ON AVERAGING AND HOMOGENIZATION IN NON-STANDARD SITUATION

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Introduction

The problem the limiting behavior of a slow component of multidimensional Markov diffusion process has many applications in physics, biology and other areas. One possible setting of this problem is as follows. Assume that we are given a Markov diffusion process $(X_\varepsilon(t), Y_\varepsilon(t))$ consisting of two components $X_\varepsilon(t)$ and $Y_\varepsilon(t)$ and depending on the parameter ε which tends to zero. Then we are interested in what happens if, as $\varepsilon \to 0$, $X_\varepsilon(t)$ changes faster and faster in time and $Y_\varepsilon(t)$ lives in the same time scale for all ε.

This problem was considered in [Kh68],[PSV77], [VF79],[Sk89], [PV1], [PV2] and many others. Roughly speaking, the main result can be described as follows. Let the generator $L_\varepsilon(x, y)$ of the process $(X_\varepsilon(t), Y_\varepsilon(t))$ is:

$$L_\varepsilon(x,y) = \varepsilon^{-2}L_1(x,y) + \varepsilon^{-1}L_2(x,y) + L_3(x,y),$$

$$L_1(x,y) = \sum_{i,j=1}^{l_1} a_{ij}^{(1)}(x,y)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{l_1} b_i^{(1)}(x,y)\frac{\partial}{\partial x_i}\,,$$

$$L_2(x,y) = \sum_{i=1}^{l_1}\sum_{j=1}^{l_2} a_{ij}^{(2)}(x,y)\frac{\partial^2}{\partial x_i \partial y_j}\,,$$

$$L_3(x,y) = \sum_{i,j=1}^{l_2} a_{ij}^{(3)}(x,y)\frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{l_2} b_i^{(3)}(x,y)\frac{\partial}{\partial y_i}\,.$$

Suppose that, for any fixed y, the Markov process $X^{x,y}(t)$ with generator $L_1(x,y)$ and satisfying $X^{x,y}(0) = x$ is ergodic. Assume the density of its stationary distribution exists and denote it by $\rho(x,y)$. Denote also for any function $g(x,y)$

$$\bar{g}(y) = \int g(x,y)\rho(x,y)dx$$

and let $\bar{Y}^y(t)$ be the Markov process starting at y with generator $\bar{L}_3$. Under some additional conditions on the coefficients, guaranteeing, in particular, compactness of the family of measures generated by $Y_\varepsilon^{x,y}(t)$ for $\varepsilon \to 0$ and weak uniqueness of the process $\bar{Y}^y(t)$, the averaging principle for the slow component was proved: $Y_\varepsilon^{x,y}(t) \xrightarrow{distr} \bar{Y}^y(t)$ for any x, y, as $\varepsilon \to 0$.

The related result also can be formulated in the PDE terms, making use of the probabilistic representation for the solution of Cauchy problem for parabolic PDE's.

Let $u_\varepsilon(t,x,y), x \in \mathbb{R}^{l_1}, y \in \mathbb{R}^{l_2}$, be a solution of the problem

$$\frac{\partial u_\varepsilon}{\partial t} = L_\varepsilon(x,y)u_\varepsilon + c(x,y)u_\varepsilon + f(x,y);\ u_\varepsilon(0,x,y) = \phi(x,y). \tag{1}$$

Assume that coefficients of the operator L_1 have the property: for any fixed y the problem

$$L_1^*(x,y)\rho(x,y) = 0;\ \int_{\mathbb{R}^{l_1}} \rho(x,y)dx = 1 \tag{2}$$

has a unique solution. (Sufficient conditions on coefficients for this property are well known, see, e.g., [PV1]). Assume also that all coefficients $a_{ij}^{(k)}, b_i^{(k)}$ are bounded and smooth enough . Then $u_\varepsilon(t,x,y) \to \bar{u}(t,y)$, as $\varepsilon \to 0$, here $\bar{u}(t,y)$ is a solution of the problem (see, e.g., [KY])

$$\frac{\partial \bar{u}}{\partial t} = \bar{L}_3(y)\bar{u}(t,y) + \bar{c}(y)\bar{u} + \bar{f}(y);\ \bar{u}(0,y) = \bar{\phi}(y). \tag{3}$$

Question: what is the limit behavior of u_ε if the fast component is not ergodic? In what follows we restrict ourself by the one-dimensional fast component, $l_1 = 1$. The fast component can be transient or null-recurrent. For the transient fast component the limit of $u_\varepsilon(t,x,y)$ can depend on the fast component x.

Assume, for instance, that the fast component can go to $+\infty$ and to $-\infty$ with positive probability, and $L_3(x,y) \to L_3^{\pm}(y), \phi(x,y) \to \phi^{\pm}(y)$ as $x \to \pm\infty$, and $L_2(x,y) = 0$. Then it is not hard to prove that a solution $u_\varepsilon(t,x,y)$ of the problem

$$\frac{\partial u_\varepsilon}{\partial t} = L_\varepsilon(x,y)u_\varepsilon;\ u_\varepsilon(0,x,y) = \phi(x,y),$$

has the following limiting behavior: for $t > \delta > 0$

$$\lim_{\varepsilon \to 0} u_\varepsilon(t,x,y) = p(x,y)u^+(t,y) + (1 - p(x,y))u^-(t,y),$$

here the function $p(x,y)$ is a solution of the problem

$$L_1(x,y)p(x,y) = 0;\ p(-\infty,y) = 0, p(+\infty,y) = 1$$

(it can be written at the explicit form) and $u^{\pm}(t,y)$ is a solution of the problem

$$\frac{\partial u^{\pm}}{\partial t} = L_3^{\pm}(y)u^{\pm};\ u^{\pm}(0,y) = \phi^{\pm}(y).$$

Averaging for null-recurrent fast component

The simplest case of the null-recurrent one-dimensional fast component is the case where $L_1(x,y) = a_{11}^{(1)}(x,y)\frac{\partial^2}{\partial x^2}$ with

$$0 < c_1 \le a_{11}^{(1)}(x,y) \le c_2 < \infty. \tag{4}$$

Thus we have considered in [KK] the diffusion process with two scales and generator

$$\begin{aligned} L_\varepsilon(x,y) &= \varepsilon^{-2} a_{00}(x,y)\frac{\partial^2}{\partial x^2} + 2\varepsilon^{-1}\sum_{i=1}^{d} a_{i0}(x,y)\frac{\partial^2}{\partial x \partial y_i} \\ &+ \sum_{i,j=1}^{d} a_{ij}(x,y)\frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{d} b_i(x,y)\frac{\partial}{\partial y_i} \\ := &\ \varepsilon^{-2}L_1(x,y) + \varepsilon^{-1}L_2(x,y) + L_3(x,y) \end{aligned} \tag{5}$$

We suppose that the following conditions concerning the coefficients of (5) are valid.

A1. *The coefficients a_{ij}, b_i are Lipschitz continuous in (x,y) and, for each x, their derivatives in y up to and including second order derivatives are bounded continuous functions of y.*

A2. *For some positive constants c_1, c_2 the condition (4) is valid and for some constant $c_3 > 0$*

$$\sum_{i=1}^{d} (a_{ii}(x,y) + b_i^2(x,y)) \le c_3(1 + |y|^2).$$

We denote below $p(x,y) = (a_{00}(x,y))^{-1}$.

B1. *The function* $p(x,y)$ *has a limit* $p^{\pm}(y)$ *in Ċesaro sense, as* $x \to \pm\infty$*:*

$$\lim_{x\to\pm\infty} x^{-1}\int_0^x p(z,y)dz = p^{\pm}(y)$$

uniformly in $y \in \mathsf{R}^d$ *and, moreover, also uniformly in* $y \in \mathsf{R}^{\mathsf{d}}$,

$$\lim_{x\to\pm\infty} x^{-1}\int_0^x \frac{\partial p(z,y)}{\partial y}dz = \frac{\partial p^{\pm}(y)}{\partial y},$$

$$\lim_{x\to\pm\infty} x^{-1}\int_0^x \frac{\partial^2 p(t,y)}{\partial y^2}dt = \frac{\partial^2 p^{\pm}(y)}{\partial y^2}.$$

We suppose further that the coefficients $a_{ij}(x,y), b_i(x,y), i = 1,...,d,\ j = 0,...,d$ and all their derivatives in y up to the second order also have averages in x, as $x \to \pm\infty$ with the weight $p(x,y)$. To write these conditions in a compact form we use the following notation: for any function $K(x,y)$ having the limit in Ċesaro sense as $x \to \pm\infty$,

$$K^+(y) := \lim_{x\to+\infty} x^{-1}\int_0^x K(t,y)dt,$$

$$K^-(y) := \lim_{x\to-\infty} x^{-1}\int_0^x K(t,y)dt;$$

$$K^{\pm}(x,y) := K^+(y)\mathbf{1}_{\{x>0\}} + K^-(y)\mathbf{1}_{\{x\le 0\}}.$$

For a function $K(x,y)$ we write $K \in \mathcal{K}$ if $K^{\pm}$ exists and

$$x^{-1}\int_0^x K(z,y)dz - K^{\pm}(x,y) = (1+|y|^2)\alpha(x,y).$$

Here (and below) the function α is bounded and satisfies

$$\lim_{|x|\to\infty}\sup_{y\in R^d}|\alpha(x,y)| = 0.$$

B2. *For* $i=1,...,d,\ j=0,...,d,$ *we have*

$$pb_i, D_y(pb_i), D_y^2(pb_i), pa_{ij}, D_y(pa_{ij}), D_y^2(pa_{ij}) \in \mathcal{K}.$$

Introduce also the notations

$$\bar{a}_{ij}(x,y) = \frac{(a_{ij}p)^{\pm}(x,y)}{p^{\pm}(x,y)}, \quad \bar{b}(x,y) = \frac{(bp)^{\pm}(x,y)}{p^{\pm}(x,y)};\ \bar{A}(x,y) = (\bar{a}_{ij}(x,y))_{i,j=0,1,...,d}.$$

Observe, in passing, that $\bar{a}_{00}(x,y) = (p^{\pm}(x,y))^{-1}$.

Consider the Markov diffusion process $\tilde{Z}(t) = (\tilde{X}_0(t), \tilde{Y}_0(t))$ with the diffusion matrix $\bar{A}(x,y)$ and the drift vector $(0,\bar{b}(x,y))^*$ and denote $\bar{L}(x,y)$ the generator of this process.

It is proven in [KK] that the slow component only does not converge to any limiting process, but the pair $(\varepsilon X_\varepsilon(t), Y_\varepsilon(t))$ converges weakly to the limit $\tilde{Z}(t)$ if the diffusion matrix $\bar{A}(x,y)$, the drift vector $(0,\bar{b}(x,y))^*$ and the initial conditions determine the distribution of $\tilde{Z}(t)$ uniquely. These coefficients are smooth functions only in each half-space $\{x>0\}$ and $\{x<0\}$ and can have jumps at the hyperplane $x=0$. This circumstance makes not self-evident the uniqueness of the process with this generator, which is essential for our approach. We have considered in [KK] some examples where this uniqueness has place, and, in general, added this uniqueness condition to our Theorem. But recently Krylov found in [Kr 03] that the weak uniqueness for the solution of the appropriate SDE's and PDE's always has place under conditions A, B. The generator of $\tilde{Z}_\varepsilon(t)$ is

$L_1(x/\varepsilon,y)+L_2(x/\varepsilon,y)+L_3(x/\varepsilon,y)=L(x/\varepsilon,y)$

In PDE's terms the result can be formulated by the following way.

Theorem 1. *Let the conditions* A, B *be fulfilled and* $c(x,y)\in\mathcal{K}, f(x,y)\in\mathcal{K}$. *Denote* $\bar{c}(x,y)=\frac{(pc)^\pm(x,y)}{p^\pm(x,y)}$; $\bar{f}(x,y)=\frac{(pf)^\pm(x,y)}{p^\pm(x,y)}$; $\bar{L}(x,y)=\frac{(pL)^\pm(x,y)}{p^\pm(x,y)}$. *Then the solution of the Cauchy problem*

$\frac{\partial u_\varepsilon}{\partial t}=L(x/\varepsilon,y)u_\varepsilon+c(x/\varepsilon,y)u_\varepsilon+f(x/\varepsilon,y);\ u_\varepsilon(0,x,y)=\varphi(x,y) \quad (6)$

converges, as $\varepsilon\to 0$, *to the unique bounded solution*

$\bar{u}(x,y,t)\in\mathcal{W}^{2,1}_{d+1,loc}$ *of the problem*

$\frac{\partial \bar{u}}{\partial t}=\bar{L}(x,y)\bar{u}+\bar{c}(x,y)\bar{u}+\bar{f}(x,y);\ \bar{u}(0,x,y)=\varphi(x,y). \quad (7)$

The natural question: is it possible to include the case of initial conditions also depending on x/ε?

Corollary 1. *Let the conditions of the Theorem 1 be valid and $\varphi(x,y) \in \mathcal{K}$ also. Denote $\bar{\varphi}(x,y) = \frac{(\varphi p)^{\pm}(x,y)}{p^{\pm}(x,y)}$. Then the solution of the Cauchy problem*

$$\frac{\partial u_\varepsilon}{\partial t} = L(x/\varepsilon, y)u_\varepsilon + c(x/\varepsilon, y)u_\varepsilon + f(x/\varepsilon, y);\ u_\varepsilon(0,x,y) = \varphi(x/\varepsilon, y) \quad (8)$$

converges for $t > \delta > 0$, as $\varepsilon \to 0$, to the unique bounded solution

$\bar{u}(x,y,t) \in \mathcal{W}^{2,1}_{d+1,loc}$ of the problem

$$\frac{\partial \bar{u}}{\partial t} = \bar{L}(x,y)\bar{u} + \bar{c}(x,y)\bar{u} + \bar{f}(x,y);\ \bar{u}(0,x,y) = \bar{\varphi}(x,y). \quad (9)$$

The proof of this Corollary can be obtained by combining the probabilistic approach of [KK] and the known results about Hölder continuity the solution of parabolic equations with bounded coefficients for $t > \delta > 0$ (see [Kr85]).

Homogenization

Homogenization problems for parabolic PDE were considered in many publications, see, for instance, [F], [PSV], [JKO], [BLP] and references there. It is supposed in the most part of these references that the coefficients of the PDE are *periodic* in all space variables. Now we first answer the question: what result concerning homogenization follows from the Theorem 1 and Corollary?

Consider the Cauchy problem for the parabolic PDE

$$\frac{\partial u_\varepsilon}{\partial t} = a_{00}(x/\varepsilon)\frac{\partial^2 u_\varepsilon}{\partial x^2} + c(x/\varepsilon)u_\varepsilon + f(x/\varepsilon); \quad u_\varepsilon(0,x) = \phi(x/\varepsilon). \quad (10)$$

Assume that $c_1 \le a_{00}(x) = p^{-1}(x) \le c_2$ $(c_1 > 0)$, $a_{00}(x)$ has two bounded derivatives, and the function $p(x)$ has limits $p^{\pm}$ in Ȁesaro sense, as $x \to \pm\infty$.

Assume also that the functions $p(x)c(x)$, $p(x)f(x)$, $p(x)\varphi(x)$ also have limits in Ȁesaro sense, as $x \to \pm\infty$. Then all conditions of the Corollary are fulfilled for the problem (10), and one can deduce from the Corollary that $u_\varepsilon(t,x) \to \bar{u}(t,x)$, as $\varepsilon \to 0$, here $\bar{u}(t,x)$ is a solution of the problem

$\frac{\partial \bar{u}}{\partial t} = \bar{a}(x)\frac{\partial^2 \bar{u}}{\partial x^2} + \bar{c}(x) + \bar{f}(x); \quad \bar{u}(0,x) = \bar{\varphi}(x).$

Here $\bar{a}(x) = 1/p^{\pm}(x), \quad \bar{c}(x) = \frac{(pc)^{\pm}(x)}{p^{\pm}(x)}, \quad \bar{f}(x) = \frac{(pf)^{\pm}(x)}{p^{\pm}(x)}, \quad \bar{\varphi}(x) = \frac{(p\varphi)^{\pm}(x)}{p^{\pm}(x)}.$

A natural question arises: is it possible to use the Theorem 1 for justification the homogenization to the parabolic equations on $\mathbb{R}^+ \times \mathbb{R}^1$ with the drift term? In other words, can we apply the Theorem 1 for the analysis of the limiting behavior of the equation

$$\frac{\partial u_\varepsilon}{\partial t} = a_{00}(x/\varepsilon)\frac{\partial^2 u_\varepsilon}{\partial x^2} + b_0(x/\varepsilon)\frac{\partial u_\varepsilon}{\partial x} + c(x/\varepsilon)u_\varepsilon + f(x/\varepsilon); \quad u_\varepsilon(0,x) = \phi(x/\varepsilon)? \quad (11)$$

Unfortunately, the answer to this question is negative: the equation of the 'fast' component in the Theorem 1 does not have a drift term. Nevertheless, it is well known, that for the case periodic coefficients homogenization has place .

It was noticed in [KhKl] that the Theorem 1 can be generalized by the following way. Instead of the process with the generator

$$L_\varepsilon(x,y) = \varepsilon^{-2}a_{00}(x,y)\frac{\partial^2}{\partial x^2} + 2\varepsilon^{-1}\sum_{i=1}^d a_{i0}(x,y)\frac{\partial^2}{\partial x \partial y_i} + \sum_{i,j=1}^d a_{ij}(x,y)\frac{\partial^2}{\partial y_i \partial y_j}$$
$$+ \sum_{i=1}^d b_i(x,y)\frac{\partial}{\partial y_i}$$

the averaging principle can be justified for the process $\hat{Z}_\varepsilon(t) = (\hat{X}_\varepsilon(t), \hat{Y}_\varepsilon(t))$ with the generator

$$\hat{L}_\varepsilon(x,y) = \varepsilon^{-2}a_{00}(x,y)\frac{\partial^2}{\partial x^2} + \varepsilon^{-1}b_0(x,y)\frac{\partial}{\partial x} + 2\varepsilon^{-1}\sum_{i=1}^d a_{i0}(x,y)\frac{\partial^2}{\partial x \partial y_i}$$
$$+ \sum_{i,j=1}^d a_{ij}(x,y)\frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^d b_i(x,y)\frac{\partial}{\partial y_i}. \quad (12)$$

Remark 1. Due to the presence the drift term b_0 in (11) the process $\tilde{X}_\varepsilon(t)$ with the generator $\varepsilon^{-2}a_{00}(x,y)\frac{\partial^2}{\partial x^2}+\varepsilon^{-1}b_0(x,y)\frac{\partial}{\partial x}$ with *fixed* y can be not necessarily null-recurrent, as in [KK], but, may be, positive recurrent or transient. Nevertheless, the averaging result in the theorem below is of the same type as for the process with null-recurrent fast component (as in [KK]) due to the fact that the drift term in this operator is small in comparison with the diffusion term.

We use here and below the same notations and assumptions concerning coefficients of the operator $\hat{L}_\varepsilon(x,y)$ as above. There are only the following differences. *We include* b_0 *in the smoothness condition* A.1, *the summation in the condition* A.2 *starts from* $i=0$, *and assume, in addition, that* $pb_0 \in \mathcal{K}$ *in the condition* B.2. Denote by $\tilde{A}, \tilde{B}$ the modified by this way conditions A, B.

As before , we consider the limiting behavior of the process $Z^{(1)}_\varepsilon(t) = (\varepsilon\hat{X}_\varepsilon(t), \hat{Y}_\varepsilon(t))$. It is easy to see from (9) that the generator of the process $Z^{(1)}_\varepsilon(t)$ has the form

$$L_{(1)}(x/\varepsilon,y) = a_{00}(x/\varepsilon,y)\frac{\partial^2}{\partial x^2} + b_0(x/\varepsilon,y)\frac{\partial}{\partial x} + 2\sum_{i=1}^d a_{i0}(x/\varepsilon,y)\frac{\partial^2}{\partial x\partial y_i}$$

$$+ \sum_{i,j=1}^d a_{ij}(x/\varepsilon,y)\frac{\partial^2}{\partial y_i\partial y_j} + \sum_{i=1}^d b_i(x/\varepsilon,y)\frac{\partial}{\partial y_i}. \quad (12)$$

Introduce also the differential operator

$\bar{L}_{(1)}(x,y) = \bar{a}_{00}(x,y)\frac{\partial^2}{\partial x^2} + \bar{b}_0(x,y)\frac{\partial}{\partial x} + 2\sum_{i=1}^d \bar{a}_{i0}(x,y)\frac{\partial^2}{\partial x \partial y_i}$

$+\sum_{i,j=1}^d \bar{a}_{ij}(x,y)\frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^d \bar{b}_i(x,y)\frac{\partial}{\partial y_i}.$ (13)

Theorem 2. *Let the conditions of Theorem 1 with replacement the conditions A, B by $\tilde{A}, \tilde{B}$ are valid. Then solution of the Cauchy problem*

$\frac{\partial u_\varepsilon}{\partial t} = L_{(1)}(x/\varepsilon, y)u_\varepsilon + c(x/\varepsilon, y)u_\varepsilon + f(x/\varepsilon, y);\ u_\varepsilon(0,x,y) = \varphi(x/\varepsilon, y)$ (14)

converges, as $\varepsilon \to 0$, to the unique bounded solution

$\bar{u}(x,y,t) \in \mathcal{W}^{2,1}_{d+1,loc}$ *of the problem*

$\frac{\partial \bar{u}}{\partial t} = \bar{L}_{(1)}(x,y)\bar{u} + \bar{c}(x,y)\bar{u} + \bar{f}(x,y);\ \bar{u}(0,x,y) = \bar{\varphi}(x,y).$ (15)

Consider now the Cauchy problem for the one-dimensional parabolic PDE

$$\frac{\partial v_\varepsilon}{\partial t} = a(x/\varepsilon)\frac{\partial^2 v_\varepsilon}{\partial x^2} + b(x/\varepsilon)\frac{\partial v_\varepsilon}{\partial x} + c(x/\varepsilon)v_\varepsilon + f(x/\varepsilon);\ v_\varepsilon(0,x) = \phi(x/\varepsilon). \quad (16)$$

The next corollary follows from this theorem.

Corollary 2. *Assume that the coefficients of (16) are Lipschitz continuous, $a(x)$ lies between two positive constants, and the functions*

$1/a(x) = p(x), b(x)p(x), c(x)p(x), f(x)p(x), \varphi(x)p(x)$ *are bounded and have limits* $p^{\pm}(x),\ (bp)^{\pm}(x),\ (cp)^{\pm}(x),\ (fp)^{\pm}(x),\ (\varphi p)^{\pm)}$ *in Çesaro sense, as $x \to \pm\infty$. Then the solution $v_\varepsilon(t,x)$ of the problem (16) converges, as $\varepsilon \to 0$, to the solution of the* *problem*

$$\frac{\partial v_0}{\partial t} = \bar{a}(x)\frac{\partial^2 v_0}{\partial x^2} + \bar{b}(x)\frac{\partial v_0}{\partial x} + \bar{c}(x)v_0 + \bar{f}(x); \quad (17)$$

$$v_0(0,x) = \bar{\varphi}(x) \qquad (18)$$

Here, as before, $\bar{a}(x) = 1/p^- \mathbf{1}_{\{x<0\}} + 1/p^+ \mathbf{1}_{\{x\geq 0\}}$, $\bar{b}(x) = \frac{(pb)^-}{p^-}\mathbf{1}_{\{x<0\}} + \frac{(pb)^+}{p^+}\mathbf{1}_{\{x\geq 0\}}$ *and the functions* $\bar{c}(x)$, $\bar{f}(x)$, $\bar{\varphi}(x)$ *are defined analogously.*

Remark 2. The coefficients and the initial condition for the problem (17, (18) are constants in each half-plane $\{x < 0\}$ and $\{x > 0\}$. This circumstance allows to write the solution of this problem explicitly.

Remark 3. For the special case $p^+ = p^-$, $(pb)^+ = (pb)^-$ and so on this result generalizes the homogenization result for the parabolic equations in $\mathbb{R}^+ \times \mathbb{R}^1$ with periodic or almost periodic coefficients. (See, e.g., [F], [BLP])

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