having the slightest idea of what these symbols mean or what melody is written there.

Nor should you be content, when you are studying a proof, with the knowledge of how each individual step follows from the one before it. Every proof has a *theme*, a master plan, that suggests to us what the individual steps should be. You have understood a proof only when you have looked into it deeply enough to perceive that theme. Sometimes, when you are reading a proof, it will take some careful digging to unearth its underlying theme, especially when the proof is very polished. One of the difficulties that faces you is that the proof you are reading is a completed article. It works. It is valid. But it doesn't always reveal all of the thoughts that led to its discovery and the thoughts that guided your teacher to write it in the form that lies before you.

To help yourself to discover the underlying theme of a proof and to anticipate the way in which it may be laid out, you may want to ask yourself the following two kinds of question:

- What are we trying to prove here? What does this statement mean to me? How else can I write this statement? What would it mean to say that this statement is not true? What other proofs do I know that are used to prove statements like this one?
- What is the given information? What does the given information tell me? How does one usually go about using this kind of given information? What other proofs do I know that use this kind of information.

Sometimes, when a proof is difficult, you will not be able to anticipate it. Read the proof one step at a time and, when you understand the individual steps, go back to the job of trying to anticipate it. You will gradually come to understand the bridge between the given information and the required statement that the proof provides. As your understanding of the proof solidifies, make sure that you understand where *all* of the given information is used. If any of this information wasn't used, then either the theorem can be improved or (more likely) you are misunderstanding something.

When you really understand a proof, you will feel able to explain it to others. In fact, you will *want* to do so; in much the same way that you might look around for someone to whom you could tell a good joke that you have just heard. One of the best ways to learn a proof is to imagine that you are going to teach it to someone else. As you study it, write it down on a blank piece of paper and imagine that you are, in fact, explaining it to another person. You have understood the proof if and only if you have the feeling that you did a really fine job of explaining it.

The way we approach the task of proving a particular mathematical statement depends on the nature of that statement and, in particular, on the way the grammatical symbols *if, and, or,* $\Rightarrow$, $\forall$, and $\exists$ appear. Exactly where these words appear and how they appear plays a major role when we devise our proof writing strategy.

3.2 Statements that Contain the Word *and*

If P and Q are mathematical statements, then, as you know, the statement $P \wedge Q$ asserts that both of the statements P and Q are true. In this section we shall discuss some strategies for proving a statement of the form $P \wedge Q$, and we shall also discuss some strategies for using information that is phrased in this form.

3.2.1 Proving a Statement of the Form $P \wedge Q$

If P and Q are mathematical statements, then, in order to prove the statement $P \wedge Q$, we have two tasks to perform: We need to show that P is true, and we also need to show that Q is true. This kind of proof can be broken down into two steps.

- Step 1: Show that the statement P is true.
- Step 2: Show that the statement Q is true.

Suppose, for example, that you want to prove that neither of the integers 1037173 nor 4087312111 is prime. We have two tasks to perform. We need to show that the number 1037173 is not prime, and then we need to show that the integer 4087312113 is not prime. To accomplish these two tasks we can observe first that

$$1037173 = 223 \times 4651,$$

and then we can observe that

$$4087312111 = 587 \times 6963053.$$

Watch for the word *and* in statements that you want to prove. The appearance of *and* is often an indication that your problem can be broken down into several simpler parts. Even if you fail to solve the problem as a whole, you may still succeed with some of these parts. Remember: A partial solution of a mathematical problem is better than no solution at all.

3.2.2 Using Information of the Form $P \wedge Q$

Suppose that we are given information of the form $P \wedge Q$ and that we want to prove a statement R. We cannot easily break this kind of proof into several parts. We have been given the truth of each of the statements P and Q, and we need to use this information to obtain the statement R.

3.2.3 Example of a Proof Using Information Containing *and*

To illustrate this kind of proof we shall prove an elementary fact about integers. As you know, if an integer a is even, then a has the form $2n$ for some integer n and if a is odd, then a has the form $2n + 1$. We shall now prove the following statement:

If a and b are odd integers, then their product ab is also odd.

Proof. Suppose that a and b are odd integers. We begin by using the fact that a is odd to write a in the form $2m + 1$ for some integer m. Next we use the fact that b is odd to write b in the form $2n + 1$ for some integer n. We observe that

$$ab = (2m + 1)(2n + 1) = 2(2mn + m + n) + 1,$$

and we conclude that ab is odd. ■

3.2.4 Some Exercises on Statements Containing *and*

1. Prove the following assertions:
   (a) If a, b, c, x, and y are positive numbers and $a^x = b$ and $b^y = c$, then $a^{xy} = c$.
   (b) If two integers m and n are both even, then mn has a factor 4.
   (c) If an integer m is even and an integer n has a factor 3, then mn has a factor 6.
2. "Ladies and gentlemen of the jury" said the prosecutor, "We shall

demonstrate beyond a shadow of doubt that, on the night of June 13, 1997, the accused, Slippery Sam Carlisle, did willfully, unlawfully and maliciously kill and murder the deceased, Archibald Bott, by striking him on the head with a blunt instrument". Outline a strategy that the prosecutor might use to prove this charge. How many separate assertions must the prosecutor prove in order to carry out his promise to the jury?
3. One of the basic laws of arithmetic tells us that if a and b are any two numbers satisfying the condition $a < b$, and if $x > 0$, then $ax < bx$. Show how this law may be used to show that if $0 < u < 1$ and $0 < v < 1$, then $0 < uv < 1$.
4. In this exercise, if we are given three nonnegative integers a, b, and c, then the integer that consists of a hundreds, b tens and c units will be written as $\lceil a, b, c \rceil$. Given nonnegative integers a, b, and c, prove the assertion $P \wedge Q \wedge R \wedge S$, where P, Q, R and S are, respectively, the following assertions:
   P. If the number $\lceil a, b, c \rceil$ is divisible by 3, then $a + b + c$ is also divisible by 3.
   Q. If the number $a + b + c$ is divisible by 3, then $\lceil a, b, c \rceil$ is also divisible by 3.
   R. If the number $\lceil a, b, c \rceil$ is divisible by 9, then $a + b + c$ is also divisible by 9.
   S. If the number $a + b + c$ is divisible by 9, then $\lceil a, b, c \rceil$ is also divisible by 9.

3.3 Statements that Contain the Word *or*

Suppose that P and Q are mathematical statements. As you know, the statement $P \vee Q$ asserts that at least one of the statements P and Q must be true. The statement $P \vee Q$ does not guarantee that both of the statements P and Q are true, although they might be. In this section we shall discuss some strategies for proving a statement of the form $P \vee Q$, and we shall also discuss some strategies for using information that is phrased in this form.

3.3.1 Proving a Statement of the Form $P \vee Q$

If P and Q are mathematical statements, then, in order to prove the statement $P \vee Q$, we need to show that at least one of the statements P and Q is true. In other words, we need to show that if either of the statements P and Q happens to be false, then the other one must be true. Among the ways in which one may approach a proof of this type, the following two are worth mentioning:

1. Try to prove that the statement P is true. If you succeed, you are done. If you can't see why P is true, or if you suspect that P may be false, try to prove that Q is true.
2. This approach is a slight refinement of the first approach. If you know that the statement P is false, then add the information that P is false to the information that you already have and try to prove that Q is true.

3.3.2 Example of a Proof of a Statement Containing *or*

We now return to the example that we saw in Subsection 3.2.3. Suppose that m and n are two given integers. The following statements all say the same thing:

1. If both m and n are odd, then mn must also be odd.
2. If mn is even, then either m is even or n is even.
3. If mn is even and m is odd, then n must be even.

In Subsection 3.2.3, we proved this statement in its first form. Now we shall prove it in its second form and you can decide which of the two proofs you prefer.

Suppose then that mn is even. We need to explain why either m or n must be even. For this purpose we shall show that if m happens to be odd, then the number n has to be even. (In other words, we are really proving the statement in its third form.) Suppose that m is odd. Using this assumption, choose an integer a such that $m = 2a + 1$ and, using the fact that mn is even, choose an integer b such that $mn = 2b$. Thus

$$(2a+1)\,n = 2b,$$

from which we deduce that

$$n = 2\,(b - an)\,,$$

and we have shown that n must be even.

3.3.3 Using Information of the Form $P \vee Q$

Suppose that we want to prove a statement R using given information of the form $P \vee Q$, where P and Q are mathematical statements. In order to construct such a proof you need to accomplish two tasks:

(a) Show that if P is true, then R is true. You can do this by assuming that P is true and showing that R is true.
(b) Show that if Q is true, then R is true. You can do this by assuming that Q is true and showing that R is true.

A proof of this type can therefore be separated into different cases.

3.3.4 Example of a Proof Using Information Containing *or*

If $x = \cos 20°$ *or* $x = -\cos 40°$ *or* $x = -\cos 80°$, *then*

$$8x^3 - 6x - 1 = 0. \tag{3.1}$$

Proof. In order to prove this statement we need to perform three tasks. We need to show that $x = \cos 20°$ is a solution of Equation (3.1). Then we need to show that $x = -\cos 40°$ is a solution of Equation (3.1). Finally we need to show that $x = -\cos 80°$ is a solution of Equation (3.1). To perform the first of these three tasks we deduce from the trigonometric identity

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

that, if $x = \cos 20°$, then

$$\begin{aligned} 8x^3 - 6x - 1 &= 2\left(4\cos^3 20° - 3\cos 20°\right) - 1 \\ &= 2\cos 60° - 1 = 0. \end{aligned}$$

We leave the second and third of these three tasks as exercises. ■

3.3.5 Some Exercises on Statements Containing *or*

1. Given that m and n are integers and that the number mn is not divisible by 4, prove that either m is odd or n is odd.
2. Given that m and n are integers, that neither m nor n is divisible by 4 and that at least one of the numbers m and n is odd, prove that the number mn is not divisible by 4.
3. Prove that if $x = -\cos 40°$ or $x = -\cos 80°$, then
$$8x^3 - 6x - 1 = 0.$$
4. A theorem in elementary calculus, known as Fermat's theorem, says that if a function f defined on an interval has either a maximum or a minimum value at a number c inside that interval, then either $f'(c) = 0$ or $f'(c)$ does not exist. Give a brief outline of a strategy for approaching the proof of this theorem.
5. A well-known theorem on differential calculus that is known as **L'Hôpital's rule** may be stated as follows:

Suppose that f and g are given functions, that c is a given number, that

$$\lim_{x\to c}\frac{f'(x)}{g'(x)} = L,$$

and that one or the other of the following two conditions holds

(a) *Both $f(x)$ and $g(x)$ approach 0 as $x \to c$.*
(b) *$g(x) \to \infty$ as $x \to c$.*

Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = L.$$

Describe how the proof of L'Hôpital's rule can be broken down into two parts. For each part of the proof, say what is being assumed and what is being proved.

3.4 Statements of the Form $P \Rightarrow Q$

As you know, if P and Q are mathematical statements, then $P \Rightarrow Q$ says that in the event that P is true, the statement Q must also be true. The assertion $P \Rightarrow Q$ can also be looked upon as a statement containing the word *or* because it says that either P is false or Q is true. In this section we shall discuss the strategy for proving a statement of the form $P \Rightarrow Q$ and the strategy for using information that is phrased in the form $P \Rightarrow Q$.

3.4.1 Proving a Statement of the Form $P \Rightarrow Q$

If P and Q are mathematical statements, then to prove the statement $P \Rightarrow Q$ we need to show that either P is false or that Q is true. We can therefore follow the procedure given in Subsection 3.3.1 and use one of the following methods:

- Assume that P is true and use this to show that Q must be true.
- Assume that Q is false and use this to show that P must be false. In other words, prove the contrapositive form of the statement $P \Rightarrow Q$.

3.4.2 Using Information of the Form $P \Rightarrow Q$

Suppose that we want to prove a statement R using given information of the form $P \Rightarrow Q$, where P and Q are mathematical statements. Since the given information $P \Rightarrow Q$ can be interpreted as saying that either P is false or Q is true, we can use the method described in Subsection 3.3.3. Another approach to this sort of proof is to reason that the condition $P \Rightarrow Q$ is interesting only when P is true. If you know (or can prove) that P is true, then you can deduce from the condition $P \Rightarrow Q$ that Q is true and use the fact that both P and Q are true to show that R is true.

On the other hand, if you know that P is false, then the statement Q is irrelevant. In this case, use the fact that P is false to show that R is true.

3.4.3 Example of a Proof Using Information Containing $\Rightarrow$

We shall take P to be the statement that $x \neq 1$ and Q to be the statement that $x = 3$. Suppose that we are given that $P \Rightarrow Q$ and that we want to prove that the equation

$$x^2 - 4x + 3 = 0$$

holds. In the event that the statement P is true we know that $x = 3$, and we can verify that the equation holds in this case. In the event that the statement P is false we have $x = 1$, and, once again, we can verify that the equation holds.

3.4.4 Exercises on the Symbol $\Rightarrow$

1. (a) Outline a strategy for proving an assertion that has the form $P \Rightarrow (Q \wedge R)$.
   (b) Write down the assertion $P \Rightarrow (Q \wedge R)$ in its contrapositive form and outline a strategy for proving it in this form.
2. (a) Outline a strategy for proving an assertion that has the form $P \Rightarrow (Q \vee R)$.
   (b) Write down the assertion $P \Rightarrow (Q \vee R)$ in its contrapositive form and outline a strategy for proving it in this form.
3. (a) Outline a strategy for proving an assertion that has the form $(P \wedge Q) \Rightarrow R$.
   (b) Write down the assertion $(P \wedge Q) \Rightarrow R$ in its contrapositive form and outline a strategy for proving it in this form.
4. (a) Outline a strategy for proving an assertion that has the form $(P \vee Q) \Rightarrow R$.
   (b) Write down the assertion $(P \vee Q) \Rightarrow R$ in its contrapositive form and outline a strategy for proving it in this form.
5. (a) Outline a strategy for proving an assertion that has the form $(P \vee (Q \Rightarrow P)) \Rightarrow R$.
   (b) Write down the assertion $(P \vee (Q \Rightarrow P)) \Rightarrow R$ in its contrapositive form and outline a strategy for proving it in this form.

3.5 Statements of the Form $\exists x\,(P(x))$

As you know, if $P(x)$ is a statement about an unknown x, then the assertion $\exists x\,(P(x))$ says that there is at least one object x for which the statement $P(x)$ is true. In this section we shall discuss the strategy for proving a statement of the form $\exists x\,(P(x))$ and the strategy for using information that is phrased in the form $\exists x\,(P(x))$.

3.5.1 Proving A Statement of the Form $\exists x\,(P(x))$

How does one prove that something exists? Legend has it that the Greek philosopher Diogenes traveled from place to place with a lantern in his hand looking for one honest man. By finding one he could certainly have proved that such a man exists. Diogenes' attempt to find an honest man demonstrates one very good way of proving that an object exists: Find one. We use this kind of proof very commonly in mathematics. Suppose, for example, that we want to prove that there exists a prime number between 80 and 90. We can achieve this proof by demonstrating that the number 83 is prime.

Although the method of giving an example is the most common way of proving existence, it has the drawback that we have to be able to find that example. This may be very hard to do. It may even be impossible. Fortunately we also have other ways of proving existence. Even though we may not know of any examples of a certain kind of object, sometimes there will be indirect evidence that the object exists. For example, if you are sitting in a crowded restaurant, you do not have to be able to see and identify a smoker in order to know that someone is smoking. All you have to do is try to breathe.

There are many important theorems in mathematics that assert the existence of an object even when it is hard or impossible to lay our hands on one. We look at two examples:

1. The well-known theorem in differential calculus that we call the **mean value theorem**, and which is illustrated in Figure 3.1, can be stated as follows:

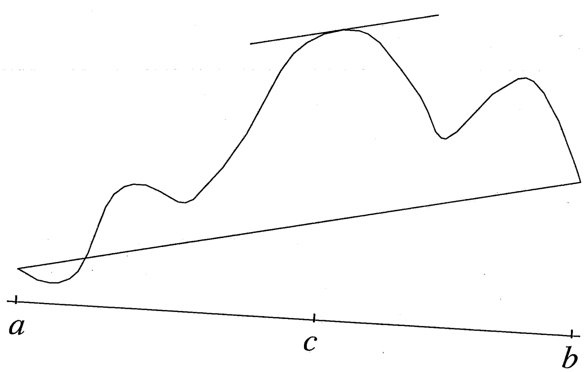


Figure 3.1

Suppose that a function f is continuous on an interval $[a, b]$ (where $a < b$) and that the derivative $f'(x)$ exists for every $x \in (a, b)$. Then there is at least one number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{3.2}$$

The importance of the mean value theorem lies in the fact that, although there must exist a number c that makes Equation (3.2) hold, we usually cannot lay our hands on such a number. Had we been able to do so, the mean value theorem would not have been nearly as interesting. As it happens, the mean value theorem makes a very deep statement about the nature of our number system. It tells us that, even though we may not be able to find the number c explicitly, it exists anyway.

2. A real number x is said to be **algebraic** if it is a solution of an equation of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots a_2 x^2 + a_1 x + a_0 = 0,$$

where n is a positive integer, and the numbers $a_0, \ldots a_n$ are integers, and $a_n \neq 0$. A number that is not algebraic is said to be **transcendental.** Although there are many common examples of transcendental numbers, including such numbers as $2^{\sqrt{2}}$, $\log_2 3$, e, π, and e^π, it is by no means easy to prove that any given number is transcendental. It is therefore hard to prove that transcendental numbers exist if the method of proof is to exhibit an example of one. In the latter part of the nineteenth century, Georg Cantor came up with a different kind of proof of the existence of transcendental numbers that is based upon his theory of sets. What Cantor showed is that the set $\mathbf{R}$ of all real numbers has a property that is called *uncountability* but that the set A of algebraic numbers does not have this property. Therefore, he reasoned, the sets $\mathbf{R}$ and A cannot be equal to each other and so there must be real numbers that are not algebraic. Thus we can use Cantor's proof to guarantee the existence of transcendental numbers even if we have never seen one.

To sum up, if we can lay our hands on an example of a certain kind of object, then we know that the object exists. This method is the most satisfying way of proving existence, but it isn't always possible. Sometimes we have to use an indirect method to prove an existence theorem and, as you may expect, the existence theorems that require indirect proofs are usually the most interesting ones.

3.5.2 Using Information of the Form $\exists x\,(P(x))$

In the last subsection we discussed proofs of statements of the type $\exists x\,(P(x))$. Now we shall assume that a statement of this type has been given and we shall discuss the possible ways of using this information. What we have assumed is that there exists at least one object x with certain properties. For this fact to be of any interest to us we must have a need for an object of this type. Our procedure is to *choose* an object x with the desired properties and to use it for whatever purpose it is needed. If the statement $\exists x\,(P(x))$ has been given, and if we need an object x satisfying the condition $P(x)$, then we need to write

choose x such that the condition $P(x)$ is true.

To illustrate this kind of proof we shall prove the following fact about differential calculus:

Suppose that f is a continuous function on an interval $[a,b]$ and that $f'(x) = 0$ for every number $x \in (a,b)$. Then $f(a) = f(b)$.

Proof. The mean value theorem guarantees that there exists at least one number $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

and so we *choose* a number $c \in (a,b)$ such that this equation holds. From the fact that $f'(x) = 0$ for every $x \in (a,b)$ we certainly have $f'(c) = 0$. Therefore

$$\frac{f(b) - f(a)}{b - a} = 0,$$

and we conclude that $f(a) = f(b)$. ■

3.5.3 A Note of Caution

When you are writing a mathematical proof, do not make the common mistake of thinking that, just because a condition of the form $\exists x\,(P(x))$ is given, such an object x has been chosen for you automatically. You do not have such an object x in your hands until you have written:

Choose x such that the condition $P(x)$ is true.

Consider the following pair of sentences:

There exists a number x such that $0 < x < 1$.
Clearly, $x^2 < x$.

The second of these two sentences is meaningless. If you were to write these sentences in an examination, the examiner would respond with something like: "*What is x?*" You cannot answer that x is "the number" introduced in your first

sentence because all the first sentence says is that there is a number between 0 and 1.

On the other hand it, would be perfectly legitimate to say:

There exists a number x such that $0 < x < 1$.
Furthermore, for every $x \in (0,1)$ we have $x^2 < x$.

Alternatively, you could say:

Using the fact that $(0,1)$ is nonempty, choose $x \in (0,1)$.
Note that $x^2 < x$.

3.6 Statements of the Form $\forall x\,(P(x))$

As you know, if $P(x)$ is a statement about an unknown x, then the assertion $\forall x\,(P(x))$ says that the condition $P(x)$ is true for *every* object x. In this section we shall discuss the strategy for proving a statement of the form $\forall x\,(P(x))$ and the strategy for using information that is phrased in the form $\forall x\,(P(x))$.

3.6.1 Inductive and Deductive Reasoning

A statement of the type $\forall x\,(P(x))$ can *never* be proved by giving an example. All an example can tell us is that there is one object x for which $P(x)$ is true. It says nothing to guarantee that $P(x)$ is true for *every* x. Thus, for example, you cannot conclude that the identity

$$\sqrt[3]{\frac{3\sqrt{3}x + (2x^2+1)\sqrt{4-x^2}}{6\sqrt{3}}} + \sqrt[3]{\frac{3\sqrt{3}x - (2x^2+1)\sqrt{4-x^2}}{6\sqrt{3}}} = x$$

holds for all numbers $x \in [-2,2]$ merely by observing that it holds when $x = 0$. Even if we were to find thousands of numbers x in $[-2,2]$ for which this identity is true, in the absence of a general proof we would still not know for sure that the identity holds for *all* $x \in [-2,2]$.

Outside of mathematics there is a type of reasoning called **inductive reasoning**, which is used to validate statements of the type $\forall x\,(P(x))$. The idea of inductive reasoning is that $P(x)$ probably holds for all x if it is known to hold for a representative selection of objects x. For example, if every time you have used a certain laundry detergent you have broken out in a rash, you could reasonably conclude that this detergent is causing your problem. Every time you use the detergent and see the rash appear you feel a little more confident that your theory is correct.

Inductive reasoning plays an important role in such walks of life as science, medicine, and even law. Doctors must use inductive reasoning to make decisions upon which our very lives depend. Courts must render decisions based on inductive reasoning. There is, in fact, a branch of mathematics, known as *statistics*, that is devoted to the problem of measuring how confidently one may assert a statement that was obtained using inductive reasoning. But inductive reasoning can never be used to make a conclusion in mathematics.

3.6.2 Proving a Statement of the Form $\forall x\,(P(x))$

In order to verify an assertion of the type $\forall x\,(P(x))$ we need to show that the condition $P(x)$ holds for every member x of some specified set. If that set is infinite, then we can't live long enough to test the conditions $P(x)$ one at a time, but, fortunately, there is a more efficient approach. We prove that the condition $P(x)$ holds for what we call an *arbitrary* value of x that we introduce using the word *suppose*.

3.6.3 Example of a Proof of a Statement Containing *for every*

For every number $x \in [-2, 2]$ we have

$$\sqrt[3]{\frac{3\sqrt{3}x + (2x^2+1)\sqrt{4-x^2}}{6\sqrt{3}}} + \sqrt[3]{\frac{3\sqrt{3}x - (2x^2+1)\sqrt{4-x^2}}{6\sqrt{3}}} = x.$$

Proof. Because we want to prove that a condition is true for every number x in the interval $[-2, 2]$, we begin our proof by writing

Suppose $x \in [-2, 2]$.

In writing this opening sentence we do not have to know that there are any numbers in the interval $[-2, 2]$. What we are saying is that, in case there are any, suppose that x is an arbitrary one of them, coming without any restrictions, so that anything we might be able to prove about x would apply just as well to any other number in the interval. In other words, let x be an arbitrary number in the interval $[-2, 2]$ that has come to challenge us to prove that

$$\sqrt[3]{\frac{3\sqrt{3}x + (2x^2+1)\sqrt{4-x^2}}{6\sqrt{3}}} + \sqrt[3]{\frac{3\sqrt{3}x - (2x^2+1)\sqrt{4-x^2}}{6\sqrt{3}}} = x.$$

Now that we have this challenger in our hands we can begin our proof. To obtain the desired result we shall first show that

$$\sqrt[3]{\frac{3\sqrt{3}x + (2x^2+1)\sqrt{4-x^2}}{6\sqrt{3}}} = \frac{x}{2} + \frac{1}{2}\sqrt{\frac{4-x^2}{3}}$$

and

$$\sqrt[3]{\frac{3\sqrt{3}x - (2x^2+1)\sqrt{4-x^2}}{6\sqrt{3}}} = \frac{x}{2} - \frac{1}{2}\sqrt{\frac{4-x^2}{3}}.$$

Once these two equations have been established, the desired result will follow at once. To establish the first of these two equations we expand and simplify the expression

$$\left(\frac{x}{2} + \frac{1}{2}\sqrt{\frac{4-x^2}{3}}\right)^3$$

to obtain

$$\left(\frac{x}{2} + \frac{1}{2}\sqrt{\frac{4-x^2}{3}}\right)^3 = \frac{3\sqrt{3}x + (2x^2+1)\sqrt{4-x^2}}{6\sqrt{3}}.$$

The second equation can be obtained similarly by showing that

$$\left(\frac{x}{2} - \frac{1}{2}\sqrt{\frac{4-x^2}{3}}\right)^3 = \frac{3\sqrt{3}x - (2x^2+1)\sqrt{4-x^2}}{6\sqrt{3}}.$$

Before leaving this topic we should notice that, although the proof we have just given is valid, it is not very satisfying. It guarantees that the equation

$$\sqrt[3]{\frac{3\sqrt{3}x + (2x^2+1)\sqrt{4-x^2}}{6\sqrt{3}}} + \sqrt[3]{\frac{3\sqrt{3}x - (2x^2+1)\sqrt{4-x^2}}{6\sqrt{3}}} = x$$

holds for every $x \in [-2, 2]$, but it doesn't tell us how we could have anticipated this equation. It doesn't motivate it. If you would like to see a better approach, go to the on-screen text and click on the icon , which will take you to some exercises in the optional document on cubic equations. Of course, those exercises should not be attempted unless you decide to read that optional document.

3.6.4 Using Information of the Form $\forall x\,(P(x))$

Information of the type $\forall x\,(P(x))$ can be useful to us only when there are certain objects x that are of particular interest to us and for which we would like to know that the condition $P(x)$ is true. We can then say that because the condition $P(x)$ is true for *every* x, certainly $P(x)$ will be true for those objects x that are of interest to us.

Suppose, for example, that I am standing in a used car lot and I want to buy a used car. The knowledge that all used car sales personnel are honest is very comforting to me. It tells me that I can place my implicit trust in the gentleman who is assisting me.

3.6.5 Example of a Proof Using Information Containing *for every*

We shall revisit the proof that appears in Subsection 3.5.2. There, we were given a function f that is continuous on an interval $[a, b]$ and for which $f'(x) = 0$ for every $x \in (a, b)$, and we proved that $f(a) = f(b)$. We began by choosing a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Then we made use of the fact that $f'(x) = 0$ for *every* $x \in (a, b)$. Among all the numbers $x \in (a, b)$, the number c is of particular interest to us. Since $f'(x) = 0$ for every $x \in (a, b)$, we know that $f'(c) = 0$. Therefore

$$\frac{f(b) - f(a)}{b - a} = 0,$$

and we deduce that $f(a) = f(b)$.

3.6.6 Exercises on Statements Containing Quantifiers

1. Physicist's proof that all odd positive integers are prime: 1 is prime.[10] 3 is prime. 5 is prime. 7 is prime. 9 is experimental error. 11 is prime. 13 is prime. We have now taken sufficiently many readings to verify the hypothesis. Comment!
2. You know that there are 1000 people in a hall. Upon inspection you determine that 999 of these people are men. What can you conclude about the 1000th person?
3. The *product rule for differentiation* says that for every number x and all functions f and g that are differentiable at x, we have

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Write down the opening line of a proof of the product rule. Your opening line should start: *Suppose that ...*

4. Given that $P(x)$ and $Q(x)$ are statements that contain an unknown x and that S is a set, outline a strategy for proving the assertion $P(x) \Rightarrow Q(x)$ for every $x \in S$. Write down the opening line of your proof.

[10] Mathematicians are divided on the issue of whether or not the number 1 should be called prime.

5. Given that $P(x)$ is a statement that contains an unknown x and that S is a set, write down an opening line of a proof of the assertion that $P(x)$ is true for every $x \in S$.
6. You are given that $P(x)$ and $Q(x)$ are statements that contain an unknown x, that S is a set, that $P(x)$ is true for every $x \in S$, and that $P(x) \Rightarrow Q(x)$. Is it possible to deduce that $Q(x)$ is true for every member x of the set S?
7. You are given that $P(x)$ and $Q(x)$ are statements that contain an unknown x, that S is a set, that $P(x)$ is true for every $x \in S$, and that $P(x) \Rightarrow Q(x)$. Is it possible to deduce that $Q(x)$ is true for at least one member x of the set S?
8. Write down the contrapositive form of the statement that for every member x of a given set S we have $P(x) \Rightarrow Q(x)$.
9. Write down the denial of the statement that for every x we have $P(x) \Rightarrow Q(x)$.
10. Prove that, for every number x in the interval $[-2, 2]$, if we define

$$u = \sqrt[3]{\frac{3\sqrt{3}x + (2x^2 + 1)\sqrt{4 - x^2}}{6\sqrt{3}}}$$

and

$$v = \sqrt[3]{\frac{3\sqrt{3}x - (2x^2 + 1)\sqrt{4 - x^2}}{6\sqrt{3}}},$$

then

$$u^2 + v^2 = 1 + uv.$$

Hint: With an eye on the proof in Subsection 3.6.3, show that $u^3 + v^3 = u + v$.

3.7 Proof by Contradiction

The idea of proof by contradiction is that if we can deduce a contradiction by assuming that a certain statement P is false, then P must be true. Proof by contradiction is usually most useful when the statement P that we are considering seems rather intangible but its denial $\neg P$ is nice and concrete. If P is this kind of statement, it is sometimes hard to find a direct proof that P is true. However, the assumption that P is false may place some very concrete information in our hands. If we can show that this information leads to a contradiction, then we have proved indirectly that P is true.

For an example of a statement P of this type, suppose that x is a given number and take P to be the statement that x is irrational.[11] Since P asserts that it is impossible to find integers m and n such that $x = \frac{m}{n}$, a direct proof of the statement P must tell us why it is impossible to find such integers m and n. How does one prove that it is impossible to do something? Now consider the statement $\neg P$. The statement $\neg P$ says that it *is* possible to find integers m and n such that $x = \frac{m}{n}$. If we are assuming the statement $\neg P$, then at least we know how to begin our proof:

Choose integers m and n such that $x = \dfrac{m}{n}$.

As you may expect from this discussion, we often use proof by contradiction to prove that a given real number is irrational.

3.7.1 Some Examples of Proof by Contradiction

1. In this example we shall show that the number $\log_2 6$ is irrational. To prove this assertion by the method of proof by contradiction we begin:

   To obtain a contradiction, assume that $\log_2 6$ is rational.

   Using the assumption that $\log_2 6$ is a positive rational number, choose positive integers m and n such that

   $$\log_2 6 = \frac{m}{n}.$$

   We observe that $2^{m/n} = 6$ and therefore

   $$2^m = 6^n.$$

   But the right side of the latter identity has a factor 3 while the left side does not, and so the two sides cannot be equal. Since the assumption that $\log_2 6$ is rational led us to a contradiction, we conclude that $\log_2 6$ must be irrational.

2. In this example we again concern ourselves with the irrationality of a number. This time we show that the number $\sqrt{2}$ is irrational. To obtain a contradiction we shall assume that $\sqrt{2}$ is rational. Choose positive integers m and n such that

   $$\sqrt{2} = \frac{m}{n}.$$

   We now cancel out any factors that are common to the numerator and denominator of the fraction m/n yielding a fraction p/q in which p and q are positive integers that have no common factor. Since $\sqrt{2} = p/q$, we have

   $$p^2 = 2q^2.$$

   Now, since q has no common factor with p we see that q has no common factor with p^2. But, on the other hand, since q is a factor of $2q^2$, we know that q is a factor of p^2. Therefore $q = 1$ and we see that $p^2 = 2$. The latter condition is certainly impossible because there is no integer whose square is 2. Since the assumption that $\sqrt{2}$ is rational leads to a contradiction, we conclude that $\sqrt{2}$ must be irrational.

[11] Recall that a number is rational if it can be expressed in the form of a ratio $\frac{m}{n}$, where m and n are integers.

3.7.2 Drawbacks of Proof by Contradiction

Although the method of proof by contradiction can be very useful when we are proving certain kinds of statements, some members of the mathematical community are less than enthusiastic about using it. They argue that it is better to give a positive reason why a statement P must be true rather than to conclude that P must be true because the statement $\neg P$ leads to a contradiction. Therefore, they feel that whenever we can see a direct way of showing that a statement P is true, we should use it.

Then there are the logical purists who point out that, if we can deduce a contradiction by assuming that a certain statement P is false, then this leaves open the remote possibility that we may also be able to deduce a contradiction by assuming that P is true. In this event, we would conclude that there is an inherent contradiction in mathematics. Therefore, the logical purists point out, if we can deduce a contradiction by assuming that a statement P is false, we need to say that, as long as there is no inherent contradiction in mathematics, P must be true.

3.7.3 Exercises on Proof by Contradiction

1. Prove that the following numbers are irrational:
   (a) $\log_{10} 5$.
   (b) $\log_{12} 24$.
   (c) $\sqrt[3]{4}$.
   (d) Any solution of the equation $8x^3 - 6x - 1 = 0$.
2. Given that m and n are integers and that mn does not have a factor 3, prove that neither m nor n can have a factor 3.
3. Suppose that we know that $x^2 - 2x < 0$ and that we wish to prove that $0 < x < 2$. Write down the first line of a proof of this assertion that uses the method of proof by contradiction. Do this in such a way that your proof

splits into two cases and complete the proof in each of these cases.

4. **H** Suppose that f is a given function defined on the interval $[0, 1]$ and suppose that we wish to prove that this function f has the property that there exists a number $\delta > 0$ such that, whenever t and x belong to the interval $[0, 1]$ and $|t - x| < \delta$, we have $|f(t) - f(x)| < 1$. Write down the first line of a proof of this assertion that uses the method of proof by contradiction.
5. Suppose that $\{x_1, x_2, \cdots x_n\}$ is a subset of a vector space[12] V and that we wish to prove that the set $\{x_1, x_2, \cdots x_n\}$ is linearly independent. Write down the first line of a proof of this assertion that uses the method of proof by contradiction. (Try to be specific. Don't just suppose that the set is linearly dependent.)

3.8 Some Further Examples

This section contains a few extra examples in which we prove or disprove statements of the form $\forall x\,(P(x))$. The proofs contained here will help you to develop skills that will be useful to you when you read some of the later chapters.

3.8.1 A Fact About Inequalities

In this subsection we discuss the two statements that appeared in Example 5 of Subsection 2.1.4. We begin by proving that the first of the statements is true. This statement is as follows:

For every number x there exists a positive number δ such that for every number t satisfying the inequality $|t - x| < \delta$ we have $|t^2 - x^2| < 1$.

Proof. Since we want to prove that a condition holds for every number x, we begin this proof by writing:

Suppose that x is any real number.

Now that we have a challenger x in our hands we need to prove that there exists a positive number δ such that for every number t satisfying $|t - x| < \delta$ we have $|t^2 - x^2| < 1$. We shall demonstrate the existence of such a number δ by finding one.

To help us find this number, we observe that if $\delta > 0$, then for any number t satisfying $|x - t| < \delta$ we have

$$\begin{aligned} |x^2 - t^2| &= |x - t|\,|x + t| = |x - t|\,|(t - x) + 2x| \\ &\le |x - t|\,(|x - t| + 2|x|) < \delta(\delta + 2|x|). \end{aligned}$$

[12] Skip this exercise if you have not had a course in linear algebra.

So, to guarantee the inequality $|t^2 - x^2| < 1$, all we have to do is find a positive number δ for which

$$\delta(\delta + 2|x|) \le 1,$$

and, with this thought in mind, we define

$$\delta = \frac{1}{1 + 2|x|}.$$

Note that $\delta \le 1$ and so

$$\delta(\delta + 2|x|) \le \frac{1}{1 + 2|x|}(1 + 2|x|) = 1.$$

Now that both x and δ have been introduced, we want to show that for every number t satisfying the inequality $|t - x| < \delta$ we have $|t^2 - x^2| < 1$. We therefore continue the proof by writing:

Suppose that t is any number satisfying $|t - x| < \delta$.

And we complete the proof by observing that

$$|t^2 - x^2| < \delta(1 + 2|x|) \le 1.$$

We shall now show that the second of the two statements that appeared in Example 5 of Subsection 2.1.4 is false. This statement is as follows:

There exists a positive number δ such that for every number x and for every number t satisfying the inequality $|t - x| < \delta$ we have $|t^2 - x^2| < 1$.

Proof. To see that this statement is false, suppose that δ is any positive number. Define $x = 1/\delta$ and

$$t = x + \frac{\delta}{2}.$$

Then, although $|t - x| = \delta/2 < \delta$, we have

$$t^2 - x^2 = x\delta + \frac{\delta^2}{4} > 1. \blacksquare$$

3.8.2 Another Fact About Inequalities

In this subsection we prove Statement 6a that appeared in Example 6 in Subsection 2.1.4:

For every number $\varepsilon > 0$ and for every number $x \in [0,1]$ there exists a positive integer N such that for every integer $n \geq N$ we have

$$\frac{nx}{1+n^2x^2} < \varepsilon.$$

Proof. Since we want to prove that a condition holds for every $\varepsilon > 0$, we begin the proof by writing:

Suppose that $\varepsilon > 0$.

Now that we have this challenger ε in our hands we need to prove that for every number $x \in [0,1]$ there exists a positive integer N such that for every integer $n \geq N$ we have

$$\frac{nx}{1+n^2x^2} < \varepsilon.$$

We therefore continue the proof by writing:

Suppose that $x \in [0,1]$.

Now that we have this challenger x in our hands we need to show that there exists a positive integer N such that for every integer $n \geq N$ we have

$$\frac{nx}{1+n^2x^2} < \varepsilon,$$

and we shall demonstrate the existence of such a number N by finding one. In the event that $x = 0$, the desired inequality holds for all values of n and we simply define $N = 1$. We see that

$$\frac{nx}{1+n^2x^2} = 0 < \varepsilon$$

for all $n \geq N$. In the event that $0 < x \leq 1$, we choose an integer $N > \frac{1}{\varepsilon x}$ and we observe that for all $n \geq N$,

$$\frac{nx}{1+n^2x^2} \leq \frac{nx}{0+n^2x^2} = \frac{1}{nx} \leq \frac{1}{Nx} < \varepsilon. \blacksquare$$

3.8.3 Yet Another Fact About Inequalities

In this subsection we show that the assertion 6c that we saw in Example 6 in Subsection 2.1.4 is false. This statement says:

For every number $\varepsilon > 0$ there exists a positive integer N such that for every number $x \in [0,1]$ and for every integer $n \geq N$ we have

$$\frac{nx}{1+n^2x^2} < \varepsilon.$$

To prove that this statement is false we shall prove that its denial is true. We therefore need to prove the following statement:

There exists a number $\varepsilon > 0$ such that for every positive integer N there exists a number $x \in [0,1]$ and there exists an integer $n \geq N$ such that

$$\frac{nx}{1+n^2x^2} \geq \varepsilon.$$

We shall demonstrate the existence of such a number ε by finding one. As a matter of fact, we shall show that the number $1/2$ is an example of such a number ε. We begin by writing:

Define $\varepsilon = 1/2$.

Now we must show that for every positive integer N there exists a number $x \in [0,1]$ and there exists an integer $n \geq N$ such that

$$\frac{nx}{1+n^2x^2} \geq \varepsilon,$$

and so we continue by writing:

Suppose that N is any positive integer.

To complete the proof we need to demonstrate the existence of a number $x \in [0,1]$ and an integer $n \geq N$ such that

$$\frac{nx}{1+n^2x^2} \geq \varepsilon,$$

and we shall do so by finding two such numbers. We define $n = N$ and $x = 1/N$, and we observe that

$$\frac{nx}{1+n^2x^2} = \frac{1}{1+1} = \varepsilon. \blacksquare$$

3.8.4 Some Additional Exercises

In each of the following exercises, decide whether the statement is true or false and then write a carefully worded proof to justify your assertion.

1. For every number $x \in [0,1]$ there exists a positive integer N such that for every number $\varepsilon > 0$ and every integer $n \geq N$ we have

$$\frac{nx}{1+n^2x^2} < \varepsilon.$$

2. For every number $x \in [0,1]$ and every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every number $t \in [0,1]$ satisfying $|t-x| < \delta$ we have $|t^2 - x^2| < \varepsilon$.

3. For every number $\varepsilon > 0$ and every number $x \in [0,1]$ there exists a number $\delta > 0$ such that for every number $t \in [0,1]$ satisfying $|t-x| < \delta$ we have $|t^2 - x^2| < \varepsilon$.

4. For every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every number $x \in [0,1]$ and every number $t \in [0,1]$ satisfying $|t-x| < \delta$ we have $|t^2 - x^2| < \varepsilon$.

5. For every number $\varepsilon > 0$ and every number x there exists a number $\delta > 0$ such that for every number t satisfying $|t-x| < \delta$ we have $|t^2 - x^2| < \varepsilon$.

6. For every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every number x and every number t satisfying $|t-x| < \delta$ we have $|t^2 - x^2| < \varepsilon$.

7. For every number $x \in (0,1]$ there exists a number $\delta > 0$ such that for every number $t \in (0,1]$ satisfying $|t-x| < \delta$ we have

$$\left|\frac{1}{t} - \frac{1}{x}\right| < 1.$$

8. There exists a number $\delta > 0$ such that for every number $x \in (0,1]$ and every number $t \in (0,1]$ satisfying $|t-x| < \delta$ we have

$$\left|\frac{1}{t} - \frac{1}{x}\right| < 1.$$

9. For every number $p \in (0,1]$ there exists a number $\delta > 0$ such that for every number $x \in [p,1]$ and every number $t \in [p,1]$ satisfying $|t-x| < \delta$ we have

$$\left|\frac{1}{t} - \frac{1}{x}\right| < 1.$$

10. (a) If either $0 < \theta < \pi$ or $\pi < \theta < 2\pi$, then

$$\arctan\left(\tan\left(\theta/2\right)\right) + \arctan\left(\tan\left(\pi/2 - \theta\right)\right) = \frac{\pi - \theta}{2}.$$

(b) Ask *Scientific Notebook* to solve the equation

$$\arctan\left(\tan\left(\theta/2\right)\right) + \arctan\left(\tan\left(\pi/2 - \theta\right)\right) = \frac{\pi - \theta}{2}.$$

Are you satisfied with the answer that it gives?

11. If x is any rational number, then

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}\left(\cos\left(n!\pi x\right)\right)^m\right) = 1.$$

12. If x is any irrational number, then

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}\left(\cos\left(n!\pi x\right)\right)^m\right) = 0.$$

Chapter 4
Elements of Set Theory

This chapter provides a brief introduction to those concepts from elementary set theory that we shall need in the rest of the book. Since many of these concepts may already be familiar to you, you should read as much or as little of this chapter as you need. If you are reading the on-screen version of this book, you can find a more extensive presentation of the set theory (that includes the concepts of countability and set equivalence, the Schröder-Bernstein equivalence theorem, and some more advanced topics) by clicking on the icon .

4.1 Introduction

In the latter part of the nineteenth century Georg Cantor revolutionized mathematics by showing that all mathematical ideas can trace their origins to a single concept: the concept of a *set*. On the face of it, Cantor's notion of a set was very simple: A set is a collection of objects. But the theory of sets that he created out of this idea was so profound that it has become the foundation stone of every branch of mathematics. Using Cantor's theory, mathematicians were able at last to provide precise definitions of important mathematical concepts such as real numbers, points in space, and continuous functions. In addition, Cantor's theory of sets provided a startling insight into the nature of infinity. Perhaps the most significant of his discoveries was the fact that some infinite magnitudes are larger than others.

Our notion of a set in this book will be the same one that Cantor used. We shall think of a set as being a collection of objects, and, when it suits us, we shall replace the word *set* by *class, family,* or *collection*. All of these words will have the same meaning. So, for example, the symbol

$$\{2, -1, 7\}$$

stands for the set whose members are the numbers 2, -1, and 7. When an object x belongs to a given set S we write $x \in S$. For example, $2 \in \{2, -1, 7\}$.

However, before we go any further with this topic we are going to pause briefly to confront some of the logical flaws that lurk in Cantor's definition of a set as a collection of objects. These are the flaws that we mentioned in Section 1.4 and which forced Frege to admit that the foundation of his monumental book had collapsed just as it went to press.

4.1.1 Bertrand Russell's Paradox

The trouble with Cantor's theory of sets is that when we are collecting objects together to make a set we have to allow for the possibility that some of the objects that we are collecting might be sets themselves. For example, the set

$$\{3, 8, \{2, 3\}\}$$

contains the number 3, the number 8, and the set $\{2, 3\}$ whose members are the numbers 2 and 3. Notice how the set $\{2, 3\}$ is a member of the set $\{3, 8, \{2, 3\}\}$. It is therefore entirely possible that a set A may be one of its *own* members. Look at the following two examples:

1. Suppose that E is the set of all cows. Since E is not a cow, it is not one of its own members.
2. Suppose that E is the set of all of those things that we could ever talk about that are not cows. Since this set E is not a cow, we see that this set E is one of its own members.

Mathematicians prefer to avoid sets that are members of themselves. They feel that if we are going to collect some objects together to make a set A, then all of those objects ought to be well known to us *before* we collect them together. Now what if one of those objects is the set A itself? We would need to know what the set A is even before we have collected its members together to define it. So mathematicians dislike the idea of sets that are members of themselves. Note, however, that this dislike is not the devastating paradox of Bertrand Russell. It is merely a warning that something nasty is going on and that a paradox may be lurking somewhere.

We shall call a set A **self-possessed** if A is one of its own members. In other words, A is self-possessed if and only if $A \in A$. Bertrand Russell's idea was to consider the set S of all of those sets A that are not self-possessed. He then asked himself a question:

Is this set S self-possessed?

He discovered that, no matter how one answers this question, the answer leads to a contradiction. In other words, not only does the assumption that S *is* self-possessed lead to a contradiction, but so does the assumption that S *isn't* self-possessed. To see how the two contradictions may be obtained one may reason as follows:

- Suppose that S is self-possessed; in other words, suppose that S belongs to S. Thus S is a self-possessed member of S. But from the definition of S we see that the members of S have to be sets that are not self-possessed, and we have arrived at a contradiction.

- Suppose that S is not self-possessed. Since every set that isn't self-possessed must belong to S, the set S must belong to S, and we conclude that S is self-possessed. Once again we have arrived at a contradiction.

It is this double contradiction that propels us into the vicious circle known as Bertrand Russell's paradox, the paradox that threw the mathematical world into an uproar in the year 1901.

4.1.2 The Zermelo-Fraenkel Axioms

The message of Russell's paradox is that it is the very existence of the set S described in Subsection 4.1.1 that forces a contradiction upon us. Russell's paradox warns us that not every collection of objects should be thought of as a *set,* especially if some of its members are already sets. For this reason, the modern theory of sets rejects Cantor's definition of a set as being simply a collection of objects, and replaces this notion by an abstract idea that is based on a system of axioms. The most common of these systems of axioms is known as the **Zermelo-Fraenkel** system.

Since the modern theory of sets is an advanced topic, we shall not present it here. We shall continue to think of a set as a collection of objects, but we shall keep at the back of our minds the warning that not every such collection should be allowed. Fortunately, the Zermelo-Fraenkel axioms provide us with all of the sets that appear in algebra, analysis, topology, and other branches of mathematics. All of the sets that we shall mention from now on in this book can be verified as legitimate in a Zermelo-Fraenkel system, and we shall therefore cease to worry about paradoxes.

4.2 Sets and Subsets

To begin this section we shall extend our language a little. As we have said, a set is a collection of objects. Any object that belongs to a given set S is said to be a **member** of S, an **element** of S, or a **point** of the set S. If x is a member of a set S, then we write $x \in S$ and say that x **belongs** to S or is **contained** in S. We also say that S **contains** x. If an object x does not belong to a set S, then we write $x \notin S$. Two sets A and B are equal when they contain precisely the same objects. The **empty set** $\emptyset$ is the set that has no members at all.[13]

4.2.1 Subsets of a Given Set

If A and B are two given sets, then we say that A is a **subset** of B or, alternatively, that A is **included** in B if every member of A is also a member of B. When A

[13] The symbol $\emptyset$ can be read aloud as "emptyset".

is a subset of B we write $A \subseteq B$, and we also write this condition as $B \supseteq A$ and say that B **includes** A. Figure 4.1 shows how we may picture the condition $A \subseteq B$. Note that two sets A and B are equal if and only if both of the conditions $A \subseteq B$ and $B \subseteq A$ hold. In the event that $A \subseteq B$ and the sets A and B are not equal, we say that A is **properly included** in B and that A is a **proper subset** of B.

If A and B are any two sets, then the denial of the condition $A \subseteq B$ says that there exists $x \in A$ such that $x \notin B$. In the event that $A = \emptyset$, we certainly can't find a member $x \in A$ such that $x \notin B$ because there are no members of A to find. We therefore conclude that $\emptyset \subseteq B$ for every set B.

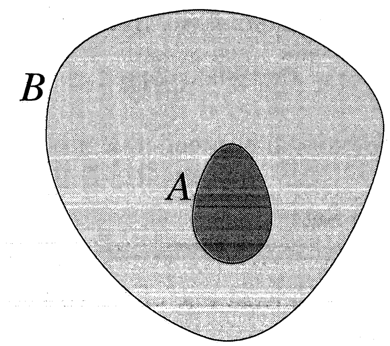


Figure 4.1

4.2.2 Equality of Sets

Two sets A and B are equal when they contain precisely the same members. Thus if A and B are two given sets, then the condition $A = B$ says that every member of A must belong to B and every member of B must belong to A. In other words, $A = B$ is equivalent to the condition that $A \subseteq B$ and also $B \subseteq A$.

4.2.3 The Power Set of a Given Set

Given any set A, the family of all subsets of A is called the power set of A and is written as $p(A)$. For a simple example, observe that

$$p(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

4.2.4 Set Builder Notation

We have already used notation such as $\{2, 7, -3\}$, which stands for the set whose members are the numbers 2, 7, and -3. This is a form of what is known as **set builder notation**. There is also another form of set builder notation. Suppose that $P(x)$ is a statement that contains a single unknown x. The notation $\{x \mid P(x)\}$

stands for the set of all of those objects x for which the statement $P(x)$ is true. This notation is particularly useful to describe subsets of a given set. The following examples illustrate this notation:

1. Writing the set of all real numbers[14] as $\mathbf{R}$ and taking $P(x)$ to be the condition $-3 \leq x < 2$, we have
$$\{x \in \mathbf{R} \mid P(x)\} = \{x \in \mathbf{R} \mid -3 \leq x < 2\} = [-3, 2).$$
2. Taking $P(x)$ to be the condition $8x^3 - 6x - 1 \geq 0$, we have
$$\{x \in \mathbf{R} \mid P(x)\} = \{x \in \mathbf{R} \mid 8x^3 - 6x - 1 \geq 0\}.$$
3. In this example we refer to the set $\mathbf{Q}$ of all rational numbers. The set
$$\{x \in \mathbf{Q} \mid x < 0 \text{ or } x^2 < 2\}$$
is the set of all of those rational numbers that are less than $\sqrt{2}$.

4.2.5 The Union of Two Sets

Suppose that A and B are given sets. The symbol $A \cup B$ stands for the set of all of those objects that belong to at least one of the two sets A and B and is called the **union** of A and B. Thus
$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

4.2.6 The Intersection of Two Sets

Suppose that A and B are given sets. The symbol $A \cap B$ stands for the set of all of those objects that belong to *both* of the sets A and B and is called the **intersection** of the sets A and B. Thus
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$
In the event that $A \cap B = \emptyset$, we say that the sets A and B are **disjoint** from each other.

4.2.7 The Difference of Two Sets

Suppose that A and B are given sets. The **difference** $A \setminus B$ of the sets is defined by
$$A \setminus B = \{x \in A \mid x \notin B\}.$$

[14] In this chapter we draw freely from the set $\mathbf{R}$ of real numbers, the set $\mathbf{Q}$ of rational numbers, the set $\mathbf{Z}$ of integers, and the set $\mathbf{Z}^+$ of natural numbers for our examples. A more careful presentation of these sets can be found in the chapters that follow.

Another way of looking at $A \setminus B$ is to say that
$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}.$$

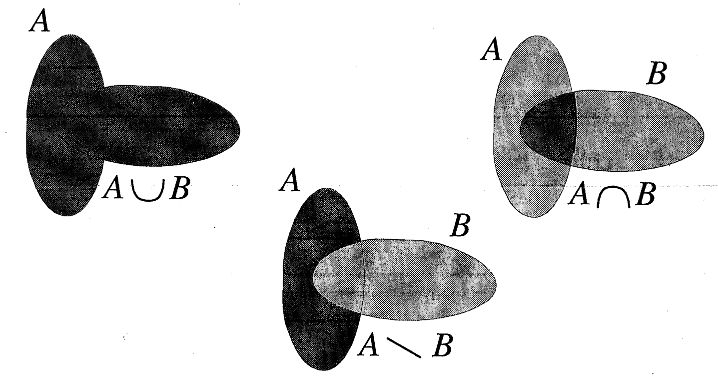


Figure 4.2

Figure 4.2 shows how we may picture the union, intersection and difference of two sets A and B.

4.2.8 The Cartesian Product of Two Sets

If A and B are any two sets, then their **Cartesian product** $A \times B$ is the set of all ordered pairs (x, y) that have $x \in A$ and $y \in B$. In other words,
$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}.$$

4.2.9 Some Common Sets

1. The set of all real numbers is usually written as $\mathbf{R}$ or $\mathbb{R}$.
2. The set of all rational numbers is usually written as $\mathbf{Q}$ or $\mathbb{Q}$.
3. The set of all integers is usually written as $\mathbf{Z}$ or $\mathbb{Z}$.
4. The set of all positive integers is usually written as $\mathbf{Z}^+$, or $\mathbb{Z}^+$, or $\mathbf{N}$, or $\mathbb{N}$.
5. If a and b are real numbers and $a \leq b$, then we shall use the standard interval notation
$$\begin{aligned} [a,b] &= \{x \in \mathbf{R} \mid a \leq x \leq b\} \\ [a,b) &= \{x \in \mathbf{R} \mid a \leq x < b\} \\ (a,b] &= \{x \in \mathbf{R} \mid a < x \leq b\} \\ (a,b) &= \{x \in \mathbf{R} \mid a < x < b\}. \end{aligned}$$

As you may know, an interval of the form $[a, b]$ is called a **closed interval** and an interval of the form (a, b) is called an **open interval.** We can also define intervals of infinite length: If a is any real number, then we define

$$\begin{aligned}
[a, \infty) &= \{x \in \mathbf{R} \mid a \leq x\}\\
(a, \infty) &= \{x \in \mathbf{R} \mid a < x\}\\
(-\infty, a] &= \{x \in \mathbf{R} \mid x \leq a\}\\
(-\infty, a) &= \{x \in \mathbf{R} \mid x < a\},
\end{aligned}$$

and finally, the symbol $(-\infty, \infty)$ is the set $\mathbf{R}$ of all real numbers. Note that if a is any real number then the interval (a, a) is the empty set $\emptyset$.

6. The set $\mathbf{R} \times \mathbf{R}$ of all ordered pairs of real numbers is called the **Euclidean plane** and is also written as $\mathbf{R}^2$.

4.2.10 Exercises on Set Notation

1. Given objects a,b, and y and given that $\{a, b\} = \{a, y\}$, prove that $b = y$.
2. Prove that if a, b, x, and y are any given objects and if $\{a, b\} = \{x, y\}$, then either $a = x$ and $b = y$, or $a = y$ and $b = x$.
3. Prove that if a, b, x, and y are any given objects and if
$$\{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\},$$
then $a = x$ and $b = y$.
4. Describe the set $p(\emptyset)$.
5. Describe the set $p(p(\emptyset))$.
6. Given that $A = \{a, b, c, d\}$, list all of the members of the set $p(A)$.
7. Given that $A = \{a, b\}$, list all of the members of the set $p(p(A))$.
8. Use the Evaluate operation in *Scientific Notebook* to evaluate the sets $\{1, 2, 3\} \cap \{2, 3, 4\}$ and $\{1, 2, 3\} \cup \{2, 3, 4\}$.

4.2.11 The DeMorgan Laws

Suppose that A, B, and C are given sets. The following simple identities are known as the **DeMorgan laws:**

1. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
2. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
3. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.
4. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

Proof. We shall prove law 2 and leave the proofs of the other three laws as

exercises. The proof that we are about to give is highly detailed. When you prove the other three laws, write your proofs first in as much detail as is given here. Then rewrite them more briefly.

To prove law 2 we shall begin by proving that

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C).$$

We want to prove that every member of the left side must belong to the right side. Since we want to prove a "for every" statement, we begin our proof with the words

Suppose that $x \in A \cap (B \cup C)$.

We know that $x \in A$ and also that $x \in B \cup C$. In other words, we know that $x \in A$ and either $x \in B$ or $x \in C$. Thus either $x \in A$ and $x \in B$ or otherwise $x \in A$ and $x \in C$. Since the first of these two possibilities says that $x \in A \cap B$ and the second possibility says that $x \in A \cap C$, we therefore know that x belongs to at least one of the two sets $A \cap B$ and $A \cap C$; in other words,

$$x \in (A \cap B) \cup (A \cap C).$$

We have therefore shown that

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C),$$

and we shall now complete the proof of law 2 by showing that

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C).$$

We want to prove that every member of the left side must belong to the right side. Since we want to prove a "for every" statement, we begin our proof with the words

Suppose that $x \in (A \cap B) \cup (A \cap C)$.

We know that x must belong to at least one of the sets $A \cap B$ and $A \cap C$. Since the given information says that one or the other of two conditions must hold, we proceed according to the method described in Subsection 3.3.3. We have two tasks to perform:

(a) We need to show that if $x \in A \cap B$, then $x \in A \cap (B \cup C)$.
(b) We need to show that if $x \in A \cap C$, then $x \in A \cap (B \cup C)$.

To perform the first of these two tasks we suppose that $x \in A \cap B$. Since $x \in A$ and $x \in B$, we know that $x \in A$ and that x belongs to at least one of the sets B and C. So in this case we certainly have $x \in A \cap (B \cup C)$.

To perform the second of these two tasks we suppose that $x \in A \cap C$. Since $x \in A$ and $x \in C$, we know that $x \in A$ and that x belongs to at least one of the sets B and C. So in this case we certainly have $x \in A \cap (B \cup C)$.

4.2.12 Exercises on Set Operations

1. Express the set $[-2, 3] \setminus (0, 1]$ as the union of two intervals.
2. Given two sets A and B, prove that the condition $A \subseteq B$ is equivalent to the condition $A \cup B = B$.
3. Given two sets A and B, prove that the condition $A \subseteq B$ is equivalent to the condition $A \cap B = A$.
4. Given two sets A and B, prove that the condition $A \subseteq B$ is equivalent to the condition $A \setminus B = \emptyset$.
5. Illustrate the identity

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$

by drawing a figure. Then write out a detailed proof.

6. Illustrate the identity

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$

by drawing a figure. Then write out a detailed proof.

7. Given that A, B, and C are subsets of a set X, prove that the condition $A \cap B \cap C = \emptyset$ holds if and only if

$$(X \setminus A) \cup (X \setminus B) \cup (X \setminus C) = X.$$

8. Given sets A, B, and C, determine which of the following identities are true.

(a) $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.
(b) $A \cup (B \setminus C) = (A \cup B) \setminus (A \cup C)$.
(c) $A \cup (B \setminus C) = (A \cup B) \cap (A \setminus C)$.
(d) $A \cup (B \setminus C) = (A \cup B) \setminus (A \cap C)$.
(e) $A \setminus (B \setminus C) = (A \setminus B) \setminus C$.
(f) $A \setminus (B \setminus C) = (A \setminus B) \setminus (A \setminus C)$.
(g) $A \setminus (B \setminus C) = (A \setminus B) \cap (A \setminus C)$.
(h) $A \setminus (B \setminus C) = (A \setminus B) \cup (A \setminus C)$.
(i) $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$.
(j) $A \setminus (B \setminus C) = (A \setminus B) \cap (A \cup C)$.
(k) $A = (A \cap B) \cup (A \setminus B)$.
(l) $p(A \cup B) = p(A) \cup p(B)$.

(m) $p(A \cap B) = p(A) \cap p(B)$.
(n) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
(o) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
(p) $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.

9. Is it true that if A and B are sets and $A = A \setminus B$, then the sets A and B are disjoint from each other?
10. Given that A is a set with 10 members, B is a set with 7 members, and that the set $A \cap B$ has 4 members, how many members does the set $A \cup B$ have?
11. Give an example of a set A that contains at least three members and that satisfies the condition $A \subseteq p(A)$.
12. For which sets A do we have $A \in p(A)$?

4.3 Functions

4.3.1 Intuitive Definition of a Function

In this brief presentation of set theory we shall be content with an intuitive view of a function f defined on a set A as a rule that assigns to each member x of the set A a unique object written as $f(x)$ and called the **value** of the function f at x.

If f is a function with domain A and if $f(x) \in B$ for every $x \in A$, then we say that f is a function **from** A **to** B and we write $f : A \to B$. When f is a function from A to B we also say that f **maps** A to B. Figure 4.3 shows how we may picture this idea.

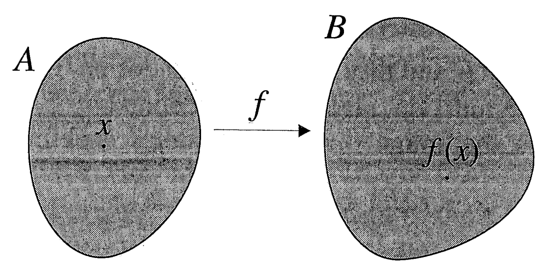


Figure 4.3

4.3.2 Domain of a Function

If f is a function defined on a set A, then the set A is said to be the **domain** of the function f. Thus the domain of a function f is the set of all objects x for which the symbol $f(x)$ is defined.

4.3.3 Range of a Function

If f is a function defined on a set A, then the set

$$\{f(x) \mid x \in A\}$$

is called the **range** of the function f. Note that if $f : A \to B$, then the range of f has to be a subset of the set B.

4.3.4 Some Examples of Functions

1. Suppose that we have defined

$$f(x) = x^2$$

for every number x. Then f is a function defined on the set $\mathbf{R}$ of all real numbers. The set $\mathbf{R}$ is the domain of f and the range of f is $[0, \infty)$.

2. Suppose that we have defined

$$f(x) = x^2$$

for every number $x \in [-1, 3]$. Then f is a function defined on $[-1, 3]$ and this interval is the domain of f. The range of f is the interval $[0, 9]$.

3. Suppose that C is a bag of jelly beans, any one of which can be yellow, green, red, or blue. Suppose that for every $x \in C$, the symbol $f(x)$ is defined to be the color of the jelly bean x. Then f is a function defined on C and the range of f is the set that contains the four words, yellow, green, red, and blue.

4.3.5 Restriction of a Function to a Set

Suppose that $f : A \to B$ and that E is any subset of A. The **restriction** of f to E is defined to be the function g from E to B defined by $g(x) = f(x)$ for every $x \in E$. Furthermore, the **image** $f[E]$ **of** E **under** f is defined by

$$f[E] = \{f(x) \mid x \in E\}.$$

Figure 4.4 shows how we may picture this idea. Note that if A is the domain of a function f, then the range of f is the set $f[A]$.

4.3.6 Preimage of A Set

Suppose that $f : A \to B$. If E is any subset of B, then the **preimage of** E **under** f is the set $f^{-1}[E]$ defined by

$$f^{-1}[E] = \{x \in A \mid f(x) \in E\}.$$

Figure 4.5 illustrates the idea of a preimage.

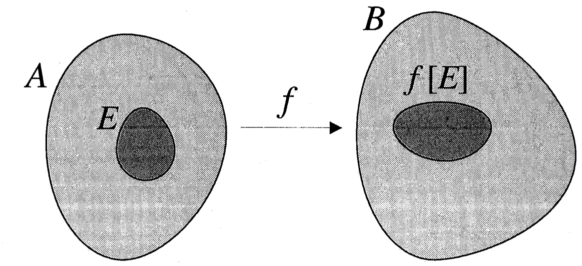


Figure 4.4

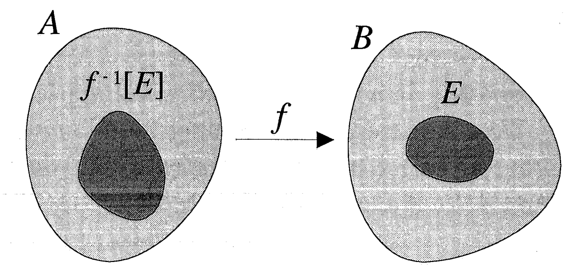


Figure 4.5

Note that the idea of a preimage does not depend on the idea of an inverse function f^{-1}, which will be defined in Subsection 4.3.12.

4.3.7 One-One Functions

A given function f is said to be **one-one** if it is impossible to find two different members x_1 and x_2 in the domain of f for which $f(x_1) = f(x_2)$. Another way of saying that a given function f is one-one is to say that whenever x_1 and x_2 belong to the domain of f and $x_1 \neq x_2$ we must have $f(x_1) \neq f(x_2)$. One-one functions are sometimes called **injective.**

Since no two different real numbers can have the same cube, we see that if $f(x) = x^3$ for every $x \in \mathbf{R}$, then the function f is one-one. However, if $f(x) = x^2$ for every $x \in \mathbf{R}$, then f is not one-one because $f(2) = f(-2)$. If f is a function from a set of real numbers into $\mathbf{R}$, then we can picture the condition that f be one-one as saying that no horizontal line can meet the graph of f more than once. Figure 4.6 illustrates the graph of a one-one function. Notice how some horizontal lines meet the graph of this function exactly once while others don't meet it at all.

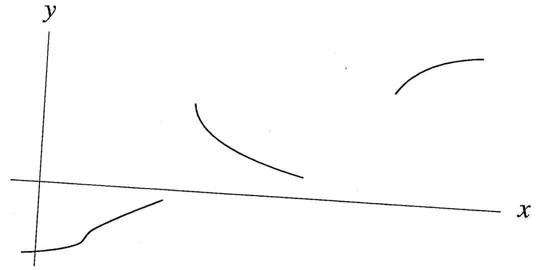


Figure 4.6

4.3.8 Monotone Functions

Suppose that S is a set of real numbers and that $f : S \to \mathbf{R}$. We say that the function f is **increasing** if whenever t and x belong to S and $t < x$ we have $f(t) \le f(x)$. We can picture an increasing function as one whose graph never falls as we move from left to right. The graph of this type of function is illustrated in Figure 4.7. If the graph of f actually rises as we move from left to right, then we say that the function f is **strictly increasing.** Thus the function f is strictly increasing if whenever t and x belong to S and $t < x$ we have $f(t) < f(x)$. Decreasing functions and strictly decreasing functions are defined similarly. A function that is either increasing or decreasing is said to be **monotone**, and a function that is either strictly increasing or strictly decreasing is said to be **strictly monotone.**

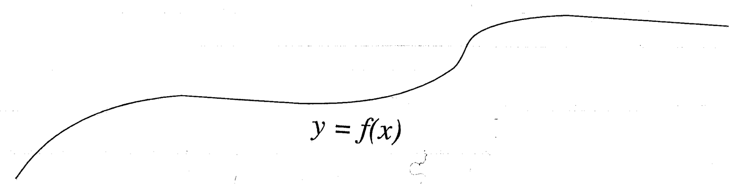


Figure 4.7

We see at once that a strictly monotone function is always one-one. However, a one-one function does not have to be strictly monotone. Figure 4.6 illustrates the graph of a one-one function that fails to be monotone.

4.3.9 Functions onto a Given Set

Suppose that f is a function from a set A to a set B. The range $f[A]$ of f is a subset of B. In the event that the range of f is the entire set B, we say that the function f is **onto** the set B. In other words, f is onto the set B if $f[A] = B$.

Another way of saying this is to say that for every member y of B there is at least one member x of A such that $y = f(x)$.

Suppose, for example, that $f(x) = x^2$ for every $x \in \mathbf{R}$. Although $f : \mathbf{R} \to \mathbf{R}$, the function f is not onto the set $\mathbf{R}$. On the other hand, f is onto the interval $[0, \infty)$.

4.3.10 A Remark About Terminology

In Subsection 4.3.7 we mentioned that a one-one function is sometimes called an *injective function.* When we say that a given function is injective we are saying that the function is of a certain special type. In other words, the word *injective* is an adjective. We can say that a particular function is injective in exactly the same way that we can say that a leaf is green.

However, unlike the concept of a one-one function, the concept that we studied in Subsection 4.3.9 does not describe a property of functions. When we say that a given function f is onto a given set B we are not only talking about the function f; we are talking about the function f and the set B. For example, if $f(x) = x^2$ for every real number x, then f fails to be onto the set $\mathbf{R}$ of real numbers even though it is onto the set $[0, \infty)$.

We can ask whether a given function f is onto a given set B but we can never ask whether or not a given function is *onto;* because such a question makes no sense. It makes no more sense than it would make if you were to ask me whether I am sitting *on.* I would have to answer such a question by asking: "Sitting on *what*?" The point of this remark is that the word *onto* is not an adjective. It is a preposition and we need to keep in mind that one of the fundamental rules of grammar prohibits the use of a preposition at the end of a sentence.

In some mathematical writing, the word *surjective* is used to describe the fact that a given function is "onto" some unspecified set. This word *surjective* is misleading because it looks like an adjective even though it is not one, and it will not be used in this book.

4.3.11 Composition of Functions

If f is a function from a set A to a set B and g is a function from B to a set C, then the **composition** $g \circ f$ of f and g is the function from A to C defined by

$$(g \circ f)(x) = g(f(x))$$

for every $x \in A$. Figure 4.8 illustrates this idea. For example, if $f(x) = x^2$ for every $x \in \mathbf{R}$ and $g(x) = 1+2x$ for every $x \in [0, \infty)$, then $(g \circ f)(x) = 1+2x^2$ for every $x \in \mathbf{R}$ and $(f \circ g)(x) = (1+2x)^2$ for every $x \in [0, \infty)$.

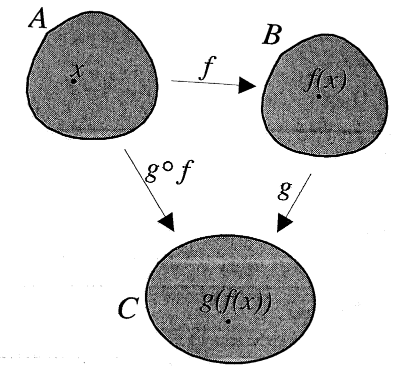


Figure 4.8

4.3.12 Inverse Functions

Suppose that f is a one-one function from a set A onto a set B. Given any member y of B there is one and only one member x of A such that $y = f(x)$. If for each member y of the set B we define $g(y)$ to be the one and only one member x of A for which $y = f(x)$, then we have defined a function $g : B \to A$. This function g is called the **inverse function** of f and is written as f^{-1}.

Note that $f^{-1}(f(x)) = x$ for all $x \in A$ and $f(f^{-1}(y)) = y$ for all $y \in B$.

Note finally that if E is any subset of B, then the expression $f^{-1}[E]$ could be interpreted both as the preimage of the set E under the function f and also as the image under the function f^{-1} of the set E. Fortunately, these two interpretations yield the same set.

4.3.13 Inverse Functions Are also One-One

Suppose that f is a one-one function from a set A onto a set B. Then the function f^{-1} is also one-one. Moreover, $(f^{-1})^{-1} = f$.

Proof. To show that f^{-1} is one-one we need to show that if y_1 and y_2 belong to B and $f^{-1}(y_1) = f^{-1}(y_2)$, then $y_1 = y_2$. Suppose that y_1 and y_2 belong to B and $f^{-1}(y_1) = f^{-1}(y_2)$. Then we have

$$y_1 = f\left(f^{-1}(y_1)\right) = f\left(f^{-1}(y_2)\right) = y_2.$$

Finally, given $x \in A$, the symbol $(f^{-1})^{-1}(x)$ stands for the member of B that f^{-1} sends to x. But since $f^{-1}(f(x)) = x$ we know that this member of B is $f(x)$. Therefore $f(x) = (f^{-1})^{-1}(x)$ and we conclude that $(f^{-1})^{-1} = f$. ■

4.3.14 Some Examples of Inverse Functions

1. If we define $f(x) = x^2$ for every $x \in [0, \infty)$, then f is a one-one function from $[0, \infty)$ onto $[0, \infty)$ and for every $y \in [0, \infty)$ we have $f^{-1}(y) = \sqrt{y}$.
2. If we define $f(x) = \tan x$ for every $x \in (-\pi/2, \pi/2)$, then f is a one-one function from $(-\pi/2, \pi/2)$ onto $\mathbf{R}$ and for every real number y we have $f^{-1}(y) = \arctan y$.
3. Suppose that $a \in \mathbf{R} \setminus \{-1, 1\}$. We begin with the observation that if x is any number unequal to $1/a$ and $y \neq -1/a$, then the equations

$$y = \frac{x - a}{1 - ax}$$

and

$$x = \frac{y + a}{1 + ay}.$$

are equivalent statements. On the other hand, it is easy to see that if either $x = 1/a$ or $y = -1/a$, then both of these equations are impossible. From this observation we deduce that the function f defined by

$$f(x) = \frac{x - a}{1 - ax}$$

for all $x \in \mathbf{R} \setminus \{1/a\}$ is a one-one function from $\mathbf{R} \setminus \{1/a\}$ onto the set $\mathbf{R} \setminus \{-1/a\}$. Furthermore, for every $y \in \mathbf{R} \setminus \{-1/a\}$, we have

$$f^{-1}(y) = \frac{y + a}{1 + ay}.$$

4. This example makes use of a little elementary calculus. Given any number a satisfying $-1 < a < 1$, we define the function f_a on the interval $[-1, 1]$ by defining

$$f_a(x) = \frac{x - a}{1 - ax}$$

for every number $x \in [-1, 1]$. From the observations that we made in Example 3 we know that each of these functions f_a is one-one. Since these functions are rational functions, they are also continuous on the interval $[-1, 1]$.

Now suppose that $-1 < a < 1$. Because f_a is one-one and because $f_a(-1) = -1$ and $f_a(1) = 1$, the number $f_a(x)$ cannot be equal to -1 or 1 for any x in the open interval $(-1, 1)$. It therefore follows from the elementary properties of continuous functions that either $f_a(x) < -1$ for every $x \in (-1, 1)$, or $-1 < f_a(x) < 1$ for every $x \in (-1, 1)$, or $f_a(x) > 1$

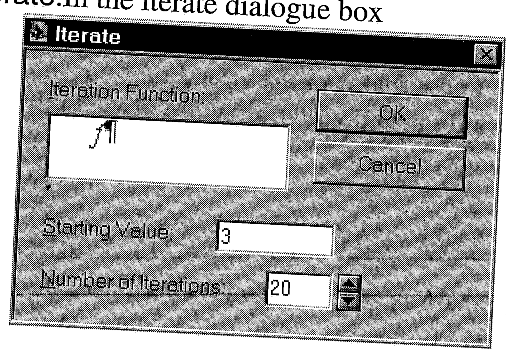
for every $x \in (-1, 1)$. But $f_a(0) = -a \in (-1, 1)$, and so we conclude that

$$f_a : [-1, 1] \to [-1, 1].$$

Finally, to see that f_a is onto the interval $[-1, 1]$ we observe that if $y \in [-1, 1]$, then $f_{-a}(y) \in [-1, 1]$ and it is easy to see that $y = f_a(f_{-a}(y))$. We deduce that for each a, the function f_a is one-one from $[-1, 1]$ onto $[-1, 1]$ and that f_{-a} is the inverse function of f_a.

4.3.15 Exercises on Functions

1. Given that $f(x) = x^2$ for every real number x, simplify the following expressions:
   (a) $f[[0, 3]]$.
   (b) $f[(-2, 3]]$.
   (c) $f^{-1}[[-3, 4]]$.
2. Point at the equation $f(x) = x^2$ and then click on the button in your computing toolbar. Then work out the expressions in parts (a) and (b) of Exercise 1 by pointing at them and clicking on the evaluate button.
3. Supply each of the definitions $f(x) = x^2$ and $g(x) = 2 - 3x$ to *Scientific Notebook* and then ask *Scientific Notebook* to solve the equation
$$(f \circ g)(x) = (g \circ f)(x).$$
4. Supply the definition
$$f(x) = \frac{x-2}{1-2x}$$
to *Scientific Notebook*. In this exercise we shall see how to evaluate the composition of the function f with itself up to 20 times starting at a variety of numbers. Open the Compute menu, click on Calculus, and move to the right and select Iterate.In the iterate dialogue box



fill in the function as f, the starting value as 3, and the number of iterations as 20. Repeat this process with different starting values. Can you draw a conclusion from what you see?

5. Given that $f(x) = x^2$ for all $x \in \mathbf{R}$ and $g(x) = 1 + x$ for all $x \in \mathbf{R}$, simplify the following expressions:
   (a) $(f \circ g)[[0, 1]]$.
   (b) $(g \circ f)[[0, 1]]$.
   (c) $(g \circ g)[[0, 1]]$.
6. (a) Given that $f(x) = (3x - 2) / (x + 1)$ for all $x \in \mathbf{R} \setminus \{-1\}$, determine whether or not f is one-one and find its range.
   (b) Point at the equation
$$y = \frac{3x-2}{x+1}$$
   and ask *Scientific Notebook* to solve for x. How many values of x are given? Is this result consistent with the answer that you gave in part (a) of the question?
7. Suppose that $f : A \to B$ and that $E \subseteq A$. Is it true that $E = f^{-1}[f[E]]$? What if f is one-one? What if f is onto B?
8. Suppose that $f : A \to B$ and that $E \subseteq B$. Is it true that $E = f[f^{-1}[E]]$? What if f is one-one? What if f is onto B?
9. Suppose that $f : A \to B$ and that P and Q are subsets of B. Prove the identities
$$f^{-1}[P \cup Q] = f^{-1}[P] \cup f^{-1}[Q]$$
$$f^{-1}[P \cap Q] = f^{-1}[P] \cap f^{-1}[Q]$$
$$f^{-1}[P \setminus Q] = f^{-1}[P] \setminus f^{-1}[Q].$$
10. Suppose that $f : A \to B$ and that P and Q are subsets of A. Which of the following statements are true? What if f is one-one? What if f is onto B?
$$f[P \cup Q] = f[P] \cup f[Q].$$
$$f[P \cap Q] = f[P] \cap f[Q].$$
$$f[P \setminus Q] = f[P] \setminus f[Q].$$

11. **H** Given that f is a one-one function from A to B and that g is a one-one function from B to C, prove that the function $g \circ f$ is one-one from A to C.
12. Given that f is a function from A onto B and that g is a function from B onto C, prove that the function $g \circ f$ is a function from A onto C.
13. Given that $f : A \to B$, that $g : B \to C$, and that the function $g \circ f$ is one-one, prove that f must be one-one. Give an example to show that the function g does not have to be one-one.
14. **H** Given that f is a function from A onto B, that $g : B \to C$, and that the function $g \circ f$ is one-one, prove that both of the functions f and g have to be one-one.
15. Given any set S, the **identity function** i_S on S is defined by $i_S(x) = x$ for every $x \in S$. Prove that if f is a one-one function from a set A onto a set B, then $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.
16. Suppose that $f : A \to B$.
    (a) Given that there exists a function $g : B \to A$ such that $g \circ f = i_A$, what can be said about the functions f and g?
    (b) Given that there exists a function $h : B \to A$ such that $f \circ h = i_B$, what can be said about the functions f and h?
    (c) Given that there exists a function $g : B \to A$ such that $g \circ f = i_A$ and that there exists a function $h : B \to A$ such that $f \circ h = i_B$, what can be said about the functions f, g, and h?
17. As in Example 4 of Subsection 4.3.14, we define

$$f_a(x) = \frac{x - a}{1 - ax}$$

whenever $a \in (-1, 1)$ and $x \in [-1, 1]$.
    (a) Prove that if a and b belong to $(-1, 1)$, then so does the number

$$c = \frac{a + b}{1 + ab}.$$

Hint: A quick way to do this exercise is to observe that $c = f_{-b}(a)$.
    (b) Given a and b in $(-1, 1)$ and

$$c = \frac{a + b}{1 + ab},$$

prove that $f_b \circ f_a = f_c$.

PART II

Elementary Concepts of Analysis

This part of the text introduces the basic principles of analysis and provides a careful introduction to the concepts of limit, continuity, derivative, integral, and infinite series. We begin with a chapter on the real number system upon which all of these concepts depend.