Spaces of Incompressible Surfaces

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The homotopy type of the space of PL homeomorphisms of a Haken 3-manifold was computed in [H1], and with the subsequent proof of the Smale conjecture in [H2], the computation carried over to diffeomorphisms as well. These results were also obtained independently by Ivanov [I1,I2]. The main step in the calculation in [H1], though not explicitly stated in these terms, was to show that the space of embeddings of an incompressible surface in a Haken 3-manifold has components which are contractible, except in a few special situations where the components have a very simple noncontractible homotopy type. The purpose of the present note is to give precise statements of these embedding results along with simplified proofs using ideas from [H3]. We also rederive the calculation of the homotopy type of the diffeomorphism group of a Haken 3-manifold.

Let M be an orientable compact connected irreducible 3-manifold, and let S be an incompressible surface in M, by which we mean:

— S is a compact connected surface, not S^2, embedded in M properly, i.e., $S \cap \partial M = \partial S$.

— The normal bundle of S in M is trivial. Since M is orientable, this just means S is orientable.

— The inclusion $S \hookrightarrow M$ induces an injective homomorphism on π_1.

We denote by $E(S, M \text{ rel } \partial S)$ the space of smooth embeddings $S \to M$ agreeing with the given inclusion $S \hookrightarrow M$ on ∂S. The group $Diff(S \text{ rel } \partial S)$ of diffeomorphisms $S \to S$ restricting to the identity on ∂S acts freely on $E(S, M \text{ rel } \partial S)$ by composition, with orbit space the space $P(S, M \text{ rel } \partial S)$ of subsurfaces of M diffeomorphic to S by a diffeomorphism restricting to the identity on ∂S. We think of points of $P(S, M \text{ rel } \partial S)$ as "positions" or "placements" of S in M. There is a fibration

$$Diff(S \text{ rel } \partial S) \longrightarrow E(S, M \text{ rel } \partial S) \longrightarrow P(S, M \text{ rel } \partial S)$$

giving rise to a long exact sequence of homotopy groups. It is known that $\pi_i Diff(S \text{ rel } \partial S)$ is zero for $i > 0$, except when S is the torus T^2 and $i = 1$, when the inclusion $T^2 \hookrightarrow Diff(T^2)$ as rotations induces an isomorphism on π_1. So the higher homotopy groups of $E(S, M \text{ rel } \partial S)$ and $P(S, M \text{ rel } \partial S)$ are virtually identical.

Theorem 1. *Let S be an incompressible surface in M. Then:*

(a) $\pi_i P(S, M \text{ rel } \partial S) = 0$ *for all* $i > 0$ *unless* $\partial S = \varnothing$ *and* S *is the fiber of a surface bundle structure on* M. *In this exceptional case the inclusion* $S^1 \hookrightarrow P(S, M)$ *as the fiber surfaces induces an isomorphism on* π_i *for all* $i > 0$.

(b) $\pi_i E(S, M \text{ rel } \partial S) = 0$ *for all* $i > 0$ *unless* $\partial S = \varnothing$ *and* S *is either a torus or the fiber of a surface bundle structure on* M. *In these exceptional cases* $\pi_i E(S, M) = 0$ *for all* $i > 1$. *In the surface bundle case the inclusion of the subspace consisting of embeddings with image a fiber induces an isomorphism on* π_1. *When* S *is a torus but not the fiber of a surface bundle structure, the inclusion of the subspace consisting of embeddings with image equal to the given* S *induces an isomorphism on* π_1.

Here it is understood that $\pi_i E(S, M \text{ rel } \partial S)$ and $\pi_i P(S, M \text{ rel } \partial S)$ are to be computed at the basepoint which is the given inclusion $S \hookrightarrow M$.

It is not hard to describe precisely what happens in the exceptional cases of (b). In the surface bundle case consider the exact sequence

$$0 \longrightarrow \pi_1 Diff(S) \longrightarrow \pi_1 E(S, M) \longrightarrow \pi_1 P(S, M) \xrightarrow{\partial} \pi_0 Diff(S)$$

By part (a) we have $\pi_1 P(S, M) \approx \mathbb{Z}$. The boundary map takes a generator of this $\mathbb{Z}$ to the monodromy diffeomorphism defining the surface bundle. If this monodromy has infinite order in $\pi_0 Diff(S)$ then the boundary map is injective so $\pi_1 E(S, M) \approx \pi_1 Diff(S)$. If the monodromy has finite order in $\pi_0 Diff(S)$, it is isotopic to a periodic diffeomorphism of this order, as Nielsen showed. Hence M is Seifert-fibered with coherently oriented fibers, and there is an action of S^1 on M rotating circle fibers, and taking fibers of the surface bundle structure to fibers. The orbit of the given embedding of S under this action then generates a $\mathbb{Z}$ subgroup of $\pi_1 E(S, M)$, which is all of $\pi_1 E(S, M)$ unless S is a torus, in which case it is easy to see that $\pi_1 E(S, M) \approx \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

Using these results we can deduce:

Theorem 2. *If* M *is an orientable Haken manifold then* $\pi_i Diff(M \text{ rel } \partial M) = 0$ *for all* $i > 0$ *unless* M *is a closed Seifert manifold with coherently orientable fibers. In the latter case the inclusion* $S^1 \hookrightarrow Diff(M)$ *as rotations of the fibers induces an isomorphism on* π_i *for all* $i > 0$, *except when* M *is the 3-torus, in which case the circle of rotations* $S^1 \hookrightarrow Diff(M)$ *is replaced by the 3-torus of rotations* $M \hookrightarrow Diff(M)$.

Proof: Suppose first that $\partial M \neq \varnothing$, hence M is automatically Haken if it is irreducible. By the theory of Haken manifolds, there exists an incompressible surface $S \subset M$ with $\partial S \neq \varnothing$, in fact with ∂S representing a nonzero class in $H_1(\partial M)$. Consider the fibration

$$Diff(M \text{ rel } \partial M \cup S) \longrightarrow Diff(M \text{ rel } \partial M) \longrightarrow E(S, M \text{ rel } \partial S)$$

The fiber can be identified with $Diff(M' \text{ rel } \partial M')$ where M' is the compact manifold, possibly disconnected, obtained by splitting M along S. If we know the theorem holds for M', then from the long exact sequence of homotopy groups of the fibration, together with part (b) of the preceding theorem, we deduce that the theorem holds for M. Again by Haken manifold theory, there is a finite sequence of such splitting operations reducing M to a disjoint union of balls. By the Smale conjecture the theorem holds for balls, so by induction the theorem holds for M.

When M is a closed Haken manifold we again consider the fibration displayed above, with S now a closed incompressible surface. If we are not in the exceptional cases that $\pi_1 E(S,M)$ is nonzero, described in the paragraph before Theorem 2, the arguments in the preceding paragraph apply since M' is a nonclosed Haken manifold, for which the theorem has already been proved.

There remain the cases that $\pi_1 E(S,M) \neq 0$.

(*i*) If S is the fiber of a surface bundle structure on M but not a torus, the result follows from the remarks preceding Theorem 2, describing how a generator of $\pi_1 E(S,M) \approx \mathbb{Z}$ is represented by the orbit of the inclusion $S \hookrightarrow M$ under the S^1 action rotating fibers of the Seifert fibering.

(*ii*) If S is a torus but not the fiber of a surface bundle, we have $\pi_1 E(S,M) \approx \mathbb{Z} \times \mathbb{Z}$, the rotations of S. Under the boundary map $\pi_1 E(S,M) \to \pi_0 Diff(M \text{ rel } S)$ each loop of rotations goes to a Dehn twist on one side of S and its inverse twist on the other side. The behavior of such twists in the mapping class group of the manifold M' is well known; see e.g. [HM]. In particular, the boundary map is injective unless M is orientably Seifert-fibered with S a union of fibers, in which case the kernel of the boundary map is $\mathbb{Z}$ represented by rotations of the circle fibers.

(*iii*) If S is the torus fiber of a surface bundle we have to consider the phenomena in both (*i*) and (*ii*). The monodromy diffeomorphism of the torus fiber defining the bundle is an element of $SL_2(\mathbb{Z})$. Note that a loop of rotations of S as in (*ii*) lies in the kernel of the boundary map iff the rotations are in the direction of a curve in S fixed by the monodromy. If the monodromy is trivial, M is the 3-torus and we have contributions to $\pi_1 Diff(M)$ from both (*i*) and (*ii*), so we get $M \subset Diff(M)$ inducing an isomorphism on π_i for $i > 0$. If the monodromy is nontrivial and of finite order, it has no real eigenvalues, so the boundary map in (*ii*) is injective and we get only the $S^1 \subset Diff(M)$ as in (*i*). If the monodromy is of infinite order and has 1 as an eigenvalue, it is a Dehn twist of the torus fiber, so we get $\pi_1 Diff(M) \approx \mathbb{Z}$ and this loop of diffeomorphisms is realized by rotating fibers of a fibering of M by circles in the eigendirection in the torus fibers of M. □

Proof of Theorem 1.

We will first prove statement (b) when $\partial S \neq \varnothing$, which suffices to deduce Theorem 2 in the nonclosed case. Then we will prove (a), and finally the remaining cases of (b).

Let $f_t : S \to M$, $t \in D^i$, be a family of embeddings representing an element of $\pi_i E(S, M \text{ rel } \partial S)$. We assume $i > 0$, so all the surfaces $f_t(S)$ are isotopic rel ∂S to the given inclusion $S \hookrightarrow M$, hence are incompressible also. Let $S \times I$ be a collar on one side of $S = S \times \{0\}$ in M. By Sard's theorem, $f_t(S)$ is transverse to $S \times \{x\}$ for almost all $x \in I$. Transversality is preserved under small perturbations, so, since D^i is compact, we can choose a finite cover of D^i by open sets U_j such that $f_t(S)$ is transverse to a slice $S_j = S \times \{x_j\} \subset S \times I$ for all $t \in U_j$. Then $f_t(S) \cap S_j$ is a finite collection of disjoint circles which vary by isotopy as t ranges over U_j. The main work will be to deform the family f_t to eliminate these circles, for all t and j simultaneously (after choosing the slices S_j and the open sets U_j a little more carefully).

Step 1. The aim here is to eliminate all the nullhomotopic circles. Let $\mathcal{C}_t^j$ be the collection of circles of $f_t(S) \cap S_j$ which are homotopically trivial in M and hence bound disks in both $f_t(S)$ and S_j, by incompressibility. Let $\mathcal{C}_t = \bigcup_j \mathcal{C}_t^j$, the union over those j's such that $t \in U_j$. We would like to construct a family of functions φ_t assigning a value $\varphi_t(C) \in (0,1)$ to each circle $C \in \mathcal{C}_t$ such that:

(1) $\varphi_t(C)$ varies continuously with $t \in U_j$ for each $C \in \mathcal{C}_t^j$.

(2) $\varphi_t(C) < \varphi_t(C')$ whenever the disk in $f_t(S)$ bounded by C is contained in the disk bounded by C'.

(3) φ_t is injective for each t, so $\varphi_t(C) \neq \varphi_t(C')$ if C and C' are distinct circles of $\mathcal{C}_t$.

Achieving (1) and (2) is not hard. For example, one can take $\varphi_t(C)$ to be the area of the disk in $f_t(S)$ bounded by C, with respect to some metric on M. To achieve (3) takes more work. First replace each S_j by $2i+1$ nearby slices $S_{jk} = S \times \{x_{jk}\}$, so that each circle of $f_t(S) \cap S_j$ is replaced by $2i+1$ nearby circles of $f_t(S) \cap S_{jk}$. Define φ_t on each of these new circles of $f_t(S) \cap S_{jk}$ to have value near the value of the original φ_t on the nearby circle of $f_t \cap S_j$, so that (1) and (2) are still satisfied for all the new circles. Perturb the new functions $t \mapsto \varphi_t(C)$ so that the solution set of each equation $\varphi_t(C) = \varphi_t(C')$ is a codimension-one submanifold of D^i, and so that these codimension-one submanifolds have general position intersections with each other, and in particular so that at most i such equations are satisfied for each t. Then for each t the perturbed φ_t is injective on the complement of a set of at most $2i$ circles. This means that if we delete from $\mathcal{C}_t$ those circles on which φ_t is not injective, there exists for each t a slice S_{jk} from which no circles have been deleted. This slice has the same property for nearby t. Let U_{jk} be the subspace of U_j consisting of points t for which no circles of $\mathcal{C}_t$ in S_{jk} are deleted. Then the open cover $\{U_{jk}\}$ of D^i and the associated slices S_{jk} satisfy (1)-(3). We relabel these as U_j and S_j.

We would like to deform the family f_t, $t \in D^i$, so as to eliminate all the circles of $\mathcal{C}_t$ without introducing new circles of $f_t(S) \cap S_j$ for $t \in U_j$. Consider first a fixed value of t and a circle $C \in \mathcal{C}_t$ for which $\varphi_t(C)$ is minimal. Thus C bounds a disk $D \subset f_t(S)$ disjoint from all other circles of $\mathcal{C}_t$. In particular, $D \cap S_j = C$, where $C \subset S_j$. By incompressibility of S_j, C also bounds a disk $D_j \subset S_j$, and the sphere $D \cup D_j$ bounds a ball $B \subset M$ since M is irreducible. We can isotope f_t to eliminate C from $f_t(S) \cap S_j$ by isotoping D across B to D_j, and slightly beyond. This isotopy extends to an ambient isotopy of M supported near B which also eliminates any circles of $f_t(S) \cap S_j$ which happen to lie inside D_j; we call such circles *secondary*, in contrast to C itself which we call *primary*. Since D and D_j are disjoint from S_k if $k \neq j$ and $t \in U_k$, so is B, so this isotopy leaves circles of $f_t(S) \cap S_k$ unchanged if $k \neq j$ and $t \in U_k$. Hence we can iterate the process, eliminating in turn each remaining circle of $\mathcal{C}_t$ with smallest φ_t value. Thus we construct an isotopy f_{tu}, $0 \leq u \leq 1$, of $f_t = f_{t0}$ eliminating each primary circle $C \in \mathcal{C}_t$ during the u-interval $[\varphi_t(C), \varphi_t(C) + \varepsilon]$, for some fixed ε, along with any secondary circles associated to C.

This isotopy f_{tu} will not depend continuously on t since as t moves from U_j to the complement of U_j, we suddenly stop performing the isotopies eliminating the circles of $f_t(S) \cap S_j$. This problem is easy to correct by the following truncation process. For each U_j choose a map $\psi_j : D^i \to [0,1]$ which is 0 outside U_j and 1 inside a slightly smaller open set U_j' in U_j, such that the U_j''s still cover D^i. Then modify the construction of f_{tu} by performing the isotopies eliminating primary circles of $f_t(S) \cap S_j$ only for $u \leq \psi_j(t)$. As observed earlier, the isotopy eliminating a primary circle of $f_t(S) \cap S_j$ does not affect circles of $f_t(S) \cap S_k$ for $k \neq j$ with $t \in U_k$, so truncating such an isotopy at $u = \psi_j(t)$ creates no problems for continuing the construction of f_{tu} for $u > \psi_j(t)$ to eliminate circles of $f_t(S) \cap S_k$.

There is one other reason why f_{tu}, as described so far, may not depend continuously on t, namely, there is a choice in the isotopy eliminating a primary circle, and these choices need to be made continuously in t. We may specify an isotopy eliminating a circle C by the following process. First enlarge the disk D slightly to a disk $D' \subset f_t(S)$, with $\partial D'$ contained in a nearby parallel copy S_j' of S_j, then choose a collar on D' containing B, i.e., an embedding $\chi : D' \times I \to M$ with $\chi \,|\, D' \times \{0\}$ the identity and $B \subset \chi(D' \times [0, 1/2]) \subset \chi(D' \times I) \subset B'$ where B' is a ball constructed like B using S_j' in place of S_j. We may also assume $\chi(\partial D' \times I) \subset S_j'$. Then an isotopy eliminating C is obtained by pushing points $x \in D'$ along the lines $\chi(\{x\} \times I)$, with this motion damped down near $\partial D'$.

The space of such collars is contractible. Namely, use one collar η to produce an isotopy $h_s : B' \to B'$ moving $\eta(D' \times \{0\})$ to $\eta(D' \times \{1\})$ in the obvious way. Then for an arbitrary collar χ, construct a deformation χ_s of χ consisting of $h_s \chi$ together with a portion of η. (To make χ_s smooth at the junction of these two pieces one can require all χ's to have the same derivative along $D' \times \{0\}$.) Then χ_1 contains the collar η, so we can deform χ_1 to η by gradually truncating the part outside η.

To make the dependence of f_{tu} on u explicit we may proceed as follows. We may assume the truncation functions $\psi_j(t)$ are piecewise linear, and we may perturb the functions $\varphi_t(C)$ to be piecewise linear functions of t also. Then we may choose a triangulation of D^i so that each simplex is contained in some U_j and so that the solution set of each equation $\psi_j(t) = \varphi_t(C)$ is a subcomplex of the triangulation. Now we construct the isotopies f_{tu} inductively over skeleta of the triangulation. Assume that f_{tu} has already been constructed over the boundary of a k-simplex Δ^k, and assume that over Δ^k itself we have already constructed f_{tu} for $t \le \varphi_t(C)$, for some primary circle $C \in \mathcal{C}_t^j$. If $\psi_j(t) \le \varphi_t(C)$ over Δ^k, the induction step is vacuous, so we may suppose $\psi_j(t) \ge \varphi_t(C)$ for $t \in \Delta^k$. By induction, for $t \in \partial\Delta^k$ we have already chosen collars χ_t for the disk D', as above, depending continuously on t, and since the space of collars is contractible we can extend these collars over Δ^k. This allows the induction step to be completed, eliminating C over Δ^k, with the elimination isotopy truncated if ψ_t so dictates.

Step 2. We show how to finish the proof of (b) if S is a disk. After Step 1, all the circles of $f_t(S) \cap S_j$ have been eliminated for all $t \in U_j$ and all j. By averaging the slices $S_j = S \times \{x_j\}$ via a partition of unity subordinate to the U_j's we can choose a continuously varying slice $S_t = S \times \{x_t\}$ disjoint from $f_t(S)$; here we use the fact that if $f_t(S)$ is disjoint from $S \times \{x\}$ and from $S \times \{y\}$ then it is disjoint from $S \times [x, y]$ since $f_t(S)$ is connected and its boundary, which is non-empty, lies in $S = S \times \{0\}$. Having $f_t(S)$ disjoint from S_t for all t, we can then by isotopy extension isotope the family f_t so that $f_t(S)$ is disjoint from $S \times \{1\}$ for all t.

The proof can now be completed as follows. The space E of embeddings $S \times I \to M$ agreeing with the given embedding on $S \times \{1\} \cup \partial S \times I$ fits into a fibration:

$$Diff(S \times I \text{ rel } \partial) \longrightarrow E \longrightarrow E(S, M - S \times \{1\} \text{ rel } \partial S)$$

It is elementary that E has trivial homotopy groups since one can canonically isotope an embedding in E so that it equals the given embedding on a neighborhood of $S \times \{1\} \cup \partial S \times I$, then gradually excise from the embedding everything but this neighborhood. The fiber

$Diff(S \times I \text{ rel } \partial)$ also has trivial homotopy groups by the Smale conjecture since S is a disk. Thus $\pi_i E(S, M - S \times \{1\} \text{ rel } \partial S)$ vanishes for $i > 0$, and the first part of the proof shows that this group maps onto $\pi_i E(S, M \text{ rel } \partial S)$, so the latter group also vanishes. When S is a disk this argument also applies for $i = 0$ since $\pi_0 E(S, M - S \times \{1\} \text{ rel } \partial S) = 0$ by the irreducibility of M.

Step 3 is to eliminate the remaining circles of $f_t(S) \cap S_j$ for all $t \in U_j$ and all j, in the case $\partial S \neq \varnothing$. The role of the ball B in Step 1 will be played by what we may call a *pinched product*. This is obtained from a product $W \times I$, where W is a nonclosed compact orientable surface, by collapsing each segment $\{w\} \times I$, $w \in \partial W$, to a point. Thus a pinched product P is a handlebody with its boundary decomposed as the union of two copies $\partial_+ P$ and $\partial_- P$ of the surface W, with $\partial_+ P$ and $\partial_- P$ intersecting only in their common boundary, a "corner" of ∂P.

If we can find a pinched product $P \subset M$ such that $\partial_+ P \subset f_t(S)$ and $\partial_- P \subset S_j$, then by isotoping f_t by pushing $\partial_+ P$ across P to $\partial_- P$, and slightly beyond, we eliminate the circles of $\partial_+ P \cap \partial_- P$ from $f_t(S) \cap S_j$, as well as any other circles of $f_t(S) \cap S_j$ which happen to lie in $\partial_- P$.

In order to locate such pinched products it is convenient to consider the covering space $p : (\widetilde{M}, \widetilde{x}_0) \to (M, x_0)$ corresponding to the subgroup $\pi_1(S, x_0)$ of $\pi_1(M, x_0)$, with respect to a basepoint $x_0 \in S$. There is then a homeomorphic copy $\widetilde{S}$ of S in $\widetilde{M}$ containing $\widetilde{x}_0$. We can associate to $\widetilde{M}$ a graph T having a vertex for each component of $\widetilde{M} - p^{-1}(S)$ and an edge for each component of $p^{-1}(S)$. This graph T is in fact a tree. For suppose γ is a loop in T based at a point in the edge corresponding to $\widetilde{S}$. This can be lifted to a loop $\widetilde{\gamma}$ in $\widetilde{M}$ based at $\widetilde{x}_0$. By the definition of $\widetilde{M}$, the loop $\widetilde{\gamma}$ is homotopic to a loop in $\widetilde{S}$, and this homotopy projects to a homotopy from γ to a trivial loop.

In the same way, the surface S_j parallel to S determines a tree T_j canonically isomorphic to T. The lift $\widetilde{S}$ lies in a component of $\widetilde{M} - p^{-1}(S_j)$ corresponding to a base vertex of T_j, and hence to a base vertex of T which is independent of j.

The family $f_t(S)$ is an i-parameter isotopy of S, so there are lifts $\widetilde{f}_t : S \to \widetilde{M}$ such that $\widetilde{f}_t(S)$ forms an i-parameter isotopy of $\widetilde{S}$. The lifts $\widetilde{f}_t(S)$ and the parameter domain being compact, we can choose a vertex v of T farthest from the base vertex among all vertices corresponding to components of $\widetilde{M} - p^{-1}(S_j)$ which meet $\widetilde{f}_t(S)$ as t varies over S^k and j varies arbitrarily. Let $\widetilde{V}_j$ be the closure of the component of $\widetilde{M} - p^{-1}(S_j)$ corresponding to v, with $\widetilde{S}_j$ its boundary component in the direction of $\widetilde{S}$. Let $\widetilde{\mathcal{C}}_t^j$ be the collection of components of $\widetilde{f}_t(S) \cap \widetilde{V}_j$ and let $\mathcal{C}_t^j$ be the (diffeomorphic) images of these components in $f_t(S)$. Let $\mathcal{C}_t = \bigcup_j \mathcal{C}_t^j$, the union over j such that $t \in U_j$.

Lemma. *For each surface $C \in \mathcal{C}_t^j$ there is a pinched product $P \subset M$ with $\partial_+ P = C$ and $\partial_- P \subset S_j$.*

Proof: This is proved in II.5 of [L], but let us sketch an argument which may be more direct. First observe that $\pi_1(\widetilde{V}_j, \widetilde{S}_j) = 0$, otherwise $\pi_1(\widetilde{S}_j) \to \pi_1(\widetilde{V}_j)$ would not be surjective and hence $\pi_1(\widetilde{S}) \to \pi_1(\widetilde{M})$ could not be an isomorphism. (This uses the fact that the components of $p^{-1}(S_j)$ are incompressible in $\widetilde{M}$.) Next, let V_j be M split along S_j, so we have a covering space $p\colon(\widetilde{V}_j, \widetilde{S}_j) \to (V_j, S_j)$. Since the inclusion $(C, \partial C) \hookrightarrow (V_j, S_j)$ lifts to $(\widetilde{V}_j, \widetilde{S}_j)$, the fact that $\pi_1(\widetilde{V}_j, \widetilde{S}_j) = 0$ implies that $\pi_1(C, \partial C) \to \pi_1(V_j, S_j)$ is zero. Thus there is a map $F\colon D^2 \to V_j$ giving a homotopy from a path α in C representing a nontrivial element of $\pi_1(C, \partial C)$ to a path β in S_j. We would like to improve F to be an embedding with $F(D^2) \cap C = \alpha$. To do this, first perturb F to be transverse to C. We can modify F to eliminate any circles of $F^{-1}(C)$, using the fact that C is incompressible in V_j. Arcs of $F^{-1}(C)$, other than α, which are trivial in $\pi_1(C, \partial C)$ can be eliminated similarly. An outermost remaining arc of $F^{-1}(C)$ can be used as a new α for which $F(D^2) \cap C = \alpha$. Now we apply the loop theorem to replace F by an embedding with the same properties. Having this embedded disk, we can isotope C by pushing α across the disk, surgering C to a simpler surface C' which, by induction on the complexity of C, splits off a pinched product P' from V_j; the induction starts with the case that C is a disk, where incompressibility of S_j and irreducibility of M gives the result. We recover C from C' by adjoining a "tunnel." If this tunnel lies outside P', then the tunnel enlarges P' to a pinched product P split off from V_j by C, as desired. The other alternative, that the tunnel lies inside P', cannot occur since C would then be compressible. □

Continuing with the main line of the proof, we would like to choose functions $\varphi_t\colon \mathcal{C}_t \to (0,1)$ satisfying:

(1) $\varphi_t(C)$ varies continuously with $t \in U_j$ for each $C \in \mathcal{C}_t^j$.

(2) $\varphi_t(C) < \varphi_t(C')$ if $C \subset C'$.

(3) φ_t is injective for each t.

As before, (1) and (2) are easy to arrange, and then (3) is achieved by replacing each S_j with a number of nearby copies of itself and rechoosing the cover $\{U_j\}$.

Having functions φ_t satisfying (1)-(3), we follow the same scheme as in Step 1 to construct an isotopy f_{tu} of f_t eliminating components $C \in \mathcal{C}_t$ during the corresponding u-intervals $[\varphi_t(C), \varphi_t(C) + \varepsilon]$. Again there are primary and secondary components, and only the primary components need be dealt with explicitly.

The end result of this family of isotopies f_{tu} is that the vertex v is no longer among the vertices of T corresponding to components of $\widetilde{M} - p^{-1}(S_j)$ meeting $\widetilde{f}_t(S)$ for $t \in U_j$, and no new such vertices have been introduced. So by iteration of the process we eventually reach the situation that $f_t(S)$ is disjoint from S_j for all $t \in U_j$ and all j.

Step 4. We can now finish the proof of (b) in the case that $\partial S \neq \varnothing$ by the argument in Step 2. The assumption that S was a disk rather than an arbitrary compact orientable surface with non-empty boundary was used in Step 2 only to deduce that $Diff(S \times I \text{ rel } \partial)$ had trivial homotopy groups, but the case $S = D^2$ suffices to show this in the present case since the handlebody $S \times I$ can be reduced to a ball by cutting along a collection of disjoint disks; see the proof of Theorem 2.

Step 5. Now we prove (a) when $\partial S \neq \varnothing$. Steps 1 and 3 work equally well in this case, with images of embeddings instead of actual embeddings. The only modification needed in steps 2 and 4 is to show that $P(S, M - S \times \{1\} \text{ rel } \partial S)$ has trivial π_i for $i > 0$. The basepoint component of $P(S, M - S \times \{1\} \text{ rel } \partial S)$ can be identified with the base space in the following fibration, where $A = S \times \{1\} \cup \partial S \times I$.

$$Diff(S \times I \text{ rel } A) \longrightarrow E(S \times I, M \text{ rel } A) \longrightarrow E(S \times I, M \text{ rel } A)/Diff(S \times I \text{ rel } A)$$

The total space $E(S \times I, M \text{ rel } A)$ is contractible, as in Step 2. And the fiber has trivial homotopy groups, as one can see from the fibration

$$Diff(S \times I \text{ rel } \partial(S \times I)) \longrightarrow Diff(S \times I \text{ rel } A) \longrightarrow Diff(S \times \{0\} \text{ rel } \partial S \times \{0\})$$

Step 6. This is to prove (a) when $\partial S = \varnothing$. Let Σ_t be a family of surfaces representing an element of $\pi_i P(S, M)$. Steps 1 and 3 work when $\partial S = \varnothing$, so we may assume Σ_t is disjoint from S_j for all $t \in U_j$ and all j. However, this does not imply Σ_t is disjoint from the regions between S_j's as it did when $\partial S \neq \varnothing$. Instead, let us look at the lifts $\widetilde{\Sigma}_t$ to $\widetilde{M}$, which are well defined at least when the parameter domain S^i is simply-connected, i.e., when $i > 1$. Let $\widetilde{V}_j$ and $\widetilde{S}_j$ be defined as in Step 3. Then $\widetilde{\Sigma}_t \subset \widetilde{V}_j$ for $t \in U_j$. If $\widetilde{S}$ is not contained in $\widetilde{V}_j$, then $\widetilde{S}$ and $\widetilde{\Sigma}_t$ are disjoint for $t \in U_j$, hence, since they are isotopic, the region between them in $\widetilde{M}$ is a product $\widetilde{S} \times I$. The surface $\widetilde{S}_j$ lies in this region, hence must be compact too. Since $\widetilde{S}_j$ is an incompressible surface in the interior of the product $\widetilde{S} \times I$, the region between $\widetilde{S}_j$ and $\widetilde{\Sigma}_t$ must be a product, projecting diffeomorphically to a product region between S_j and Σ_t. We can use such products in place of the pinched products in Step 3, and thus isotope the family Σ_t to a new family for which $\widetilde{V}_j$ does contain $\widetilde{S}$. Then we can easily isotope the family Σ_t to be disjoint from $S \times \{1\}$ for all t and apply the argument in Step 5 to finish the proof that $\pi_i P(S, M) = 0$ for $i > 1$.

When $i = 1$ we can replace the parameter domain S^1 by I and then the lifts $\widetilde{\Sigma}_t$ exist, with $\widetilde{\Sigma}_0 = \widetilde{S}$. The difficulty is that $\widetilde{\Sigma}_1$ may be a different lift of S from $\widetilde{S}$. If this happens, then in the notation of the preceding paragraph, $\widetilde{V}_j$ will not contain $\widetilde{S}$. In this case the product $\widetilde{S} \times I$ will contain another lift of S. The region between $\widetilde{S}$ and such a lift will also be a product $\widetilde{S} \times I$, which means that M must be a surface bundle with S as fiber. Then we have a surjective homomorphism $\pi_1 P(S,M) \to \mathbb{Z}$ which measures which lift of S to $\widetilde{M} = S \times \mathbb{R}$ the surface $\widetilde{\Sigma}_1$ is. On the kernel of this homomorphism the arguments of the preceding paragraph apply, and we deduce that this kernel is zero. Thus $\pi_1 P(S,M)$ is $\mathbb{Z}$, represented by fibers of the surface bundle. When M is not a surface bundle with fiber S we must have $\widetilde{\Sigma}_1 = \widetilde{S}$, and $\pi_1 P(S,M) = 0$.

Step 7. When $\partial S = \varnothing$ we can deduce (b) from (a) by looking at the long exact sequence of homotopy groups for the fibration $Diff(S) \to E(S,M) \to P(S,M)$. This is immediate except when M is a surface bundle with fiber S. In this special case, exactness at $\pi_1 E(S,M)$ implies that this fundamental group is represented by loops of embeddings $S \hookrightarrow M$ with image a fiber. □

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