MATH 304 **Notes and Problems** **Due April 22, 2008**

Recall from Solow that $N_\epsilon(x_0) = \{x \in \mathbb{R} \mid |x - x_0| < \epsilon\}$ is called an *ϵ-neighborhood* of $x_0 \in \mathbb{R}$.

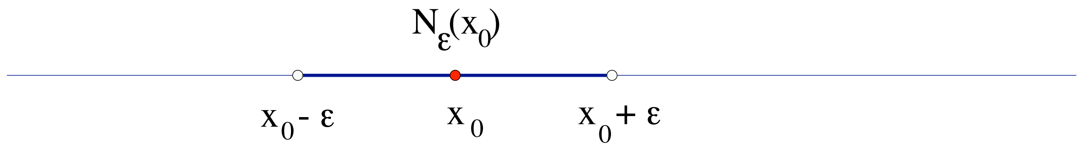

Figure 1: An example of a neighborhood of x_0 in $\mathbb{R}$.

A set $U \subset \mathbb{R}$ is called an *open set* if for every $x_0 \in U$, there is a neighborhood of x_0, $N_\epsilon(x_0) \subset U$. If a set $X \subset \mathbb{R}$, an $A \subset X$, we say that A is *relatively open in* X if there is an open set $U \subset \mathbb{R}$ such that $A = U \cap X$. For example, the interval $A = [0, 1/2)$ is relatively open in the interval $X = [0, 1]$ because $A = [0, 1/2) = (-1/2, 1/2) \cap [0, 1) = U \cap X$, where $U = (-1/2, 1/2)$ is open in $\mathbb{R}$.

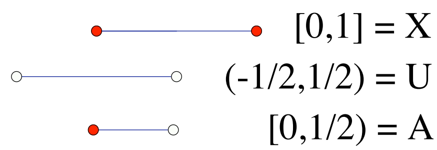

Figure 2: Intervals showing a relatively open set.

For any function $f : X \to Y$, $A \subset X$, and $B \subset Y$, we define $f(A) = \{f(a) \mid a \in A\}$, and $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$.

In class we showed:

Theorem 0.1 *A function $f : [a, b] \to \mathbb{R}$ is continuous if and only if for every open set $U \subset \mathbb{R}$, $f^{-1}(U)$ is relatively open in the interval $[a, b]$.*

Problem 1. Prove or find a counterexample to the statement: For any function $f : X \to Y$, $A \subset X$, and $B \subset Y$, $f(A) = B$ if and only $f^{-1}(B) = A$.

Problem 2. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \sin(\pi/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Calculate $f^{-1}(\mathbb{R} - \{0\})$.

Problem 3. Prove that the set $[0, 1] \cap (\mathbb{R} - \{x \in \mathbb{R} \mid x = 1/n,\ n \geq 1,\ n \in \mathbb{Z}\} \cup \{0\})$ is relatively open in in the interval $[0, 1]$. (Hint: You can use that the function in Problem 2 is continuous, if you like.)

Problem for the Quiz on Tuesday, April 22 (To be written in class): Describe all the relatively open subsets of the set $\{x \in \mathbb{R} \mid x = 1/n,\ n \geq 1,\ n \in \mathbb{Z}\} \cup \{0\}$.