Links We Almost Missed Between Delannoy Numbers and Legendre Polynomials

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Delannoy numbers

$$d_{i,j} = d_{i-1,j} + d_{i,j-1} + d_{i-1,j-1}$$

$d_{i,j} :=$

i \ j	0	1	2	3	4
0	1	1	1	1	1
1	1	3	5	7	9
2	1	5	13	25	41
3	1	7	25	63	129
4	1	9	41	129	321

They count the number of lattice paths from $(0,0)$ to (m,n) using only steps $(1,0)$, $(0,1)$, and $(1,1)$.

$\Rightarrow d_{n,n} = \sum_{j=0}^{n} \binom{n}{j}\binom{n+j}{j}$.

(Defined by Henri Delannoy (1895), Sulanke has ≥ 29 interpretations.)

A mysterious relation with the Legendre polynomials

Good (1958), Lawden (1952), Moser and Zayachkowski (1963) observed that

$$d_{n,n} = P_n(3),$$

where $P_n(x)$ is the n-th Legendre polynomial.

There has been a consensus that this link is not very relevant.

Banderier and Schwer (2004): "there is no "natural" correspondence between Legendre polynomials and these lattice paths."

Sulanke (2003): "the definition of Legendre polynomials does not appear to foster any combinatorial interpretation leading to enumeration".

("Your nose is big." (Rostand: Cyrano de Bergerac))

Jacobi and Legendre polynomials

Usual definition of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$:

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= (-2)^{-n}(n!)^{-1}(1-x)^{-\alpha}(1+x)^{-\beta} \\ &\quad \frac{d^n}{dx^n}\left((1-x)^{n+\alpha}(1+x)^{n+\beta}\right). \end{aligned}$$

$\alpha, \beta > -1$ "for integrability purposes", $\alpha = \beta = 0$ gives Legendre.

The formula below extends to all $\alpha, \beta \in \mathbb{C}$ (see Szegő (4.21.2)):

$$P_n^{(\alpha,\beta)}(x) = \sum_j \binom{n+\alpha+\beta+j}{j}\binom{n+\alpha}{n-j}\left(\frac{x-1}{2}\right)^j.$$

Substitute $\alpha = \beta = 0$:

$$P_n^{(0,0)}(x) = \sum_j \binom{n+j}{j}\binom{n}{j}\left(\frac{x-1}{2}\right)^j$$

is the n-th Legendre polynomial.

Part I: Balanced simplicial complexes

Asymmetric Delannoy numbers

$\widetilde{d}_{m,n} :=$

$m \backslash n$	0	1	2	3	4
0	1	2	4	8	16
1	1	3	8	20	48
2	1	4	13	38	104
3	1	5	19	63	192
4	1	6	26	96	321

$\widetilde{d}_{m,n}$ is the number of lattice paths from $(0,0)$ to $(m, n+1)$ having steps $(x,y) \in \mathbb{N} \times \mathbb{P}$.

(Variant of A049600 in the On-Line Encyclopedia of Integer Sequences.)

Lemma 1 *The asymmetric Delannoy numbers satisfy*

$$\widetilde{d}_{m,n} = \sum_{j=0}^{n} \binom{n}{j}\binom{m+j}{j}.$$

Proof: We are enumerating sequences $(0,0) = (x_0, y_0), (x_1, y_1), \ldots, (x_j, y_j), (x_{j+1}, y_{j+1}) = (m, n+1)$, where $0 \le j \le n$, $0 = x_0 \le x_1 \le \cdots \le x_j \le x_{j+1} = m$, and $0 = y_0 < y_1 < \cdots < y_j < y_{j+1} = n+1$. For a given j there are $\binom{m+j}{j}$ ways to choose $0 = x_0 \le x_1 \le \cdots \le x_j \le x_{j+1} = m$ and $\binom{n}{j}$ ways to choose $0 = y_0 < y_1 < \cdots < y_j < y_{j+1} = n+1$. ◇

Since

$$P_n^{(0,\beta)}(x) = \sum_j \binom{n+\beta+j}{j}\binom{n}{j}\left(\frac{x-1}{2}\right)^j,$$

we get

$$\widetilde{d}_{m,n} = P_n^{(0,m-n)}(3) \quad \text{for } m \ge n$$

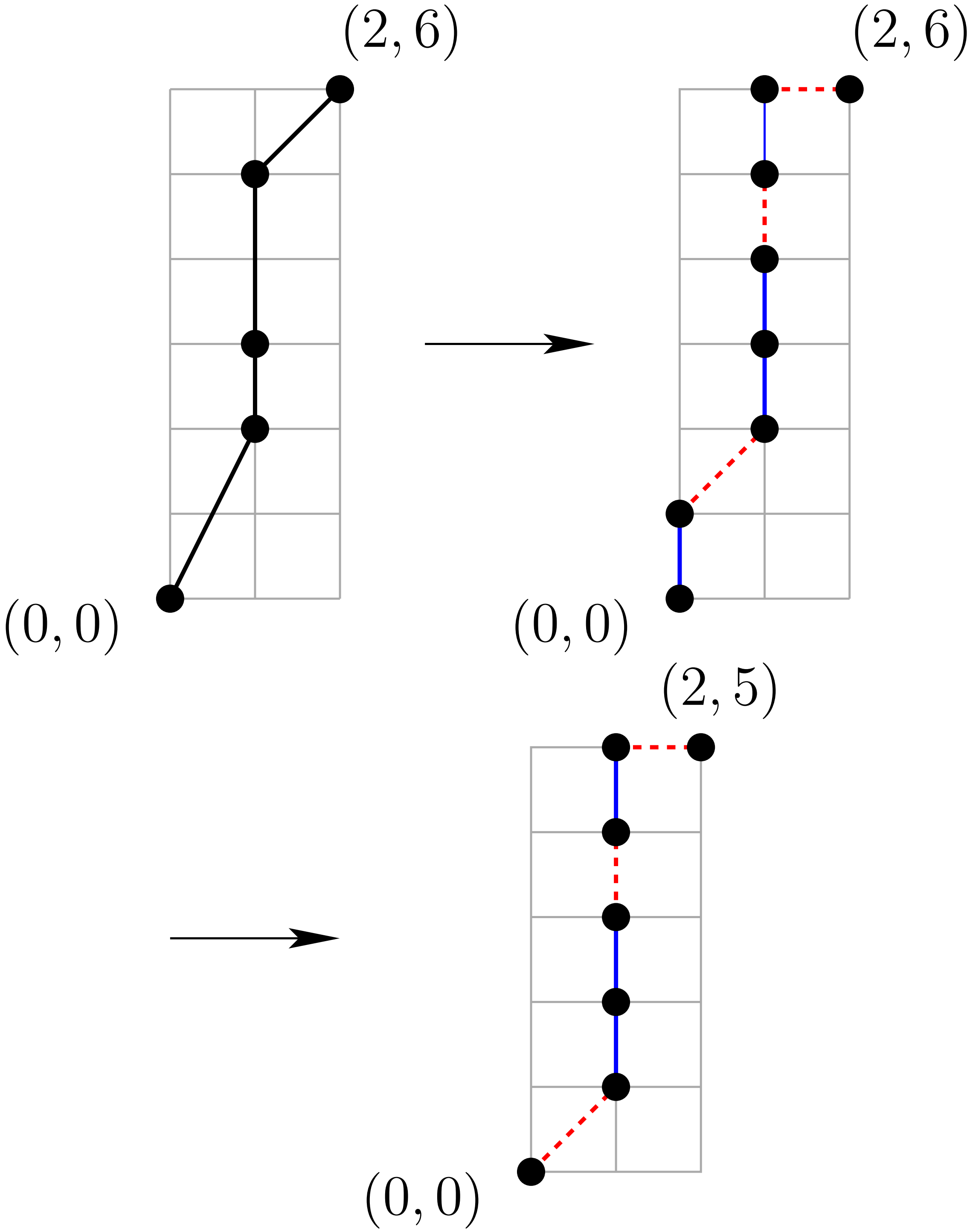
because $\dfrac{3-1}{2} = 1$.

A related 2-colored path enumeration problem

Proposition 1 *For any* $(m,n) \in \mathbb{N} \times \mathbb{N}$ *the number* $\widetilde{d}_{m,n}$ *also enumerates all* 2*-colored lattice paths from* $(0,0)$ *to* (m,n) *satisfying the following:*

(*i*) *Each step is either a blue* $(0,1)$ *or a red* $(x,y) \in \mathbb{N} \times \mathbb{N} \setminus \{(0,0)\}$.

(*ii*) *At least one of any two consecutive steps is a blue* $(0,1)$.



($m = 2$ and $n = 5$.)

Simplicial complexes

simplicial complex $\triangle$*:* family of subsets of V, such that $\{v\} \in \triangle$ for all $v \in V$ and every subset of a $\sigma \in \triangle$ belongs to $\triangle$.
face: $\sigma \in \triangle$, *dimension:* $\dim(\sigma) = |\sigma| - 1$.
f-vector: $(f_{-1}, \ldots, f_{n-1})$, where f_{j-1} is the number of $(j-1)$-dim faces.
h-vector: $(h_0, \ldots, h_n)$ given by

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{n-j}{i-j} f_{j-1}.$$

balanced complex: $(n-1)$-dimensional, vertices may be colored using n-colors.
flag f-vector: $(f_S : S \subseteq \{1, 2, \ldots, n\})$
flag h-vector: $(h_S : S \subseteq \{1, 2, \ldots, n\})$, given by

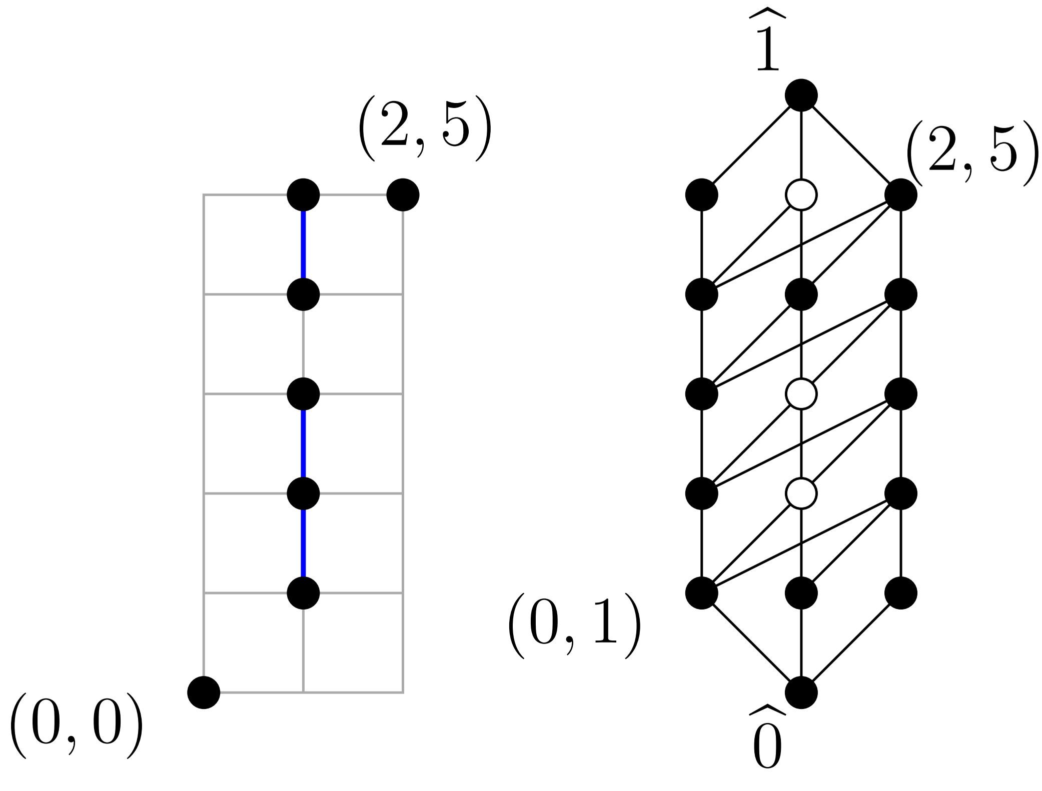
$$h_S = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T.$$

Theorem (Björner-Frankl-Stanley):
$(\beta_S : S \subseteq \{1, 2, \ldots, n\})$ is the flag h-vector of some $(n-1)$-dimensional balanced Cohen-Macaulay complex $\Leftrightarrow$ it is the flag f-vector a colored simplicial complex.

The order complex of a Jacobi poset

(i) For each $q \in \{1, \dots, n\}$, P_n^β has $n+\beta+1$ elements of rank q, they are labeled $(0,q)$, $(1,q)$, $\dots$, $(n+\beta,q)$.

(ii) Given (p,q) and (p',q') in $P_n^\beta \setminus \{\widehat{0}, \widehat{1}\}$ we set $(p,q) < (p',q')$ iff. $p \le p'$ and $q < q'$.



The Jacobi poset P_5^{-3}

Why the name Jacobi?

$$f_{j-1}\left(\triangle\left(P_n^\beta \setminus \{\widehat{0},\widehat{1}\}\right)\right) = \binom{n}{j}\binom{n+\beta+j}{j}.$$

$$\sum_{j=0}^{n} f_{j-1}\left(\triangle\left(P_n^\beta \setminus \{\widehat{0},\widehat{1}\}\right)\right) \cdot \left(\frac{x-1}{2}\right)^j = P^{(0,\beta)}(x)$$

for $\beta \geq 0$.

Connection with the asymmetric Delannoy numbers

balanced join operation: Let $\triangle_1$ and $\triangle_2$ be pure, balanced, of the same dimension. Assume the balanced colorings λ_1 and λ_2 use the same set of colors. Then, for $\lambda = (\lambda_1, \lambda_2)$,

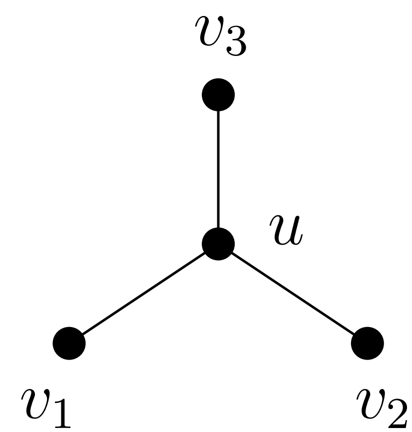
$$\triangle_1 *_\lambda \triangle_2 = \{\sigma \cup \tau : \sigma \in \triangle_1, \tau \in \triangle_2, \lambda_1(\sigma) \cap \lambda_2(\tau) = \emptyset\}$$

is the *balanced join* of $\triangle_1$ and $\triangle_2$ with respect to λ.

Theorem 1 *$\widetilde{d}_{m,n}$ is the number of facets in the balanced join $\triangle\left(P_n^{m-n} \setminus \{\widehat{0},\widehat{1}\}\right) *_\lambda \triangle^{n-1}$.*

Properties of the balanced join operation

It depends on the coloring chosen:



$\triangle *_\lambda \triangle$ has $1 \cdot 3 + 3 \cdot 1 = 6$ edges if $\lambda_1 = \lambda_2$, but $1 \cdot 1 + 3 \cdot 3 = 10$ edges if $\lambda_1 \neq \lambda_2$.

Theorem 2 *The flag h-vector of the balanced join $\triangle *_\lambda \triangle^{n-1}$ of a balanced $(n-1)$-dimensional simplicial complex with an $(n-1)$-simplex $\triangle^{n-1}$ satisfies*

$$h_S\left(\triangle *_\lambda \triangle^{n-1}\right) = f_S\left(\triangle\right).$$

Balanced join and the C-M property

Theorem 3 *If* $\triangle_1$ *and* $\triangle_2$ *are balanced Cohen-Macaulay and their balanced join exists, then* $\triangle_1 *_\lambda \triangle_2$ *is also Cohen-Macaulay.*

The proof uses Reisner's Criterion:

Theorem 4 (Reisner) $\triangle$ *is Cohen-Macaulay if and only if for all* $\sigma \in \triangle$ *and* $i < \dim \mathrm{lk}_\triangle(\sigma)$ *we have* $\widetilde{H}_i(\mathrm{lk}_\triangle(\sigma), k) = 0$*. Here* $\widetilde{H}_i$ *denotes the i-th reduced homology group of the appropriate oriented chain complex.*

link: $\mathrm{lk}_\triangle(\tau) := \{\sigma \in \triangle : \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \triangle\}$.
oriented j-simplex: $[v_0, \ldots, v_j]$, $\{v_0, \ldots, v_j\} \in \triangle$, permuting elements induces multiplying with the sign of the permutation. These generate $C_j(\triangle)$.
boundary map: $\partial_j : C_j(\triangle) \to C_{j-1}(\triangle)$, given by

$$\partial_j[v_0, \ldots, v_j] = \sum_{i=0}^{j} (-1)^i [v_0, \ldots, \widehat{v}_i, \ldots, v_j].$$

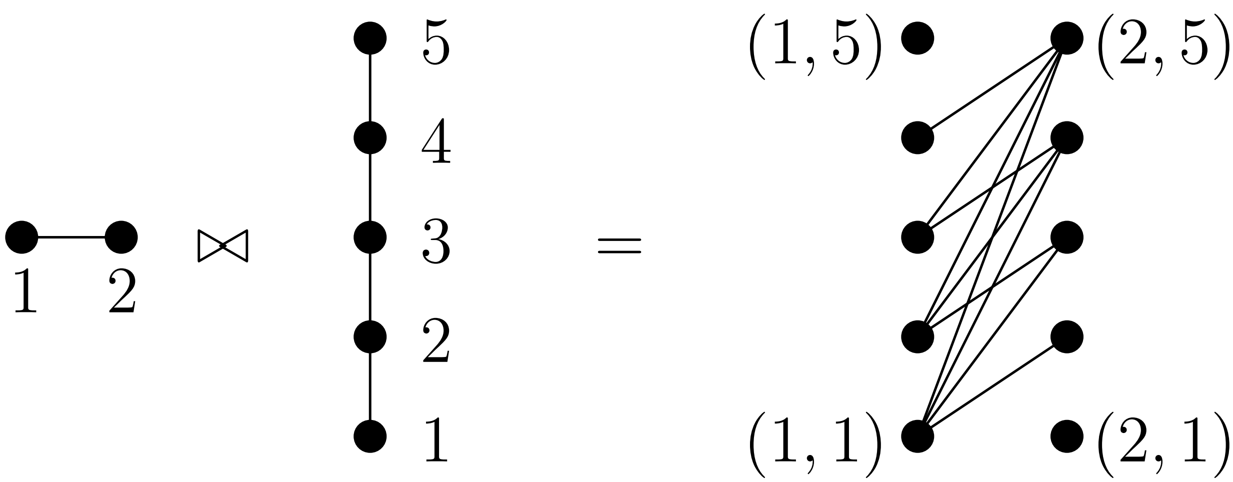
More on Jacobi posets

Proposition 2 *The order complex* $\triangle(P_n^\beta \setminus \{\widehat{0}, \widehat{1}\})$ *associated to the Jacobi poset* P_n^β *is Cohen-Macaulay.*

Proof uses *EL-labelings*.

Proposition 3 *The flag h-vector of* $\triangle(P_n^\beta \setminus \{\widehat{0}, \widehat{1}\})$*, colored by the rank function, equals the flag f-vector of* $\triangle(C_{n+\beta-1} \bowtie C_{n-1})$*, with respect to the coloring induced by the rank function of the second coordinate.*

strict direct product $P \bowtie Q$*:* the set $P \times Q$, ordered by $(p, q) < (p', q')$ if $p < p'$ and $q < q'$.



The half-strict dilemma

right-strict direct product $P \rtimes Q$ the set $P \times Q$, ordered by $(p, q) < (p'q')$ if $p \leq p'$ and $q < q'$.

Proposition 4 *$P_n^{\beta} \setminus \{\widehat{0}, \widehat{1}\}$ is isomorphic to $C_{n+\beta} \rtimes C_{n-1}$.*

Proposition 5 *Assume P is an arbitrary poset and Q is a graded poset of rank $n+1$. Then $P \rtimes (Q \setminus \{\widehat{0}, \widehat{1}\})$ may be turned into a graded poset of rank $n+1$ by adding a unique minimum element $\widehat{0}$ and a unique maximum element $\widehat{1}$. The rank function may be taken to be the rank function of Q applied to the second coordinate.*

Conjecture 1 *If P is a poset with a Cohen-Macaulay order complex and Q is a graded Cohen-Macaulay poset then $P \rtimes (Q \setminus \{\widehat{0}, \widehat{1}\}) \cup \{\widehat{0}, \widehat{1}\}$ is a graded Cohen-Macaulay poset.*

The half-strict fact

Theorem 5 *Assume that P is any poset whose order complex has a non-negative h-vector and that Q is a graded posets with a non-negative flag h-vector. Then the flag h-vector of $P \rtimes (Q \setminus \{\widehat{0}, \widehat{1}\}) \cup \{\widehat{0}, \widehat{1}\}$ is non-negative.*

In fact:

$$h_S(P \rtimes (Q \setminus \{\widehat{0}, \widehat{1}\}) \cup \{\widehat{0}, \widehat{1}\}) =$$

$$\sum_{T \subseteq S} h_T(Q) \sum_{i=0}^{\min(d,|R|)} h_i(P) \binom{d + |T| - i - 1}{|S| - i}.$$

Part II: The “Legendrotope” of dimension n

Central sections of centrally symmetric polytopes

$P \subset \mathbb{R}^n$ is a centrally symmetric polytope, H is given by $\sum_{i=1}^{n} \lambda_i x_i = \langle \lambda \mid x \rangle = 0$. $Q := P \cap H$ is a *non-degenerate central section* if $Q \cap V(P) = \emptyset$.

Then $V(P) = V_+(P) \uplus V_-(P)$ where
$V_+(P) := \{(x_1, \ldots, x_n) \in V(P) \,:\, \langle \lambda \mid x \rangle > 0\}$ and
$V_-(P) := \{(x_1, \ldots, x_n) \in V(P) \,:\, \langle \lambda \mid x \rangle < 0\}$. Each vertex of Q is of the form $H \cap [u, -v]$.

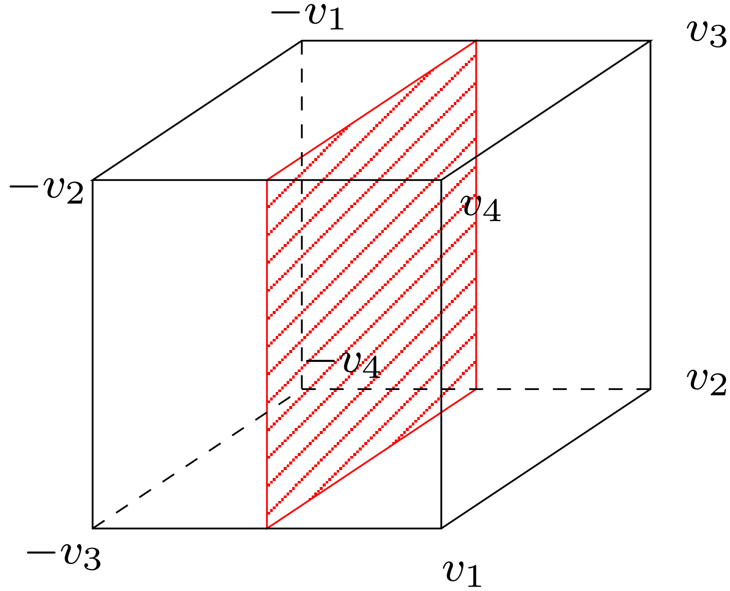
Define the graph $G = G(P, H)$ on $V(G) := V_+(P)$ by letting (u, v) be a directed edge in G exactly when $[u, -v] \cap H$ is a vertex of Q. This graph contains no loops.

Definition 1 *Let G be a directed graph with no multiple edges. Let S and T be disjoint subsets of $V(G)$. The* directed restriction of G to (S, T) *is the digraph with vertex set $S \cup T$ with edge set*
$\{(s, t) \in E(G) \,:\, s \in S, t \in T\}$.

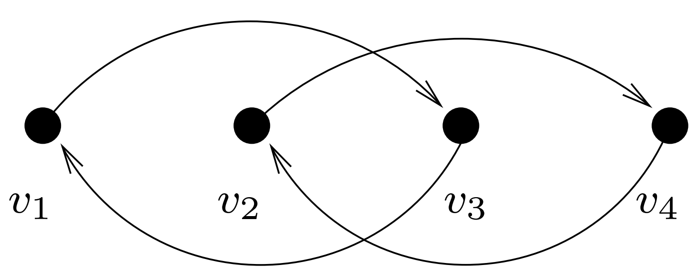
Proposition 6 *Assume the vertices of $Q = P \cap H$ are represented by the edges of the graph $G = G(P, H)$. Given a face F of P, the vertices contained in the face $F \cap H$ of Q are represented by the edges in the directed restriction of $G(P, H)$ to $(V_+(F), -V_-(F))$.*

An Example

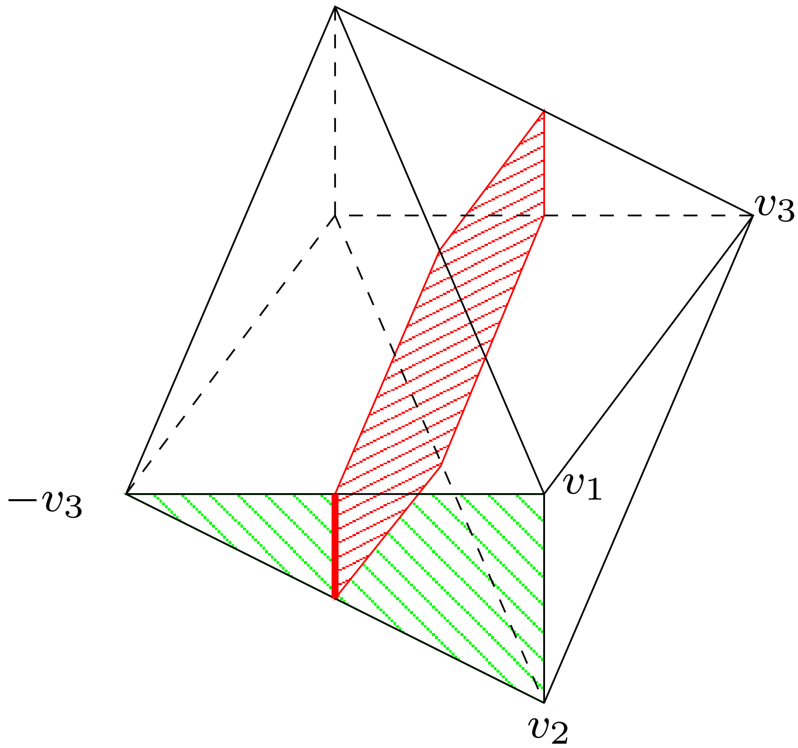
P is the 3-dimensional cube, H contains the midpoints of some parallel edges:

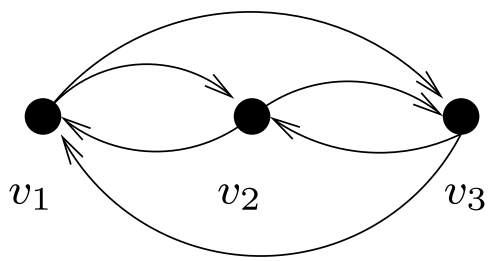


Here is the associated graph:

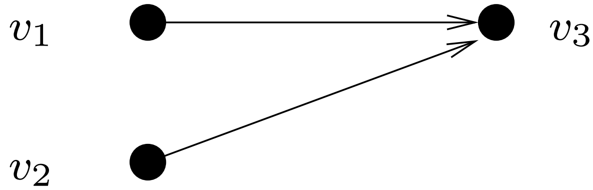


The case of a simplicial P





The directed restriction of $G(P, H)$ to $(V_+(F), -V_-(F))$ is a complete bipartite graph, with each edge directed towards its endpoint in $V_-(F)$.



Face structure

The following Lemma generalizes a result of Kapranov, Postnikov, and Zelevinski. (Stated for undirected graphs, trees, convex hulls of midpoints of edges.)

Lemma 2 *Let $F \subset P$ be a face of P. A subset S of the edges of the directed restriction of $G(P, H)$ to $(V_+(F), -V_-(F))$ represents a simplex if and only if, disregarding the orientation of the edges, the set S contains no circuit.*

The Legendre polytope

Definition 2 *We define the* Legendre polytope $\mathcal{L}_n$ *as the non-degenerate central section of $2\mathcal{O}_{n+1}$ with the hyperplane*

$$H_n := \left\{ (x_0, \ldots, x_n) \; : \; \sum_{i=0}^{n} x_i = 0 \right\}.$$

Lemma 3 *A set $S \subset \{(e_i, e_j) \; : \; i \neq j\}$ of edges represents all vertices in a facet of the boundary $\partial\mathcal{L}_n$ of $\mathcal{L}_n$ if and only if there is a proper subset A of $A \subset \{e_0, \ldots, e_n\}$ such that S consists of all edges starting in A and ending in $\{e_0, \ldots, e_n\} \setminus A$.*

Connection to root polytopes

- The convex hull of $\mathbf{0}$ and $\{e_i - e_j \ : \ i < j\}$ is the *root polytope* $P_{A_n^+}$ (Gelfand, Graev and Postnikov [9])
- $P_{A_n^+}$ is also known as the "Catalanotope" (Stanley).
- More general root polytopes were studied by Postnikov [17].
- Extensions to the root systems B_n, C_n, and D_n were studied by Wungkum Fong [8].

Since all vertices of $\mathcal{L}_n$ not belonging to $P_{A_n^+}$ form $V(-P_{A_n^+}) \setminus \{\mathbf{0}\}$, we may think of $\mathcal{L}_n$ as the convex hull of $P_{A_n^+}$ and $-P_{A_n^+}$. The set of facets of $\partial\mathcal{L}_n$ may be partitioned into three classes:

1. the facets of $P_{A_n^+}$ not containing the origin;
2. the facets of $-P_{A_n^+}$ not containing the origin;
3. facets which contain at least one vertex of $P_{A_n^+}$ and one vertex of $-P_{A_n^+}$.

Pulling triangulations of $\partial\mathcal{L}_n$

Fact: Facets in any triangulation of $P_{A_n^+}$ are associated with admissible trees on the vertex set $\{e_0, \ldots, e_n\}$.

Translation: Facets correspond to maximal cycle free sets of edges in directed restrictions (S, T) such that $S \uplus T = V$.

Two special triangulations introduced by Gelfand, Graev and Postnikov are induced by the *standard* and *antistandard* trees. They correspond to pulling triangulations with respect to the lex and the revlex order.

Definition 3 *The* revlex order *on* $V(P_{A_n^+}) \setminus \{\mathbf{0}\}$ *is defined by setting* $(e_i, e_j) < (e_k, e_l)$ *if* $j < l$ *or* $j = l$ *and* $i > k$. *The* lexicographic order *on* $V(P_{A_n^+}) \setminus \{\mathbf{0}\}$ *is defined by setting* $(e_i, e_j) < (e_k, e_l)$ *if* $i < k$ *or* $i = k$ *and* $j < l$.

Proposition 7 (Stanley) *Suppose that one of the vertices of P is the origin and that the matrix whose rows are the vertices of P is totally unimodular. Let $<$ be any ordering on $V(P)$ such that the origin is the least vertex with respect to $V(P)$. Then $<$ is compressed.*

Theorem 6 (Heller) *The incidence matrix of a directed graph is totally unimodular.*

Consequences

Corollary 1 *Not only the (anti-)standard triangulation, but all pulling triangulations of* $P_{A_n^+}$ *starting with pulling at the origin, are compressed.*

Corollary 2 *Not only the (anti-)standard triangulation, but all pulling triangulations of* $\partial\mathcal{L}_n$ *are compressed.*

Theorem 7 *The number of* $(j-1)$*-dimensional faces in any pulling triangulation of* $\partial\mathcal{L}_n$ *that uses only the vertices of* $\mathcal{L}_n$ *is*

$$f_{j-1}\left(\triangle_<\left(\partial\mathcal{L}_n\right)\right) = \binom{n+j}{j}\binom{n}{j}.$$

This is easy to show for the lexicographic order.

$$F_\triangle(x) := \sum_{j=0}^{d} f_{j-1}\left(\frac{x-1}{2}\right)^j.$$

Corollary 3 *The* F*-polynomial of any pulling triangulation of* $\partial\mathcal{L}_n$ *that uses only the vertices of* $\mathcal{L}_n$ *is* $P_n(x)$*, the* n*-th Legendre polynomial.*

How about the "Catalanotope"?

Theorem 8 *The number of $(j-1)$-dimensional faces in any pulling triangulation of $P_{A_n^+} \cap \partial\mathcal{L}_n$ is*

$$f_{j-1}\left(\triangle_<(P_{A_n^+} \cap \partial\mathcal{L}_n)\right) = \frac{1}{j+1}\binom{n+j}{j}\binom{n}{j}.$$

Proof uses revlex order.

Note:

$$P_n^{(0,0)}(x) \quad = \quad \sum_{j=0}^{n} \binom{n+j}{n}\binom{2j}{j}\left(\frac{x-1}{2}\right)^j$$

whereas

$$\begin{aligned} & \sum_{j=0}^{n} \frac{1}{j+1}\binom{n+j}{j}\binom{n}{j}\left(\frac{x-1}{2}\right)^j \\ = \quad & \sum_{j=0}^{n} \binom{n+j}{n} C_j \left(\frac{x-1}{2}\right)^j \end{aligned}$$

The relation between the root polytope $P_{A_n^+}$ and the Legendre polytope $\mathcal{L}_n$ is thus a "geometric enhancement" of the relation between the Catalan numbers and central binomial coefficients.

Delannoy numbers in the "Legendrotope"

Theorem 9 *For* $i > 0$ *the Delannoy number* $d_{n,n-i}$ *is the number of all faces* F *in the lexicographic pulling triangulation of* $\partial\mathcal{L}_n$ *that contain at least one point in the generalized orthant*
$\{(x_0, \ldots, x_n) : x_0 < 0, x_1 < 0, \ldots, x_i < 0\}$.

This is true because:

Theorem 10 *The number of those* $(k+i-1)$*-dimensional faces in the lexicographic pulling triangulation of* $\partial\mathcal{L}_n$ *which contain at least one vertex of the form* $e_s - e_t$ *for each* $t \in \{0, 1, \ldots, i-1\}$ *is* $\binom{n+k}{k+i}\binom{n-i}{k}$.

“Peek under the hood:”

Obviously,

$$d_{n,n-i} = \sum_{k=0}^{n-i} \binom{n+k}{k+i} \binom{n-i}{k}.$$

Exotically,

$$\binom{n+k}{k+i}\binom{n-i}{k}$$

$$= \sum_{u=1}^{n-i+1} \binom{n-i+1}{u} \binom{k+i-1}{k+i-u} \sum_{v} \binom{n-i+1-u}{v} \binom{u-1}{k-v}$$

$$= \sum_{u,v} \binom{n-i+1}{u, v, n-i+1-u-v} \binom{k+i-1}{k+i-u, k-v, u+v-k-1}.$$

Part III: Lattice path enumeration

Shifted Legendre polynomials

$$\widetilde{P}_n(x) := P_n(2x-1).$$

Easy to show:

$$\widetilde{P}_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{n-k} \binom{2k}{k} x^k.$$

Definition 4 *Let u, v, w be commuting variables. We define the* weighted Delannoy numbers $d_{m,n}^{u,v,w}$ *as the total weight of lattice paths from $(0,0)$ to (m,n), with steps $(0,1)$, $(1,0)$, and $(1,1)$, where each step $(0,1)$ has weight u, each step $(1,0)$ has weight v, and each step $(1,1)$ has weight w. The weight of a lattice path is the product of the weights of its steps.*

Surprise! (?)

$$d_{n,n}^{u,v,w} = (-w)^n \widetilde{P}_n\left(-\frac{uv}{w}\right)$$

Now

$$d_{n,n} = d_{n,n}^{1,1,1} = (-1)^n \widetilde{P}_n(-1) = (-1)^n P_n(-3) = P_n(3)$$

since $(-1)^n P_n(-x) = P_n(x) \ldots$

Lattice path model for the shifted Legendre polynomials

$$\widetilde{P}_n(x) = d_{n,n}^{1,x-1,1} = d_{n,n}^{1,x,-1}$$

Amusing detail:

$$S_{n+1,n+1}^{1,x,w} = \int_0^x d_{1,t,w}\ dt$$

is the total weight of lattice paths from $(0,0)$ to $(n+1, n+1)$ staying strictly below the line (t,t) (see “royal paths”, “weighted Schröder numbers”).

A combinatorial proof of the orthogonality of the Legendre polynomials

Assume $m < n$.

$$\begin{aligned}&(n+m+1)! \int_0^1 x^m \widetilde{P}_n(x)\, dx \\ = \quad & \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{n-k} \binom{2k}{k} \frac{(n+m+1)!}{m+k+1}\end{aligned}$$

Weight the steps in the lattice paths with $1, 1, -1$. Number r, $c_1, \ldots, c_m$ and the n rows with a permutation, such that the label of r is less than the label of any c_i and less than the label of any row containing a $(0,1)$ step. Cancel the diagonal steps with the $((1,0),(0,1))$ sequences, when possible. You will be left with pairs of lattice paths and permutations such that

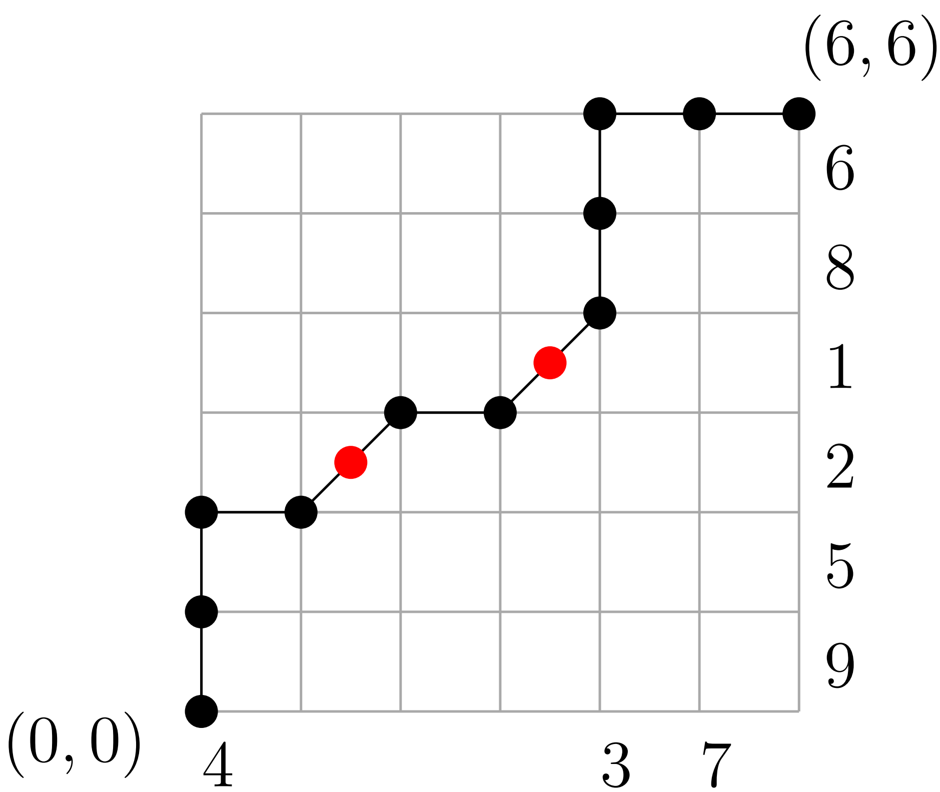
(a) $((1,0),(0,1))$ is forbidden;

(b) label of any row containing a diagonal step is less then the label of r

(b) makes the label of r unique, (a) makes the lattice path depend on the position of the diagonal steps only ($\sim$ "rook placements").

Example

$n = 6$, $m = 2$.



This also proves the orthogonality of Laguerre polynomials

Left to show:

$$p_n(m) := \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}^2 (n-k)!(m+k)! = \delta_{m,n} \cdot n!$$

Here $(-1)^n p_n(-m) = (n+m-1)_n$ (by some certainly known coloring story).

Going off-diagonal

Define *shifted Jacobi polynomials by*

$$\widetilde{P}_n^{(\alpha,\beta)}(x) := P_n^{(\alpha,\beta)}(2x-1).$$

Then we have

$$\widetilde{P}_n^{(0,\beta)}(x) = d_{n+\beta,n}^{1,x,-1}$$

As a consequence of the well known-relation $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$ we get

$$\widetilde{P}_n^{(\alpha,0)}(x) = d_{n+\alpha,n}^{1,x-1,1}$$

Corollary 4

$$d_{n+\alpha,n} = \widetilde{P}^{(\alpha,0)}(2) = P_n^{(\alpha,0)}(3).$$

There is a combinatorial proof for $\beta \in \mathbb{Z}$ that the polynomials $\widetilde{P}_n^{(0,\beta)}(x)$ form an orthogonal basis with respect to the inner product

$$\langle f, g \rangle := \int_0^1 f(x) \cdot g(x) \cdot x^\beta \, dx$$

The proof merges into the proof of orthogonality of generalized Laguerre polynomials $L_n^{(\beta)}(x)$ with respect to

$$\langle f, g \rangle := \int_0^\infty f(x) \cdot g(x) \cdot x^\beta e^{-x} \, dx$$

Concluding remarks

Happy Birthday Lou!

References

[1] C. Banderier and S. Schwer, Why Delannoy numbers ?, *J. Statist. Plan. Inference* **135** (1) (2005), 40–54.

[2] A. Björner, P. Frankl, and R. Stanley, The number of faces of balanced Cohen-Macaulay complexes and a generalized Macaulay theorem, *Combinatorica* **7** (1) (1987), 23–34.

[3] A. Björner and M. Wachs, Bruhat order of Coxeter groups and shellability, *Adv. in Math.* **43** (1982), 87–100.

[4] A. Björner and M. Wachs, On lexicographically shellable posets, *Trans. AMS* **277** (1983), 323–341.

[5] T.S. Chihara, "An Introduction to Orthogonal Polynomials," Gordon and Breach Science Publishers, New York-London-Paris, 1978.

[6] L. Comtet, "Advanced Combinatorics," D. Reidel Publishing Co., 1974.

[7] H. Delannoy, Employ d'échiquier pour la résolution de certains problèmes de probabilités, *Assoc. Franc. Bordeaux* **24** (1895), 70–90.

[8] Wungkum Fong, Triangulations and combinatorial properties of convex polytopes, PhD. Thesis, MIT, 2000.

[9] I. M. Gelfand, M. I. Graev, A. Postnikov, Combinatorics of hypergeometric functions associated with positive roots, in Arnold-Gelfand Mathematical Seminars: Geometry and Singularity Theory, Birkhauser, Boston, 1996, 205–221.

[10] I.J. Good, Legendre polynomials and trinomial random walks, *Proc. Cambridge Philos. Soc.* **54** (1958), 39–42.

[11] I. Heller, On linear systems with integral valued solutions, *Pacific J. Math* **7** (1957), 1351-1364.

[12] I. Heller and A. J. Hoffman, On unimodular matrices, *Pacific J. Math* **4** (1962), 1321-1327.

[13] G. Hetyei, Central Delannoy numbers and balanced Cohen-Macaulay complexes, Annals of Combinatorics **10** (2006) 443–462.

[14] G. Hetyei, Delannoy orthants of Legendre polytopes, preprint 2007, www.math.uncc.edu/preprint/2007/2007_11.pdf.

[15] D.F. Lawden, On the solution of a linear difference equation, *Mah. Gaz.* **36** (1952), 193–196.

[16] L. Moser and W. Zayachkowski, Lattice paths with diagonal steps, *Scripta Math.* **26** (1963), 223–229.

[17] A. Postnikov, Permutohedra, Associahedra, and Beyond, preprint 2005, arXiv: math.CO/0507163.

[18] M. Skandera, Dumont's statistic on words, *Electron. J. Combin* **8** (2001), #R11.

[19] N.J.A. Sloane, "On-Line Encyclopedia of Integer Sequences," www.research.att.com/~njas/sequences

[20] R.P. Stanley, Balanced Cohen–Macaulay complexes, *Transactions of the AMS* **Vol. 249** (1979), No. 1, 139-157.

[21] R.P. Stanley, "Combinatorics and Commutative Algebra," Second Edition, Birkhäuser, Boston, 1996.

[22] R. P. Stanley, Decompositions of rational convex polytopes, *Annals of Discrete Mathematics* **6** (1980), 333-342.

[23] R.P. Stanley, "Enumerative Combinatorics, Volume II," Cambridge University Press, Cambridge, 1999.

[24] R.A. Sulanke, Objects counted by the central Delannoy numbers, *J. Integer Seq.* **6** (2003), Article 03.1.5, 19 pp.

[25] Gábor Szegő, "Orthogonal Polynomials", AMS Colloquium Publications Vol. 23, Providence, Rhode Island 2000.

[26] H.S. Wilf, "generatingfunctionology," Academic Press, 1994.