# Relations on solid angles of low-dimensional polytopes

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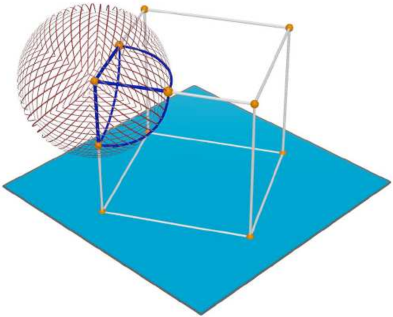
# Interior Angles

Let $P$ be a $d$-polytope. The interior angle $\alpha(F, P)$ measures the fraction of directions that one can move from a face $F$ into $P$.

That is,

$$\alpha(F, P) = \frac{vol(S_\varepsilon(x) \cap P)}{vol(S_\varepsilon(x))}$$

where $x$ is in the interior of $F$ and $S_\varepsilon(x)$ is the $(d-1)$-sphere of radius $\varepsilon$ centered at $x$ for $\varepsilon$ sufficiently small.



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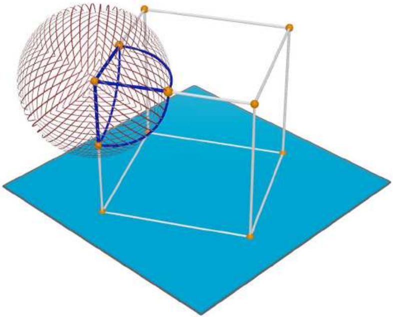

# Angle Sums

The angle sums of $P$ for $0 \leq i \leq d$ are

$$\alpha_i(P) = \sum_{i-\text{faces } F \subseteq P} \alpha(F, P).$$

In particular, note that $\alpha_{d-1}(P) = \frac{1}{2} f_{d-1}(P)$ and $\alpha_d(P) = 1$.
We say $\alpha_{-1}(P) = 0$.

The $\alpha$-vector of $P$ is

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# Relations on the $\alpha$-vector

## Theorem (Gram Relation (1874, 1967))

*For any $d$-polytope $P$,*

$$\sum_{i=0}^{d-1} (-1)^i \alpha_i(P) = (-1)^{d-1}.$$

## Theorem (Perles Relations, 1967)

*For any simplicial polytope $P$ and $-1 \leq k \leq d-1$,*

$$\sum_{j=k}^{d-1} (-1)^j \binom{j+1}{k+1} \alpha_j(P) = (-1)^d (\alpha_k(P) - f_k(P)).$$

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# The $\gamma$-vector

Likewise, we define the $\gamma$-vector as an analog to the $h$-vector: $\gamma(P) = (\gamma_0(P), \gamma_1(P), \dots, \gamma_d(P))$ where

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# The Perles relations and the $\gamma$-vector

**Theorem (Perles Relations)**

*For a simplicial $d$-polytope $P$, $\gamma_i(P) + \gamma_{d-i}(P) = h_i(P)$ for $0 \leq i \leq d$.*

Questions:

- Are $\gamma$-vectors non-negative?
- Are $\gamma$-vectors unimodal?
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# Simplices

**Theorem**

*The $\gamma$-vectors of a $d$-simplex $\Delta^d$, $d \leq 4$, are non-decreasing and therefore non-negative.*

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# Simplices: Bounds on $\gamma_1$

We know that $\gamma_1(P) = \alpha_0(P)$ for all polytopes $P$.

**Lemma**

*For all simplices $\Delta$, $\gamma_1(\Delta) \leq \frac{1}{2}$.*

This is based on two facts:

Given a realization of a simplex $\Delta$ in a fixed coordinate system,

- No vector is in the interior of two vertex angles; the vertex angles are disjoint.
- If a vector is in the union of the interiors of the vertex angles, its negation is not.

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# Non-decreasing $\gamma$-vectors for Simplices

Let $\Delta^d$ be a $d$-simplex.

Recall that $\gamma_0(\Delta^d) = 0$ and $\gamma_d(\Delta^d) = 1$.

Since $0 \leq \gamma_1(\Delta^d) < \frac{1}{2}$ and $\gamma_1(\Delta^d) + \gamma_{d-1}(\Delta^d) = h_1(\Delta^d) = 1$, $\frac{1}{2} < \gamma_{d-1}(\Delta^d) \leq 1$.

Also, we know that $\gamma_2(\Delta^4) = \frac{1}{2}$ by the Perles relations.

It follows that the $\gamma$-vectors of $d$-simplices, $d \leq 4$, have non-decreasing $\gamma$-vectors, and are therefore non-negative.

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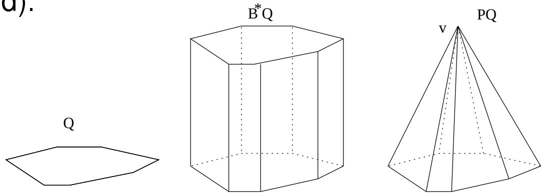
# Pyramid and Prism Constructions

We will consider the effect of pyramid and prism constructions on the $\gamma$-vector. We begin with a $(d-1)$-polytope $Q$ placed in the $x_d = 0$ plane in $\mathbb{R}^d$ and choose a height $x_d = k$ for the second copy of $Q$ (directly above the first) or the apex (above the centroid).

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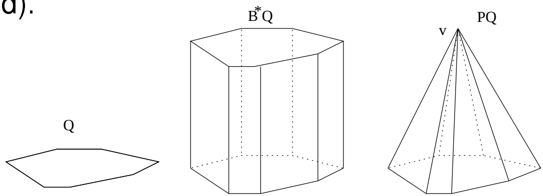


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# Effect of Constructions on the $\gamma$-vector

We can show the following facts based on the effect of the constructions on the $\alpha$-vector.

$$\gamma(B^* Q) = (\gamma(Q), 1).$$

$$\gamma(P_0 Q) = (0, \frac{1}{2} h_0(Q), \frac{1}{2} h_1(Q), \ldots, \frac{1}{2} h_{d-2}(Q), 1).$$

$$\gamma(P_\infty Q) = \frac{1}{2} \left[ (0, \gamma(Q)) + (\gamma(Q), 1) \right].$$

We can get $\gamma$-vectors arbitrarily close to these idealized $\gamma$-vectors.

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# Observations about Pyramid and Prism Constructions

- Based on the constructions, by taking a very short pyramid over a simplicial polytope $Q$, we can have $\gamma$-vectors $(0, \frac{1}{2}h_0(Q), \frac{1}{2}h_1(Q), \ldots, \frac{1}{2}h_{d-2}(Q), 1)$, which are clearly non-negative and peak in the middle.
- By taking prisms over such short pyramids, we can make the peak of the $\gamma$-vector move as far forward of the middle as we wish.
- Using any of the three constructions, we would preserve non-negativity and unimodality of the $\gamma$-vector of the base.

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# 3-Dimensional Pyramids and Prisms

*(Joint work with Robert Zima)*

**Lemma**

*For all 3-dimensional pyramids over an n-gon $P_n$*
$\gamma_1(P_n) + \gamma_2(P_n) = \frac{n-1}{2}$.

**Theorem**

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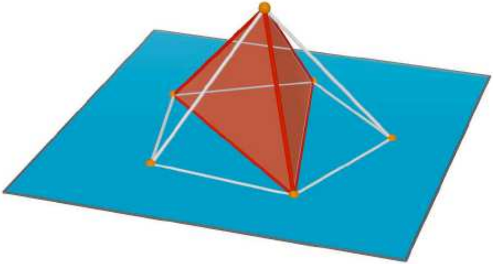
# Non-negativity of $\gamma(P_n)$

To prove this, we subdivide $P_n$ into $n-2$-simplices $T_i$, $i = 1, \ldots, n-2$ and use earlier results about the non-negativity of simplices.



We know $\gamma_0(P_n) = 0$, $\gamma_1(P_n) = \alpha_0(P_n) > 0$, and $\gamma_3(P_n) = 1$.

We compute

- $\gamma_1(P_n)+\gamma_2(P_n) = \alpha_1(P_n)-\alpha_0(P_n) = \alpha_2(P_n)-\alpha_3(P_n) = \frac{n-1}{2}$.
- $\gamma_1(P_n) = \alpha_0(P_n) = \sum_{i=1}^{n-2} \alpha_0(T_i) < \frac{n-2}{2}$.
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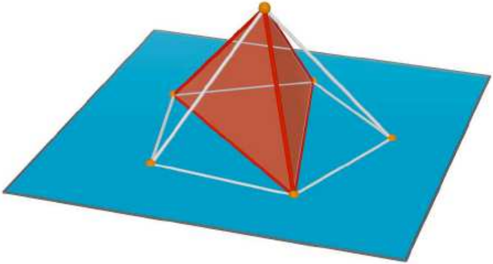

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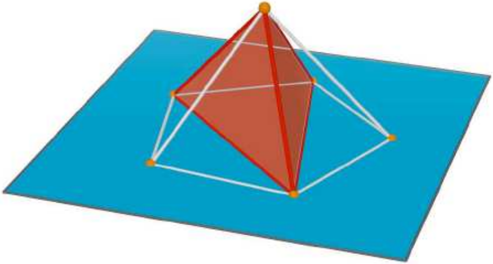

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- $\gamma_1(P_n)+\gamma_2(P_n) = \alpha_1(P_n)-\alpha_0(P_n) = \alpha_2(P_n)-\alpha_3(P_n) = \frac{n-1}{2}$.
- $\gamma_1(P_n) = \alpha_0(P_n) = \sum_{i=1}^{n-2} \alpha_0(T_i) < \frac{n-2}{2}$.
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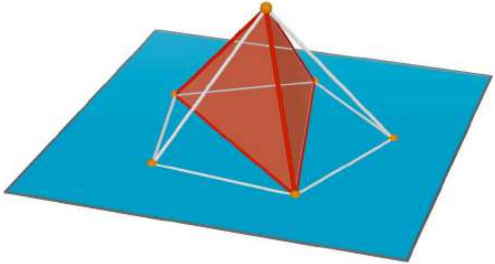
# Non-negativity of $\gamma(P_n)$

To prove this, we subdivide $P_n$ into $n-2$-simplices $T_i$, $i = 1, \ldots, n-2$ and use earlier results about the non-negativity of simplices.



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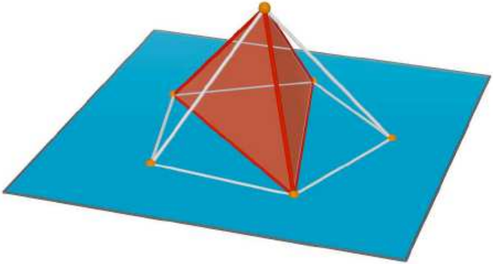

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# Unimodality and Prisms

$\gamma(P_n)$ is unimodal.

- If $\gamma_2(P_n) \leq 1$, then the $\gamma$-vector is non-decreasing.
- If $\gamma_2(P_n) > 1$, then the $\gamma$-vector is unimodal with peak at $\gamma_2$.

We have not yet found a subdivision of a prism that gives a conclusive result for non-negativity of general prisms. The best result is that $\gamma_2(B_n) > -n + 3$, so all triangular prisms have non-negative $\gamma$-vectors.

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# Non-negativity and Unimodality Summary

- 3- and 4-simplices have non-decreasing, non-negative $\gamma$-vectors. It is conjectured that all simplices have non-decreasing, non-negative $\gamma$-vectors.
- Prisms and pyramids do not all have non-decreasing $\gamma$-vectors. However, all 3-pyramids have non-negative $\gamma$-vectors.
- Since the prism and pyramid constructions preserve non-negativity and unimodality, repeated prisms and pyramids over 3-simplices, 4-simplices, triangular prisms or 3-pyramids have non-negative, unimodal $\gamma$-vectors.

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# Dimension of spaces of $\gamma$-vectors.

Using the pyramid and prism constructions in a manner similar to Bayer and Billera (1984), we can span the spaces of vectors defined by the Gram and Perles relations.

**Theorem**

*The affine span of the $\gamma$-vectors of $d$-simplices has dimension $\left\lfloor \frac{d-1}{2} \right\rfloor$.*

**Theorem**

*The affine span of the $\gamma$-$h$-vectors of simplicial $d$-polytopes has dimension $d - 1$.*

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*The affine span of the $\gamma$-$h$-vectors of general $d$-polytopes has dimension $2d - 3$ for $d \geq 2$.*

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# Open Questions

- Do all simplices have non-decreasing $\gamma$-vectors?
- Do all polytopes have non-negative $\gamma$-vectors? If not, for which class of polytopes do we have non-negativity?
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