EINSTEIN FOUR-MANIFOLDS OF THREE-NONNEGATIVE CURVATURE OPERATOR

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Abstract. We prove that Einstein four-manifolds with three-nonnegative curvature operator are either flat, isometric to $(S^4,\ g_0)$, $(\mathbb{C}P^2,\ g_{FS})$, $(S^2 \times S^2, g_0 \oplus g_0)$, or their quotients by finite groups of fixed point free isometries, up to rescaling. We also prove that Einstein four-manifolds with four-nonnegative curvature operator and positive intersection form are isometric to $(\mathbb{C}P^2,\ g_{FS})$ up to rescaling.

1. Introduction

A Riemannian manifold $(M^n,\ g)$ is Einstein if its Ricci curvature is a multiple of the metric, i.e.,

$$\mathrm{Ric}_g = \lambda g,$$

for some constant λ. The study of Einstein manifolds are important in both differential geometry and (mathematical) physics. In dimensions two and three, Einstein manifolds have been completely classified, they are all space forms. However, it becomes much more complicated in dimension four and higher.

On one hand, several topological obstructions have been discovered for Einstein four-manifolds of positive or nonnegative sectional curvature. It was proved by M. Berger [2] that the Euler characteristic χ of Einstein four-manifolds with nonnegative sectional curvature and positive scalar curvature are bounded by $1 \le \chi(M) \le 9$; N. Hitchin [11] proved that χ and the signature τ must satisfy $|\tau(M)| \le (\frac{2}{3})^{3/2}\chi(M)$; M. Gursky and C. LeBrun [10] further improved Hitchin's result to $|\tau(M)| \le \frac{4}{15}\chi(M)$, provided that (M, g) is not half conformally flat.

On the other hand, there have also been several classification results. Hitchin's classical theorem (see [3, Theorem 13.30]) states that half conformally-flat Einstein four-manifolds with positive scalar curvature are isometric to either $(S^4,\ g_0)$ or $(\mathbb{C}P^2,\ g_{FS})$ up to rescaling; Gursky and LeBrun [10] proved that compact Einstein four-manifolds of nonnegative sectional curvature and positive intersection form are isometric to $(\mathbb{C}P^2,\ g_{FS})$ up to rescaling; D. Yang [16] proved that oriented Einstein four-manifolds with $\mathrm{Ric} = g$ and sectional curvature $K \ge (\sqrt{1249} - 23)/120$ are isometric to $(S^4,\ g_0)$ or $(\mathbb{C}P^2,\ g_{FS})$ up to rescaling; É. Costa [8] later relaxed Yang's condition to $K \ge (2 - \sqrt{2})/6$. Moreover, S. Tachibana [14] proved that Einstein

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manifolds with positive curvature operator are isometric to space forms; S. Brendle [6] recently proved that Einstein manifolds with positive isotropic curvature are isometric to space forms, and with nonnegative isotropic curvature are locally symmetric.

Inspired by the celebrated work [4] of C. Böhm and B. Wilking on the classification of Riemannian manifolds with 2-positive curvature operator, in his Ph. D. thesis [15], the second author studied the relations between k-positive curvature operators, positive isotropic curvature, and sectional curvature on Einstein four-manifolds using Berger curvature decomposition [1]. Here by k-positive (k-nonnegative), we mean that the sum of the lowest k eigenvalues of the curvature operator is positive (nonnegative). More precisely, assuming $\mathrm{Ric} = g$, it was shown that 2-positive curvature operator is equivalent to positive isotropic curvature; sectional curvature $K > \frac{1}{12}$ implies 3-positive curvature operator; 3-positive curvature operator implies $K > \frac{1}{30}$; and 4-positive curvature operator is equivalent to sectional curvature bounded above by 1. Similar results hold for 3-nonnegative, 4-nonnegative curvature operators, as well as nonnegative isotropic curvature, see Proposition 2.4 for details.

In this paper, we classify Einstein four-manifolds with 3-nonnegative curvature operator, which improves the rigidity results of Yang [16], Costa [8], and Brendle [6] in dimension 4. This also answers a question in [15].

Theorem 1.1. *Let (M, g) be an Einstein four-manifold with 3-nonnegative curvature operator, then it is either flat, isometric to $(S^4,\ g_0)$, $(\mathbb{C}P^2,\ g_{FS})$, $(S^2 \times S^2, g_0 \oplus g_0)$, or their quotients by finite groups of fixed point free isometries, up to rescaling.*

For Einstein four-manifolds with 4-nonnegative curvature operator, we obtain a generalization of [10, Theorem A] by Gursky and LeBrun.

Theorem 1.2. *Let (M, g) be an Einstein four-manifold with 4-nonnegative curvature operator and positive intersection form, then it is isometric to $(\mathbb{C}P^2,\ g_{FS})$ up to rescaling.*

The proof of Theorem 1.1 is an improvement of Yang's method in [16] plus more delicate analysis on $W^{\pm}$, where $W^{\pm}$ are self-dual and anti-self-dual part of Weyl tensor. The proof of Theorem 1.2 follows from Gursky and LeBrun's argument [10] and a point-wise estimate for $|W^+|^2 + |W^-|^2$.

The rest of this paper is organized as follows. In Section 2, we describe two curvature decompositions for Einstein four-manifolds, the duality decomposition and Berger's decomposition, and the relation between 3-positive, 4-positive curvature operators, positive isotropic curvature, and sectional curvature. In Section 3, we prove Theorems 1.1 and 1.2.

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2. Curvature decomposition for Einstein four-manifolds

The Hodge star operator $\star : \wedge^* TM \to \wedge^* TM$ induces a natural decomposition of the vector bundle of 2-forms $\wedge^2 TM$ on an oriented four-manifold $(M,\ g)$,

$$\wedge^2 TM = \wedge^+ M \oplus \wedge^- M,$$

where $\wedge^{\pm} M$ are the eigenspaces of ± 1 respectively. Sections of $\wedge^+ M$ and $\wedge^- M$ are called self-dual and anti-self-dual 2-forms respectively. It further induces a decomposition for the curvature operator $\mathfrak{R} : \wedge^2 TM \to \wedge^2 TM$:

$$\mathfrak{R} = \begin{pmatrix} \frac{s}{12} g + W^+ & \mathrm{Ric} - \frac{s}{4} g \\ \mathrm{Ric} - \frac{s}{4} g & \frac{s}{12} g + W^- \end{pmatrix},$$

where s denotes the scalar curvature. If $(M^4,\ g)$ is Einstein, then we get

$$\mathfrak{R} = \begin{pmatrix} \mathfrak{R}^+ & 0 \\ 0 & \mathfrak{R}^- \end{pmatrix} = \begin{pmatrix} \frac{s}{12} g + W^+ & 0 \\ 0 & \frac{s}{12} g + W^- \end{pmatrix}. \tag{2.1}$$

The duality decomposition for Einstein four-manifold implies that $\mathfrak{R},\ \mathfrak{R}^{\pm},\ W,\ W^{\pm}$ are all harmonic. Using the harmonicity of $W^{\pm}$, A. Derdziński [9] derived the following Weitzenböck formula (also see A. Besse [3, Prop. 16.73]),

Proposition 2.1. *Let $(M,\ g)$ be an Einstein four-manifold, then*

$$\Delta |W^{\pm}|^2 = 2|\nabla W^{\pm}|^2 + s|W^{\pm}|^2 - 36 \det W^{\pm}. \tag{2.2}$$

Later, Gursky and LeBrun [10] and Yang [16] further proved the following refined Kato inequality, which was proven to be optimal by T. Branson [5] and by D. Calderbank, P. Gauduchon and M. Herzlich [7],

Proposition 2.2. *Let $(M,\ g)$ be an Einstein four-manifold, then*

$$|\nabla W^{\pm}|^2 \geq \frac{5}{3} |\nabla |W^{\pm}||^2. \tag{2.3}$$

On the other hand, Berger [1] has another beautiful curvature tensor decomposition for Einstein four-manifolds (see also [13]),

Proposition 2.3. *Let $(M,\ g)$ be an Einstein four-manifold with $Ric = \lambda g$. For any $p \in M$, there exists an orthonormal basis $\{e_i\}_{1 \leq i \leq 4}$ of $T_p M$, such that relative to the corresponding basis $\{e_i \wedge e_j\}_{1 \leq i < j \leq 4}$ of $\wedge^2 T_p M$, $\mathfrak{R}$ takes the form*

$$\mathfrak{R} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \tag{2.4}$$

where $A = diag\{a_1,\ a_2,\ a_3\}$, $B = diag\{b_1,\ b_2,\ b_3\}$. Moreover, we have the following properties,

(1) $a_1 = K(e_1, e_2) = K(e_3, e_4) = \min\{K(\sigma) : \sigma \in \wedge^2 T_pM, ||\sigma|| = 1\}$,
$a_3 = K(e_1, e_4) = K(e_2, e_3) = \max\{K(\sigma) : \sigma \in \wedge^2 T_pM, ||\sigma|| = 1\}$,
$a_2 = K(e_1, e_3) = K(e_2, e_4)$, *and* $a_1 + a_2 + a_3 = \lambda$*;*
(2) $b_1 = R_{1234},\ b_2 = R_{1342},\ b_3 = R_{1423}$*;*
(3) $|b_2 - b_1| \le a_2 - a_1,\ |b_3 - b_1| \le a_3 - a_1,\ |b_3 - b_2| \le a_3 - a_2$.

By diagonalizing the matrix in Berger's decomposition, we obtain eigenvalues of the curvature operator in order (see [15]),

$$\begin{cases} a_1 + b_1 \le a_2 + b_2 \le a_3 + b_3, \\ a_1 - b_1 \le a_2 - b_2 \le a_3 - b_3. \end{cases} \tag{2.5}$$

Therefore, 2-positive curvature operator is equivalent to $(a_1 + a_2) \pm (b_1 + b_2) > 0$ and $a_1 > 0$; positive isotropic curvature implies $(a_1 + a_2) \pm (b_1 + b_2) > 0$; 3-positive curvature operator is equivalent to $2a_1 + a_2 \pm b_2 > 0$; and 4-positive curvature operator is equivalent to $a_1 + a_2 > 0$ and $1 + (a_1 \pm b_1) > 0$.

It coincides that, for Einstein four-manifolds, diagonalization of (2.4) becomes (2.1), as eigenvectors of $a_i + b_i$ $(1 \le i \le 3)$ are self-dual 2-forms, and eigenvectors of $a_i - b_i$ $(1 \le i \le 3)$ are anti-self-dual 2-forms. Combining the duality decomposition with Berger's decomposition, we get explicit expression for both self-dual and anti-self-dual Weyl tensors,

$$\begin{cases} W^+(\omega_i^+, \omega_j^+) = [(a_i + b_i) - \frac{s}{12}]\delta_{ij}, \\ W^-(\omega_i^-, \omega_j^-) = [(a_i - b_i) - \frac{s}{12}]\delta_{ij}, \end{cases}$$

where $\{\omega_i^+\}_{1\le i\le 3}$ and $\{\omega_i^-\}_{1\le i\le 3}$ are the corresponding orthonormal bases of $\wedge^+ M$ and $\wedge^- M$ in Berger's decomposition.

We have the following interesting relations between k-positive (k-nonnegative) curvature operators, positive isotropic curvature, and sectional curvature.

Proposition 2.4. *Let* $(M,\ g)$ *be an Einstein four-manifold with* $Ric = g$.

(1) $\mathfrak{R}$ *is 2-positive (2-nonnegative) if and only if the isotropic curvature is positive (nonnegative).*
(2) *If* $\mathfrak{R}$ *is 3-positive, then* $K > \frac{1}{30}$*; if* $\mathfrak{R}$ *is 3-nonnegative, then either* $\min K = 0$ *or* $K \ge \frac{1}{30}$*. On the other hand, if* $K > (\ge) \frac{1}{12}$*, then* $\mathfrak{R}$ *is 3-positive (3-nonnegative).*
(3) $\mathfrak{R}$ *is 4-positive (4-nonnegative) if and only if* $a_1 + a_2 > (\ge)\ 0$*, or* $K < (\le)\ 1$*. Moreover, this implies* $K > (\ge)\ \frac{1}{28}(7 - \sqrt{105})$*.*

Proof. We follow the argument in [15]. We prove 3-positive and 4-positive cases, and the nonnegative cases are similar. We start with some easy observations. It is well known that 2-positive curvature operator implies positive isotropic curvature. If $K > \frac{1}{12}$, then by Berger's decomposition, $2a_1 + a_2 - |b_2| \ge 2a_1 + a_2 - \frac{1}{3}(a_3 - a_1) \ge 4a_1 - \frac{1}{3} > 0$. If $\mathfrak{R}$ is 4-positive, it is obvious that $a_1 + a_2 > 0$, hence $a_3 < 1$.

Recall that for Einstein manifolds (see [6]),

$$\Delta R(e_i, e_j, e_k, e_l) + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) = 2R_{ijkl}, \tag{2.6}$$

where $B_{ijkl} = g^{mn}g^{pq}R_{imjp}R_{knlq}$. Applying Berger's curvature decomposition, we get explicitly that

$$\Delta R(e_1, e_2, e_1, e_2) + 2(a_1^2 + b_1^2 + 2a_2a_3 + 2b_2b_3) = 2a_1.$$

Suppose that the minimum of the sectional curvature of (M^4, g) is attained at p by the tangent plane spanned by $\{e_1, e_2\}$. For any $v \in T_pM$ and the geodesic $\gamma(t)$ with $\gamma(0) = p,\ \gamma'(0) = v$, let $\{e_1,\ e_2,\ e_3,\ e_4\}$ be a parallel orthornormal frame along $\gamma(t)$, then we have

$$(D^2_{v,v}R)(e_1, e_2, e_1, e_2)(p) = D^2_{v,v}(R(e_1, e_2, e_1, e_2))(p) \geq 0.$$

Hence it follows that $(\Delta R)(e_1, e_2, e_1, e_2)(p) \geq 0$, therefore at p, we get

$$a_1^2 + b_1^2 + 2(a_2a_3 + b_2b_3) \leq a_1. \tag{2.7}$$

First we prove that 2-positive curvature operator is equivalent to positive isotropic curvature. It suffices to show that $(a_1 + a_2) \pm (b_1 + b_2) > 0$ implies $a_1 > 0$. In fact if $(a_1 + a_2) \pm (b_1 + b_2) > 0$, then

$$\begin{cases} a_2 \pm b_2 > 0, \\ a_3 \pm b_3 > 0. \end{cases}$$

Therefore by (2.7), we have

$$a_1(p) \geq a_1^2 + b_1^2 + 2(a_2a_3 + b_2b_3) > a_1^2 + b_1^2 \geq 0.$$

Next we prove that 3-positive curvature operator implies positive sectional curvature. If $\mathfrak{R}$ is 3-positive, then

$$\begin{cases} a_2 \pm b_2 > -2a_1, \\ a_3 \pm b_3 > -2a_1. \end{cases}$$

Assuming that $a_1(p) \leq 0$, then $a_2 \pm b_2 > 0$ and $a_3 \pm b_3 > 0$, hence we have again

$$a_1(p) \geq a_1^2 + b_1^2 + 2(a_2a_3 + b_2b_3) > a_1^2 + b_1^2 \geq 0,$$

which contradicts to (2.7). Therefore $a_1(p) > 0$, i.e., $(M,\ g)$ has positive sectional curvature.

Now we derive a lower bound for a_1 when $\mathfrak{R}$ is 3-positive. Let $a_2(p) = ka_1(p),\ k \geq 1$.

(1) If $b_2b_3 \geq 0$, then it follows from (2.7) that

$$a_1 \geq a_1^2 + 2a_2a_3 \geq a_1^2 + 2a_1(1 - 2a_1) = 2a_1 - 3a_1^2,$$

this implies that $a_1 \geq 1/3$.

(2) If $b_2b_3 < 0$, without loss of generality, we assume that $b_2 < 0,\ b_3 > 0$. We can derive two estimates for a_1.

(a) First, by 3-positivity of the curvature operator, $|b_2| < a_2 + 2a_1$, therefore $b_1^2 + 2b_2b_3 = b_2^2 + b_3^2 + 4b_2b_3 = (b_3 + 2b_2)^2 - 3b_2^2 \geq -3b_2^2 > -3(k+2)^2a_1^2$. Plugging this into the inequality (2.7), we have

$$\begin{aligned} a_1 &\geq a_1^2 + b_1^2 + 2(a_2a_3 + b_2b_3) \\ &> a_1^2 + 2ka_1[1 - (k+1)a_1] - 3(k+2)^2a_1^2 \\ &= 2ka_1 - (5k^2 + 14k + 11)a_1^2, \end{aligned}$$

this leads to

$$a_1 > \frac{2k-1}{5k^2 + 14k + 11}. \tag{2.8}$$

(b) Second, it follows from Berger's curvature decomposition that, $b_3 - b_2 = b_3 + |b_2| \leq a_3 - a_2 = 1 - (2k+1)a_1$, then

$$b_1^2 + 2b_2b_3 = b_2^2 + b_3^2 + 4b_2b_3 \geq -\frac{1}{2}(b_3 - b_2)^2 \geq -\frac{1}{2}[1 - (2k+1)a_1]^2,$$

therefore,

$$\begin{aligned} a_1 &\geq a_1^2 + b_1^2 + 2(a_2a_3 + b_2b_3) \\ &\geq a_1^2 + 2ka_1[1 - (k+1)a_1] - \frac{1}{2}[1 - (2k+1)a_1]^2 \\ &= -(4k^2 + 4k - \frac{1}{2})a_1^2 + (4k+1)a_1 - \frac{1}{2}, \end{aligned}$$

this leads to

$$a_1 \leq \frac{4k - \sqrt{8k^2 - 8k + 1}}{8k^2 + 8k - 1} \quad \text{or} \quad a_1 \geq \frac{4k + \sqrt{8k^2 - 8k + 1}}{8k^2 + 8k - 1}. \tag{2.9}$$

Suppose that $a_1 \geq \frac{4k+\sqrt{8k^2-8k+1}}{8k^2+8k-1}$, then $a_1 = \frac{1}{3}$ if $k = 1$; it follows from a direct computation that if $k > 1$,

$$a_2 - a_3 = (2k+1)a_1 - 1 \geq (2k+1)\frac{4k + \sqrt{8k^2 - 8k + 1}}{8k^2 + 8k - 1} - 1 > 0,$$

which contradicts the facts that $a_2 \leq a_3$. Hence it is clear from (2.8) and (2.9) that either $a_1 = \frac{1}{3}$, or

$$\frac{2k-1}{5k^2 + 14k + 11} < a_1 \leq \frac{4k - \sqrt{8k^2 - 8k + 1}}{8k^2 + 8k - 1}.$$

The above inequality holds only if $1 \leq k \leq 4$, so we arrive at

$$a_1 > \min_{1 \leq k \leq 4} \frac{2k-1}{5k^2 + 14k + 11} = \frac{1}{30}.$$

This completes the proof that if $\mathfrak{R}$ is 3-positive, then $K > \frac{1}{30}$.

Similarly if $\mathfrak{R}$ is 3-nonnegative, then $a_1 \geq 0$, hence either $\min K = 0$ or if $a_1 > 0$ then $K \geq \frac{1}{30}$.

Finally we prove that $a_1+a_2>0$ implies $\mathfrak{R}$ is 4-positive. It suffices to prove that $a_1+a_2>0$ implies $1+(a_1\pm b_1)>0$. Since $|b_1|\le\frac{1}{3}-a_1$, so $1+(a_1\pm b_1)>0$ provided $a_1>-\frac{1}{3}$. We will now show that if $a_1+a_2>0$ then $a_1>\frac{1}{4}(1-\sqrt{\frac{15}{7}})$.

Assuming $a_1(p)=\min a_1<0$ (otherwise there is nothing to prove), set $P=\frac{1}{2}(a_3-a_2)\ge 0$, since $a_1+a_2>0$, we have

$$P=\frac{1}{2}(1-2a_2-a_1)<\frac{1}{2}(1+a_1).$$

Rewrite $a_3=\frac{1}{2}(1-a_1)+P,\ a_2=\frac{1}{2}(1-a_1)-P$, then $2a_2a_3=\frac{1}{2}(1-a_1)^2-2P^2$.

Let $Q=|b_2+\frac{1}{2}b_1|=|b_3+\frac{1}{2}b_1|\ge 0$, then $2b_2b_3=2(-\frac{b_1}{2}\pm Q)(-\frac{b_1}{2}\mp Q)=\frac{1}{2}b_1^2-2Q^2$. Without loss of generality, we may assume $b_1\ge 0$, by Berger's curvature decomposition, $|b_i-b_j|\le a_i-a_j$, for $1\le j\le i\le 3$, so we have

$$\begin{aligned}Q&\le\min\{\frac{3}{2}b_1+(a_2-a_1),-\frac{3}{2}b_1+(a_3-a_1),\frac{1}{2}(a_3-a_2)\}\\&=\min\{\frac{1}{2}(1-3a_1+3b_1)-P,\frac{1}{2}(1-3a_1-3b_1)+P,P\}\\&=\min\{\frac{1}{2}(1-3a_1+3b_1)-P,P\},\end{aligned}$$

here we use the fact that a_1+b_1 is the lowest eigenvalue of W^+, hence less than or equal to $\frac{1}{3}$. Plugging P and Q into (2.7) we get

$$\begin{aligned}a_1&\ge a_1^2+b_1^2+2(a_2a_3+b_2b_3)\\&=a_1^2+\frac{1}{2}(1-a_1)^2+\frac{3}{2}b_1^2-2P^2-2Q^2.\end{aligned}$$

We divide the rest proof into two cases:

(1) If $\frac{1}{2}(1-3a_1+3b_1)-P\le P$, i.e.,

$$\frac{1}{4}(1-3a_1+3b_1)\le P<\frac{1}{2}(1+a_1),$$

then $b_1<\frac{1}{3}(1+5a_1)$ and $Q\le\frac{1}{2}(1-3a_1+3b_1)-P$. So we have

$$\begin{aligned}a_1&\ge a_1^2+\frac{(1-a_1)^2}{2}+\frac{3}{2}b_1^2-2P^2-2(\frac{1}{2}(1-3a_1+3b_1)-P)^2\\&>a_1^2+\frac{(1-a_1)^2}{2}+\frac{3}{2}b_1^2-\frac{1}{2}(1+a_1)^2-\frac{1}{2}(-4a_1+3b_1)^2\\&=-7a_1^2-2a_1-3b_1^2+12a_1b_1\\&>-7a_1^2-2a_1-\frac{1}{3}(1+5a_1)^2+4a_1(1+5a_1),\end{aligned}$$

this leads to $a_1>\frac{1}{4}(1-\sqrt{\frac{15}{7}})$.

(2) If $P<\frac{1}{2}(1-3a_1+3b_1)-P$, then $Q\le P$ and

$$P<\min\{\frac{1}{2}(1+a_1),\frac{1}{4}(1-3a_1+3b_1)\},$$

so we have

$$a_1 \geq a_1^2 + \frac{1}{2}(1-a_1)^2 + \frac{3}{2}b_1^2 - 4P^2.$$

(a) If $\frac{1}{2}(1+a_1) > \frac{1}{4}(1-3a_1+3b_1)$, similar to Case (1), we get $a_1 > \frac{1}{4}(1-\sqrt{\frac{15}{7}})$.

(b) If $\frac{1}{2}(1+a_1) \leq \frac{1}{4}(1-3a_1+3b_1)$, or $b_1 \geq \frac{1}{3}(1+5a_1)$, in this case we get

$$\begin{aligned} a_1 &> a_1^2 + \frac{1}{2}(1-a_1)^2 + \frac{3}{2}b_1^2 - (1+a_1)^2 \\ &\geq a_1^2 + \frac{1}{2}(1-a_1)^2 + \frac{1}{6}(1+5a_1)^2 - (1+a_1)^2, \end{aligned}$$

which leads to $a_1 > \frac{1}{28}(17-\sqrt{345})$.

Hence if $a_1+a_2 > 0$ then $a_1 > \frac{1}{4}(1-\sqrt{\frac{15}{7}})$. This completes the proof that $a_1+a_2>0$ implies 4-positive curvature operator. □

3. **Proof of Main Theorems**

In this section, we shall prove Theorems 1.1 and 1.2. Notice that in Theorem 1.1, if the scalar curvature $s=0$, then (M,g) is flat; while in Theorem 1.2, the case that scalar curvature $s=0$ can not happen, since (M,g) is flat implies that $|W^+| = |W^-| \equiv \frac{1}{3}$, then the signature

$$\tau(M) = b_+(M) - b_-(M) = \frac{1}{12\pi^2}\int_M [|W^+|^2 - |W^-|^2] dv_g = 0,$$

which contradicts to the assumption that M has positive intersection form. In the rest of this section, without loss of generality, we may assume that $\mathrm{Ric} = g$, hence $s=4$. By Hitchin's theorem [11], we can further assume that $(M,\ g)$ is not half conformally flat.

We first prove Theorem 1.1. For some $\alpha > 0$ to be determined later, and any $\epsilon > 0$, there exists $t = t(\alpha,\epsilon) \in \mathbb{R}^+$, such that

$$\int_M (|W^+|^2+\epsilon)^{\frac{\alpha}{2}} - t(|W^-|^2+\epsilon)^{\frac{\alpha}{2}} dv = 0,$$

Applying Weitzenböck formula (2.2) and refined Kato inequality (2.3) onto $|W^\pm|$, we conclude that

(3.10)

$$\begin{aligned} &\Delta[(|W^+|^2+\epsilon)^\alpha + t^2(|W^-|^2+\epsilon)^\alpha] \\ =&\alpha(|W^+|^2+\epsilon)^{\alpha-2}[(|W^+|^2+\epsilon)(2|\nabla W^+|^2 + s|W^+|^2 - 36\det W^+) + (\alpha-1)|\nabla|W^+|^2|^2] \\ &+t^2\alpha(|W^-|^2+\epsilon)^{\alpha-2}[(|W^-|^2+\epsilon)(2|\nabla W^-|^2 + s|W^-|^2 - 36\det W^-) + (\alpha-1)|\nabla|W^-|^2|^2] \\ \geq&[(4-\frac{2}{3\alpha})|\nabla(|W^+|^2+\epsilon)^{\frac{\alpha}{2}}|^2 + \alpha(|W^+|^2+\epsilon)^{\alpha-1}(s|W^+|^2 - 36\det W^+)] \\ &+t^2[(4-\frac{2}{3\alpha})|\nabla(|W^-|^2+\epsilon)^{\frac{\alpha}{2}}|^2 + \alpha(|W^-|^2+\epsilon)^{\alpha-1}(s|W^-|^2 - 36\det W^-)]. \end{aligned}$$

Using the Poincaré inequality, we have

$$
\begin{aligned}
&(4-\frac{2}{3\alpha})\int_M(|\nabla(|W^+|^2+\epsilon)^{\frac{\alpha}{2}}|^2+t^2|\nabla(|W^-|^2+\epsilon)^{\frac{\alpha}{2}}|^2)dv\\
&\geq(2-\frac{1}{3\alpha})\int_M|\nabla[(|W^+|^2+\epsilon)^{\frac{\alpha}{2}}-t(|W^-|^2+\epsilon)^{\frac{\alpha}{2}}]|^2dv\\
&\geq(2-\frac{1}{3\alpha})\lambda_1\int_M[(|W^+|^2+\epsilon)^{\frac{\alpha}{2}}-t(|W^-|^2+\epsilon)^{\frac{\alpha}{2}}]^2dv,
\end{aligned}
$$

where λ_1 is the lowest positive eigenvalue of the Laplace operator. In our case that $\mathrm{Ric}=g$, we have $\lambda_1\geq\frac{4}{3}$ (see, for example, [12]).

Picking $\alpha=\frac{1}{3}$, which maximizes the value of $\frac{1}{\alpha}(2-\frac{1}{3\alpha})$, and integrating the above inequality (3.10), we arrive at

$$
\begin{aligned}
0\geq\frac{1}{3}\int_M&4[(|W^+|^2+\epsilon)^{\frac{\alpha}{2}}-t(|W^-|^2+\epsilon)^{\frac{\alpha}{2}}]^2+t^2(|W^-|^2+\epsilon)^{\alpha-1}(s|W^-|^2-36\det W^-)\\
&+(|W^+|^2+\epsilon)^{\alpha-1}(s|W^+|^2-36\det W^+).
\end{aligned}
$$

Now let $\epsilon\to 0$, we get

$$
\begin{aligned}
0\geq\int_M&t^2|W^-|^{2\alpha-2}(s|W^-|^2-36\det W^-)+4(|W^+|^\alpha-t|W^-|^\alpha)^2\\
&+|W^+|^{2\alpha-2}(s|W^+|^2-36\det W^+).
\end{aligned}
\tag{3.11}
$$

The integrand is a quadratic function of t, with positive leading coefficient and discriminant

$$
D=64|W^+|^{2\alpha-2}|W^-|^{2\alpha-2}[|W^+|^2|W^-|^2-(2|W^+|^2-9\det W^+)(2|W^-|^2-9\det W^-)].
$$

We need the following technical lemma,

Lemma 3.1. *If $\mathfrak{R}$ is 3-nonnegative, then $D\leq 0$. Furthermore, the equality holds if and only if $W^+=0$, $W^-=0$, or W^+ and W^- have the same eigenvalues $\{-\frac{1}{3},-\frac{1}{3},\frac{2}{3}\}$.*

Proof. To show $D\leq 0$, it is equivalent to show

$$
D'=|W^+|^2|W^-|^2-6|W^+|^2\det W^--6|W^-|^2\det W^++27\det W^+\det W^-\geq 0.
$$

Denote that $a\leq b\leq c$ to be eigenvalues of W^+, $x\leq y\leq z$ to be eigenvalues of W^-, then $a\leq 0,\ x\leq 0,\ c\geq 0,\ z\geq 0,\ b=-a-c,\ y=-x-z$; we also have

$$
\begin{aligned}
|W^+|^2&=a^2+b^2+c^2=2(a^2+ac+c^2),\\
|W^-|^2&=x^2+y^2+z^2=2(x^2+xz+z^2),\\
\det W^+&=abc=-ac(a+c),\\
\det W^-&=xyz=-xz(x+z).
\end{aligned}
$$

The condition of 3-nonnegative curvature operator implies that $c-x \le 1$, $z-a \le 1$, $a+x \ge -\frac{2}{3}$, and $c+z \le \frac{4}{3}$. Denote D' as a function of a, c, x, z, i.e.,

(3.12)
$$\begin{aligned} D' = f(a,c,x,z) =& 4(a^2+ac+c^2)(x^2+xz+z^2) + 12(a^2+ac+c^2)(x^2z+xz^2) \\ &+ 12(a^2c+ac^2)(x^2+xz+z^2) + 27(a^2c+ac^2)(x^2z+xz^2). \end{aligned}$$

We shall show that $f \ge 0$ in the region

$$\Omega = \{(a,c,x,z) : a \le 0,\ x \le 0,\ c \ge 0,\ z \ge 0,\ -2c \le a \le -\frac{c}{2},\ -2z \le x \le -\frac{z}{2},$$
$$c \le x+1,\ z \le a+1,\ a+x \ge -\frac{2}{3},\ c+z \le \frac{4}{3}\}.$$

Notice that in particular we have $c \le 1,\ z \le 1$ in Ω.

Taking the first derivatives of f, we have

$$f_a = (2a+c)[(4+12c)(x^2+xz+z^2) + (12+27c)(x^2z+xz^2)],$$
$$f_x = (2x+z)[(4+12z)(a^2+ac+c^2) + (12+27z)(a^2c+ac^2)].$$

Observe that $g_{c,z}(x) = (4+12c)(x^2+xz+z^2) + (12+27c)(x^2z+xz^2)$ is a quadratic function of x with positive leading coefficient and discriminant

$$z^2[(12+27c)z - (12+36c)][(12+27c)z + (4+12c)],$$

which is nonpositive since $z \le 1$. Therefore $g_{c,z}(x) \ge 0$ in Ω, hence $f_a \le 0$. Similarly $g_{c,z}(a) = (4+12z)(a^2+ac+c^2) + (12+27z)(a^2c+ac^2) \ge 0$, and $f_x \le 0$.

Therefore the minimum of f is attained at $a = -\frac{c}{2}$ and $x = -\frac{z}{2}$. Plugging this into (3.12) we get

$$f = \frac{9}{4}c^2z^2(1-c-z+\frac{3}{4}cz),$$

where

$$(c,z) \in \Omega' = \{(c,z) : 0 \le c \le 1, 0 \le z \le 1, c+2z \le 2,\ 2c+z \le 2\},$$

with boundary components:

(1) $c=0$ and $0 \le z \le 1$,
(2) $z=0$ and $0 \le c \le 1$,
(3) $c+2z=2$ and $\frac{2}{3} \le z \le 1$,
(4) $2c+z=2$ and $\frac{2}{3} \le c \le 1$.

The only critical point of f in Ω' is $c = z = \frac{2}{9}(5-\sqrt{7})$, where $f = \frac{16}{19683}(1022\sqrt{7} - 2671) > 0$. On the boundary components:

(1) If $c=0$ or $z=0$, then $f=0$.
(2) If $2c+z=2$ and $\frac{2}{3} \le c \le 1$, then $f = \frac{9}{2}c^2(1-c)^3(3c-2) \ge 0$, with equality holds if and only if $c=1$ and $z=0$, or $c=z=\frac{2}{3}$.
(3) If $c+2z=2$ and $\frac{2}{3} \le z \le 1$, we get a similar conclusion.

Therefore $f = D' \geq 0$ in Ω, with equality holds if and only if $|W^+||W^-| = 0$ or $(a,c,x,z) = (-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}, \frac{2}{3})$.

□

We can now finish the proof of Theorem 1.1. Since both $|W^+|^2$ and $|W^-|^2$ are real analytic, then either the discriminant $D < 0$ in an open dense set, this contradicts (3.11) because of the integrand is strictly positive; or if $D \equiv 0$, we have $W^+ \equiv 0$, $W^- \equiv 0$, or W^+ and W^- have the same eigenvalues $\{-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\}$ at every point. The first two cases contradict our assumption that (M,g) is not half conformally flat. For the third case, using Weitzenböck formula (2.2), it leads to $\nabla W^{\pm} \equiv 0$, so (M,g) is locally symmetric, whose curvature operator has eigenvalues $\{0,0,1,0,0,1\}$. Therefore, by Cartan's classification of symmetric spaces, (M,g) is isometric to $(S^2 \times S^2, g_0 \oplus g_0)$ or its quotient.

□

Next we prove Theorem 1.2. The proof basically follows from Gursky and LeBrun [10] plus the following point-wise estimate:

Lemma 3.2. *Let (M,g) be an Einstein four-manifold with 4-nonnegative curvature operator, then*

$$|W^+|^2 + |W^-|^2 < \frac{25}{9}. \tag{3.13}$$

Proof. We use a similar argument as in Lemma 3.1. Notice that 4-nonnegative curvature operator implies that $a + x \geq \frac{1}{2}(1 - \sqrt{\frac{15}{7}}) - \frac{2}{3} > -\frac{9}{10}$ and $c + z \leq \frac{4}{3}$. Denote that

$$f(a,c,x,z) = |W^+|^2 + |W^-|^2 = 2(a^2 + ac + c^2 + x^2 + xz + z^2). \tag{3.14}$$

We shall prove that $f \leq \frac{1249}{450}$ in the region

$$\Omega = \{(a,c,x,z):\ a \leq 0,\ x \leq 0,\ c \geq 0,\ z \geq 0,\ -2c \leq a \leq -\frac{c}{2},$$
$$-2z \leq x \leq -\frac{z}{2},\ a + x \geq -\frac{9}{10},\ c + z \leq \frac{4}{3}\}.$$

Computing the first derivatives of f:

$$f_a = 2(2a + c), \qquad f_c = 2(2c + a), \qquad f_x = 2(2x + z), \qquad f_z = 2(2z + x),$$

we have $f_a \leq 0$ and $f_x \leq 0$, $f_c \geq 0$ and $f_z \geq 0$, therefore the maximum of f is attained at $a + x = -\frac{9}{10}$ and $c + z = \frac{4}{3}$. Plugging into (3.14) we get

$$\begin{aligned} f(a,c) &= 2[a^2 + ac + c^2 + (\frac{9}{10} + a)^2 - (\frac{9}{10} + a)(\frac{4}{3} - c) + (\frac{4}{3} - c)^2] \\ &= 4a^2 + 4ac + 4c^2 + \frac{14}{15}a - \frac{53}{15}c + \frac{1249}{450}, \end{aligned} \tag{3.15}$$

where (a,c) lies in

$$\Omega' = \{(a,c):\ -\frac{9}{10} \leq a \leq 0,\ 0 \leq c \leq \frac{4}{3},\ -2c \leq a \leq -\frac{c}{2},\ 2a + c \geq -\frac{7}{15},\ a + 2c \leq \frac{53}{30}\},$$

with boundary components:

(1) $a=-2c$ and $0\le c\le \frac{7}{45}$,
(2) $c=-2a$ and $-\frac{53}{90}\le a\le 0$,
(3) $a+2c=\frac{53}{30}$ and $\frac{53}{45}\le c\le \frac{4}{3}$,
(4) $2a+c=-\frac{7}{15}$ and $-\frac{9}{10}\le a\le -\frac{14}{45}$.

The only critical point of f in Ω' is at $a=-\frac{9}{20}$, $c=\frac{2}{3}$, where $f=\frac{1249}{900}$. For boundary components, we can compute case by case and conclude that $f\le\frac{1249}{450}$, with equality holds if and only if $a=c=0$, or $a=-\frac{9}{10}$ and $c=\frac{4}{3}$. Therefore $f\le\frac{1249}{450}$ in Ω, with equality holds if and only if $(a,c,x,z)=(-\frac{9}{10},\frac{4}{3},0,0)$, or $(a,c,x,z)=(0,0,-\frac{9}{10},\frac{4}{3})$. □

Proof of Theorem 1.2. Applying Lemma 4.1 to Gauss-Bonnet formula, we derive that

$$\chi(M)=\frac{1}{8\pi^2}\int_M\Big(|W^+|^2+|W^-|^2+\frac{s^2}{24}\Big)dv<\frac{31}{6\times 8\pi^2}\int_M\frac{s^2}{24}dv. \tag{3.16}$$

By Meyer's theorem and Bishop's volume comparison theorem, we have

$$\begin{aligned}\chi(M)&<\frac{31}{6\times 8\pi^2}\int_M\frac{s^2}{24}dv\le\frac{31}{6\times 8\pi^2}\int_{S^4}\frac{s^2}{24}dv\\&=\frac{31}{6}\chi(S^4)=\frac{31}{3}.\end{aligned}$$

Since $\chi(M)\in\mathbb{Z}$, we get an upper bound for the Euler characteristic,

Lemma 3.3. *Let (M,g) be an Einstein four-manifold with 4-nonnegative curvature operator, then $\chi(M)\le 10$.*

Combining (3.16) with the following gap theorem of Gursky and LeBrun for Einstein four-manifolds which are not half conformally flat,

$$2\chi(M)-3\tau(M)\ge\frac{3}{4\pi^2}\int_M\frac{s^2}{24}d\mu, \tag{3.17}$$

we can conclude that $2\chi(M)-3\tau(M)>\frac{36}{31}\chi(M)$, which implies

Lemma 3.4. *Let (M,g) be an Einstein four-manifold with 4-nonnegative curvature operator which is not half conformally flat, then*

$$\frac{93}{26}|\tau|<\chi\le 10. \tag{3.18}$$

For $\mathrm{Ric}=g$, notice that $\tau(M)=b_+(M)-b_-(M)$, $\chi(M)=2+b_+(M)+b_-(M)$. Moreover, if (M,g) has positive intersection form, then $b_+(M)>0$, $b_-(M)=0$, and

$$\frac{93}{26}\tau(M)=\frac{93}{26}b_+\ge\frac{67}{26}+b_+>2+b_+=\chi, \tag{3.19}$$

this contradicts to (3.18), so (M,g) must be half conformally flat, hence $|W^-|\equiv 0$ and $|W^+|\not\equiv 0$, since $b_+(M)>0$. Therefore by Hitchin's theorem, (M,g) is isometric to $(\mathbb{C}P^2,g_{FS})$, up to scaling, this completes our proof of Theorem 1.2.

□

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