Theorem 0.1 *(Intermediate Value) Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) \leq c \leq f(b)$. Then there is an $x^* \in [a, b]$ such that $f(x^*) = c$.*

A function $f : [a, b] \to \mathbb{R}$ is *strictly monotone increasing* if for all $a \leq x < y \leq b$, $f(x) < f(y)$.

Theorem 0.2 *(Inverse function) Let $f : [a, b] \to \mathbb{R}$ be a continuous strictly monotone increasing function. Then $f([a, b]) = [f(a), f(b)]$ and the inverse function $f^{-1} : f([a, b]) \to [a, b]$ is continuous.*

Problem 1. Prove that $f(x) = x^3 - 15x + 1$ has three real roots. (Hint: Look at $f(-1)$ and $f(1)$.)

Problem 2. True or false? If a real valued function $f(x)$ has a point x_0, where its derivative $f'(x_0) = 0$, then $f(x)$ cannot have an inverse. (Prove it or find a counterexample.)

Problem 3. Prove that a continuous strictly monotone function $f : \mathbb{R} \to \mathbb{R}$ has the property that for every open set $U \subset \mathbb{R}$, $f(U)$ is open.

Problem 4. Construct a set starting with a trapezoid T_1 as in Figure 1 on the left. The base is twice the length of each of the top three sides, and T_1 can be subdivided into four other congruent trapezoids. Remove the bottom trapezoid to obtain T_2, which is the union of the three remaining smaller trapezoids. Then iterate the construction with each of the new smaller trapezoids. Let T_∞ be the intersection of all of these T_n. What is the fractal dimension of T_∞?

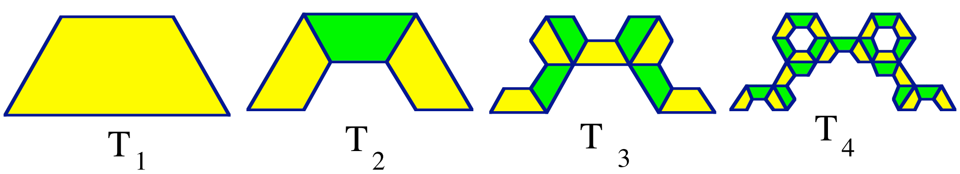

Figure 1: For each colored trapezoid of T_n remove the trapezoid on its base to get T_{n+1}.