# Two-batch liar games on a general bounded channel

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BilleraFest

# Basic liar game setting

Two-person game:

1. Carole picks a number $x \in [n] := \{1, \ldots, n\}$
2. Paul asks $q$ questions to determine $x$:
   given $[n] = A_1 \dot\cup A_2 \dot\cup \cdots \dot\cup A_t$,
   for what $i$ is $x \in A_i$?

Playing optimally, Carole answers with an adversarial strategy; it's a perfect information game.

Catch: Carole is allowed to lie up to $k$ times.

# Example ternary game

$t = 3$ (Ternary coding).

- Paul partitions $[n] = A_1 \dot{\cup} A_2 \dot{\cup} A_3$ and asks "for what $i$ is $x \in A_i$?"
- Carole answers 1, 2, or 3

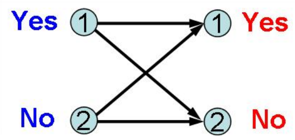
Example. $n = 6$, $q = 4$, $t = 3$, $k = 1$

| | Paul | | | | Lies | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Rnd** | $A_1$ | $A_2$ | $A_3$ | **Carole** | **1** | **2** | **3** | **4** | **5** | **6** |
| 1 | $\{1,2\}$ | $\{3,4\}$ | $\{5,6\}$ | 2 | ✓ | ✓ | | | ✓ | ✓ |
| 2 | $\{3\}$ | $\{4\}$ | $\{1,2,5,6\}$ | 3 | | | ✓ | ✓ | | |
| 3 | $\{1,2\}$ | $\{3,4\}$ | $\{5,6\}$ | 3 | ✓ | ✓ | ✓ | ✓ | | |
| 4 | $\{5\}$ | $\{6\}$ | $\emptyset$ | 1 | | | | | | ✓ |

Therefore $x = 5$.

# Binary symmetric case

- $t = 2$ binary case $\leftrightarrow$ "is $x \in A_1$?"
- symmetric lies: Carole may
  - lie with Yes when truth is No
  - lie with No when truth is Yes




**Question.** Given $q$, what is the maximum $n$ for which Paul has a winning strategy to find $x$?

- $k = 0$, binary search, $n = 2^q$

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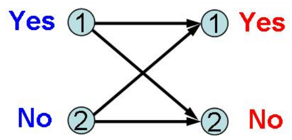


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- $k = 0$, binary search, $n = 2^q$
- $k = 1$, Pelc (87); $k = 2$, Guzicki (90); $k = 3$, Deppe (00)
- $k < \infty$, Spencer (1992) (*up to bounded additive error*)

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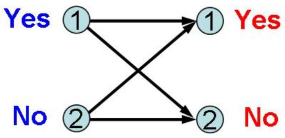


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- $k < \infty$, Spencer (1992) (*up to bounded additive error*)
- $k/q \to f \in (0, 1/2)$, Berlekamp (1962+), Zingangirov

# Binary symmetric case, $k = 1$

**Question.** Given $q$, what is the maximum $n$ for which Paul has a winning strategy to find $x$?

- Let $k = 1$, Carole chooses $y \in [n]$
- $q + 1$ possible responses if $y$ is the distinguished element:

| | Game response string $w \in [2]^q$ | | | | | |
|---|---|---|---|---|---|---|
| 0 lies | $w_1$ | $w_2$ | $w_3$ | $\cdots$ | $w_{q-1}$ | $w_q$ |
| 1 lie | $\overline{w}_1$ | $*$ | $*$ | $\cdots$ | $*$ | $*$ |
| | $w_1$ | $\overline{w}_2$ | $*$ | $\cdots$ | $*$ | $*$ |
| | | $\vdots$ | | | $\vdots$ | |
| | $w_1$ | $w_2$ | $w_3$ | $\cdots$ | $w_{q-1}$ | $\overline{w}_q$ |

**Sphere Bound** $y, y'$ can't both be $x \implies n \leq 2^q / \binom{q}{\leq 1}$ where $\binom{q}{\leq 1} = \binom{q}{0} + \binom{q}{1}$.

# Binary symmetric case, $k < \infty$

- $\binom{q}{\le k}$ response strings corresponding to $y \in [n]$ being the distinguished element

**Sphere Bound** $n \le 2^q / \binom{q}{\le k}$

$X_i :=$ elements of $[n]$ with $i$ accumulated lies

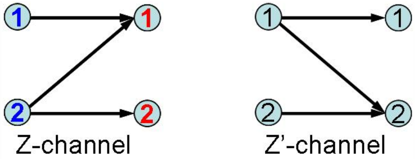
Paul balances $A_1 \dot\cup A_2$ by solving each round

$$|A_1 \cap X_i| \doteq \frac{|X_i|}{2}, \qquad \text{for } 0 \le i \le k.$$

# Asymmetric lying

- asymmetric lies: Carole may
  - lie with Yes (1) when truth is No (2)
  - But not vice versa!

Called the $Z$-channel




- $k < \infty$, Dumitriu & Spencer (2004)
- $k < \infty$ w/improved asymptotics, Spencer & Yan (2003)

Asymmetric strategy: still based on balancing.

# A motivating question

(Linial 2005): What if Paul knows that Carole is lying according to one of the $Z$-channels, but not which one?

Our answer: Yes! We generalize the "channel" constraining Carole's lies as much as possible.

# A closer look: game lie strings

| | Paul | | | Carole | 6's lie string | |
|---|---|---|---|---|---|---|
| **Rnd** | $A_1$ | $A_2$ | $A_3$ | $w$ | $a$ | $b$ |
| 1 | $\{1,2\}$ | $\{3,4\}$ | $\{5,6\}$ | 2 | 3 | 2 |
| 2 | $\{3\}$ | $\{4\}$ | $\{1,2,5,6\}$ | 3 | | |
| 3 | $\{1,2\}$ | $\{3,4\}$ | $\{5,6\}$ | 3 | | |
| 4 | $\{5\}$ | $\{6\}$ | $\emptyset$ | 1 | 2 | 1 |

| | | | | | |
|---|---|---|---|---|---|
| Truthful string for $y=6$ | $w' =$ | 3 | 3 | 3 | 2 |
| Lie string for $y=6$ | $u =$ | 3 <br> 2 | | | 2 <br> 1 |
| Game response string | $w =$ | 2 | 3 | 3 | 1 |

Write $u = (3,2)(2,1)$; we say $\boxed{w' \xrightarrow{u} w}$

# The general bounded $t$-ary channel

- Lies: $L(t) := \{(a,b) \in [t] \times [t] \ : \ a \neq b\}$ (truth$=$ $a$, Carole: $b$)
- Lie strings: $L(t)^j := \{(a_1,b_1)\cdots(a_j,b_j) \ : \ (a_i,b_i) \in L(t)\}$
- Empty string: $L(t)^0 := \{\epsilon\}$

**Definition (General bounded channel)**

Fix $k \geq 0$. A channel $C$ of order $k$ is an arbitrary subset

$$C \subseteq \bigcup_{j=0}^{k} L(t)^j,$$

such that $C \cap L(t)^k \neq \emptyset$.

# Element survival and winning for Paul

**Definition**

An element $y \in [n]$ survives the game iff its lie string is in $C$.

**Definition**

Paul wins the original liar game iff at most one element survives after $q$ rounds.

Paul wins the pathological liar game iff at least one element survives after $q$ rounds.

$$\left.\begin{array}{r} A_C(q) := \max n \\ A^*_C(q) := \min n \end{array}\right\} \text{ such that Paul can win the } \left.\begin{array}{c} \text{original} \\ \text{pathological} \end{array}\right\} \text{ liar}$$

game with $n$ elements.

# Example channels

- Binary, symmetric, two lies. $(t = 2,\ k = 2)$

$$C = \{\epsilon,\ (1,2),\ (2,1),\ (1,2)(1,2),\ (1,2)(2,1),\ (2,1)(2,1),\ (2,1)(1,2)\}$$

$$\frac{2^q}{\binom{q}{\le 2}} - O(1) = A_C(q) \le A^*_C(q) = \frac{2^q}{\binom{q}{\le 2}} + O(1)$$

Guzicki ('90); Ellis, Ponomarenko, Yan ('05)

- Binary, $Z$-channel, two lies. $(t = 2,\ k = 2)$

$$C = \{\epsilon,\ (2,1),\ (2,1)(2,1)\}$$

$$A_C(q), A^*_C(q) \sim \frac{2^{q+\mathbf{2}}}{\binom{q}{\le 2}},$$ Spencer, Yan ('03); here

# Example channels (con't)

- Binary, unidirectional, two lies. $(t = 2, k = 2)$

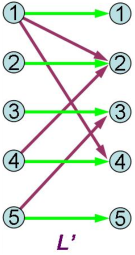
$$C = \{\epsilon, (1,2), (2,1), (1,2)(1,2), (2,1)(2,1)\}$$

$$A_C(q), A_C^*(q) \sim \frac{2^{q+1}}{\binom{q}{\leq 2}},$$ here

- Selective lies.
  - Pick arbitrary $L' \subseteq L(t)$.
  - Let $C = \bigcup_{j=0}^{k} (L')^j$.

$$A_C(q), A_C^*(q) \sim \frac{t^{q+k}}{|L'|^k \binom{q}{\leq k}}$$

Dumitriu, Spencer ('05); here



# Example channels (con't)

Example (weighted lies).

- Weight the lies of $L(t)$, normalized to minimum weight 1.
- Let $k$ bound the total allowable weight of a game lie string.
- Let $C = \{u \in L(t)^{\geq 0} \ : \ \mathrm{weight}(u) \leq k\}$.

$A_{C,t}(q)$ was solved asymptotically by Alshwede,Cicalese,&Deppe (2006+); slightly improved here.

Example (Model-based channel).

- Select a communication model (probability map $p : L(t)^{\geq 0} \to [0, 1]$).
- Select a probability threshold $p_0$.
- Let $C = \{u \in L(t)^{\geq 0} \ : \ p(u) > p_0\}$.

Paul must handle all likely errors/lie strings.

# The proposed sphere bound

- Select Paul's strategy tree to be random partitions so the truthful response string is random.

- Carole picks a lie string $u \in C$, and places to put the lies.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Truthful string for $y$ | $w' =$ | $w'_1$ | $\cdots$ | $w'_{i_1}$ | $\cdots$ | $w'_{i_\ell}$ | $\cdots$ | $w'_{i_j}$ | $\cdots$ | $w'_q$ |
| Lie string for $y$ | $u =$ | | | $a_1$<br>$b_1$ | | $a_\ell$<br>$b_\ell$ | | $a_j$<br>$b_j$ | | |
| Response string | $w =$ | $w_1$ | $\cdots$ | $w_{i_1}$ | $\cdots$ | $w_{i_\ell}$ | $\cdots$ | $w_{i_j}$ | $\cdots$ | $w_q$ |

- Compatibility: $\Pr(w'_{i_\ell} = a_\ell) = t^{-1}$

# The proposed sphere bound

- The expected number of response strings for which $y$ survives is:

$$\sum_{u \in C} \binom{q}{|u|} t^{-|u|} \sim |C \cap L(t)^k| \binom{q}{k} t^{-k}.$$

## Definition (Asymptotic Sphere Bound)

For $q$ rounds, base $t$, and an order $k$ channel $C$, the sphere bound is

$$\mathrm{SB}_C(q) := \frac{t^{q+k}}{|C \cap L(t)^k| \binom{q}{k}}.$$

# Carole's bound

**Theorem (Carole's bound)**

$$\begin{aligned} A_C(q) &\leq \mathrm{SB}_C(q)(1+o(1)), \\ A^*_C(q) &\geq \mathrm{SB}_C(q)(1-o(1)). \end{aligned}$$

Proof idea.

- Get lower and upper bounds on the number of response strings for which an element $y$ survives.
- If $n$ is too large, the response string sets collide. If Carole responds with a string in the intersection, Paul cannot be sure which element Carole was thinking of.
- If $n$ is too small, the response strings fail to cover $[t]^q$.

# Paul's bound

## Theorem (Paul's bound)

$$\begin{aligned} A_C(q) &\geq \mathrm{SB}_C(q)(1-o(1)), \\ A_C^*(q) &\leq \mathrm{SB}_C(q)(1+o(1)). \end{aligned}$$

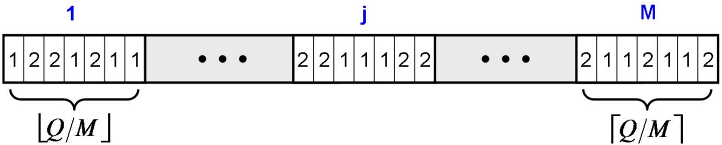
*Furthermore, we may restrict Paul to two nonadaptive batches of questions of sizes $q_1$ and $q_2$, with*

$$\begin{aligned} q_1 + q_2 &= q \qquad \textit{and} \\ (\log_t q)^{3/2} << \quad q_2 &\leq \mathrm{cst} \cdot q^{k/(2k-1)}, \end{aligned}$$

Remark. Proof builds on techniques of Dumitriu&Spencer.

# $(M, r)$-balanced strings in $[t]^Q$

$$\frac{1}{t}\left\lceil \frac{Q}{M} \right\rceil - r(t-1) - \Theta(1) \;\le\; \#1\text{'s}, \#2\text{'s} \;\le\; \frac{1}{t}\left\lceil \frac{Q}{M} \right\rceil + r$$




- By counting the number of ways to place lies in sections we can bound $|\{w' : w' \xrightarrow{u} w\}|$.

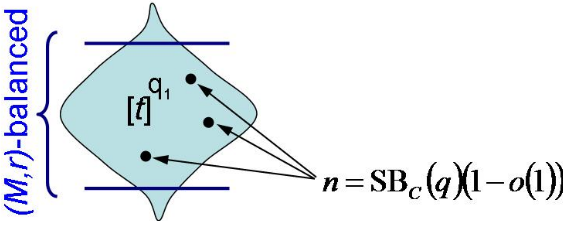
**Lemma**

*Let $u = (a_1, b_1) \cdots (a_j, b_j)$, and $w \in [t]^Q$ be $(M, r)$-balanced. Then*

$$\binom{M}{j}\left(\frac{1}{t}\left\lceil \frac{Q}{M} \right\rceil - r(t-1) - \Theta(1)\right)^j \le |\{w' : w' \xrightarrow{u} w\}| \le \binom{M+j-1}{j}\left(\frac{1}{t}\left\lceil \frac{Q}{M} \right\rceil + r\right)^j$$

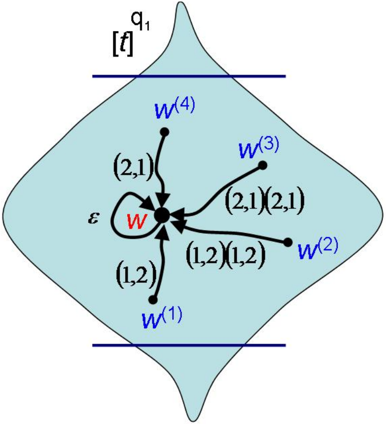
# First batch of $q_1$ questions

(Proof illustrated with $C = \{\epsilon, (1,2), (2,1), (1,2)(1,2), (2,1)(2,1)\}$.)




- Paul maps $n$ evenly to $(M, r)$-balanced vertices of $[t]^{q_1}$
- Paul asks: What is the $i^{th}$ coordinate in your element's length-$q_1$ string?

# Carole's first batch response

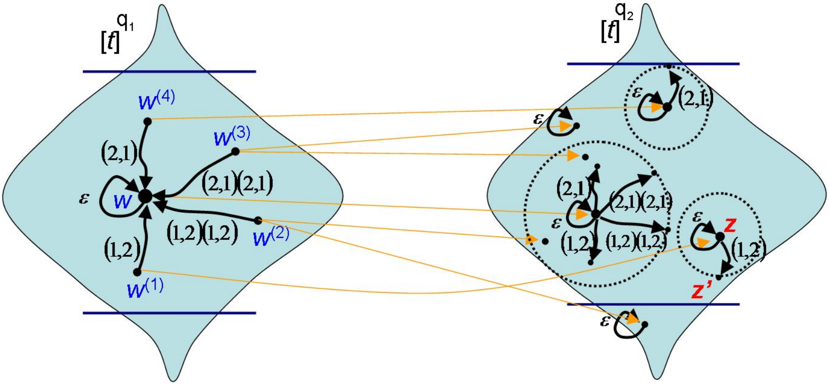



Suppose Carole responds with balanced $w \in [t]^{q_1}$.
Which $y \in [n]$ survive?

Any $y$ identified with $w'$ such that:

- $u \in C$, and
- $w' \xrightarrow{u} w$

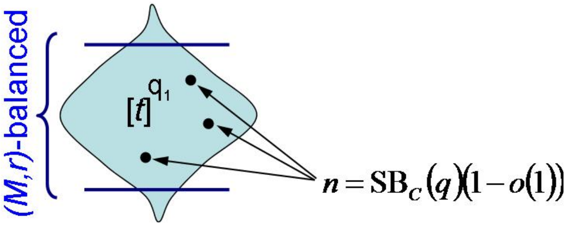
# Paul's second batch of $q_2$ questions




- $y$'s survive in various ways
- Fit $y$'s which can take more lies inside disjoint Hamming balls
- $(M, r)$-balance $\Rightarrow$ control on $|\{w^{(i)} : w^{(i)} \xrightarrow{u} w\}|$, $|\{z : z \xrightarrow{v} z'\}|$
- Greedily pack other $y$'s in unoccupied space

# First batch, pathological case

(Proof illustrated with $C = \{\epsilon, (1,2), (2,1), (1,2)(1,2), (2,1)(2,1)\}$.)



# First batch, pathological case

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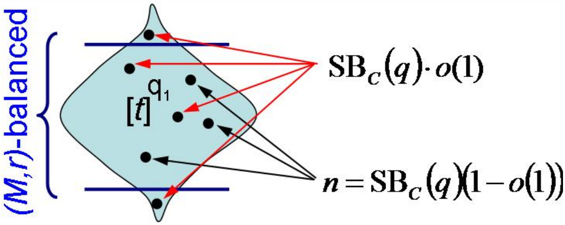
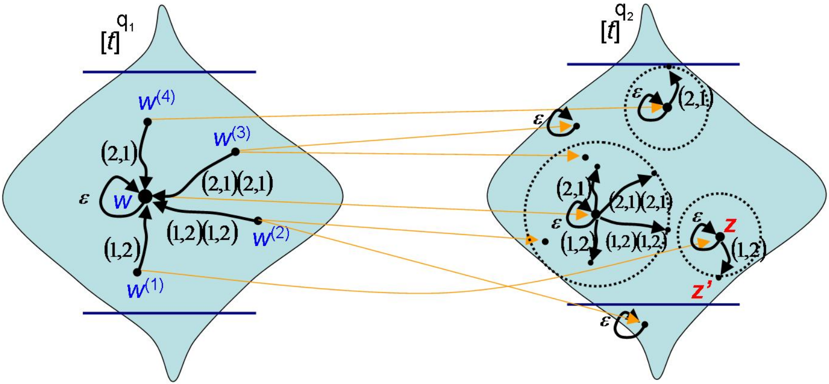



- Paul adds negligibly many elements evenly over $[t]^{q_1}$

# Paul's second batch, pathological case



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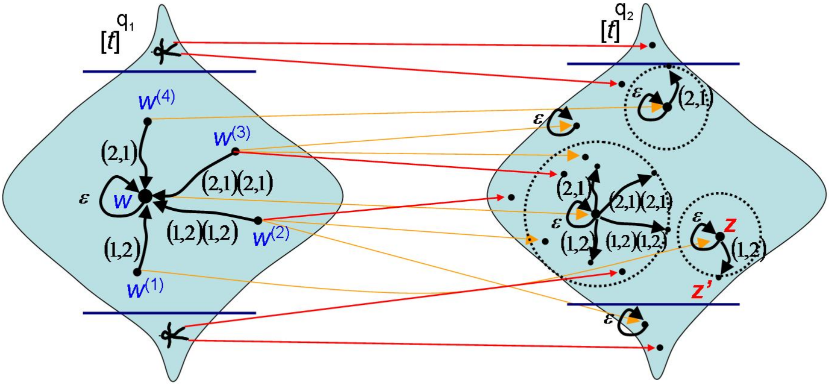


- Count only additional $y$'s for which Carole may not lie again
- Greedily convert packing into covering in $[t]^{q_2}$

# Summary

## Theorem

$$\mathrm{SB}_C(q)(1+o(1)) \geq A_C(q) \geq \mathrm{SB}_C(q)(1-o(1)),$$
$$\mathrm{SB}_C(q)(1-o(1)) \leq A_C^*(q) \leq \mathrm{SB}_C(q)(1+o(1)).$$

*Furthermore, **(1)** we may restrict Paul to two nonadaptive batches of questions of sizes $q_1$ and $q_2$, with*

$$\begin{array}{rcll} & q_1+q_2 & = q & \text{and} \\ (\log_t q)^{3/2} << & q_2 & \leq \mathrm{cst}\cdot q^{k/(2k-1)}, & \end{array}$$

***(2)** the response sets for $A_C(q)$ are a subset of those for $A_C^*(q)$.*

# Concluding remarks and open questions

Open Questions.

- Can we further reduce or eliminate completely the adaptiveness?
- Can these techniques be used to improved the asymptotic best known packings and coverings of $[t]^q$ with fixed-radius Hamming balls (not tight for radius $\geq 2$)?
- Will these techniques work for coin-weighing, fault-testing, and related search problems?

Happy Birthday, Lou!

....and thank you!