Nonlinear Discrete Optimization

Shmuel Onn

Technion – Israel Institute of Technology

http://ie.technion.ac.il/~onn

Billerafest 2008 - conference in honor of Lou Billera's 65th birthday

(Update on Lecture Series given at CRM Montréal)

Based on several papers joint with several co-authors including
Berstein, De Loera, Hemmecke, Lee, Rothblum, Weismantel, Wynn

Framework for Nonlinear Discrete Optimization

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The problem can be interpreted as multi-objective discrete optimization, where the goal is to optimize the "balancing" by f of d linear criteria:

$$\min \text{ or } \max \{f(w_1x, \ldots, w_dx) : x \text{ in } S\} .$$

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It is generally intractable even for fixed $d=1$ and f the identity on R (e.g., it may be NP-hard or even require exponential oracle-time).

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Some Applications and Examples Where Our Theory Provides Polynomial Time Algorithms

- Vector Partitioning and Clustering:

- Shaped partition problems (SIAM Opt.)
- Partition problems with convex objectives (Math. OR)

Vector Partitioning

Partition m items evaluated by k criteria to p players, to maximize social utility that is function of the sums of vectors of items each player gets.

Example: Consider m=6 items, k=2 criteria, p=3 players

The criteria-item matrix is:

items

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 9 & 16 & 25 & 36 \end{bmatrix}$$ criteria

Each player should get 2 items

The nonlinear function on k x p matrices is $f(X) = \sum X_{ij}^3$

The matrix of a partition such as π = (34, 56, 12) is:

players

$$A^\pi = \begin{bmatrix} 7 & 11 & 3 \\ 25 & 61 & 5 \end{bmatrix}$$ criteria

The social utility of π is $f(A^\pi) = 244432$

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All 90 partitions π of items {1, …,6} To 3 players where each player gets 2 items

The optimal partition is:

π = (34, 56, 12)

with optimal utility:

players

$$A^{\pi} = \begin{bmatrix} 7 & 11 & 3 \\ 25 & 61 & 5 \end{bmatrix}$$ criteria

$f(A^{\pi}) = 244432$

[12 34 56]	[12 35 46]	[12 36 45]	[12 45 36]	[12 46 35]	[12 56 34]
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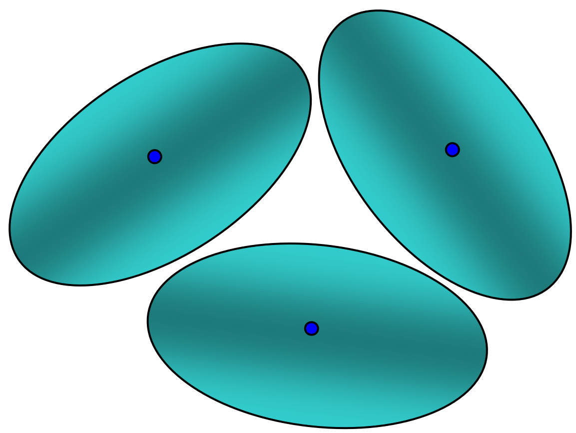
Minimum Variance Clustering

Given m points $v^1, \ldots, v^m$ in R^k, group them into p (balanced) clusters so as to minimize the sum of cluster variances .



P=3, k=3, m large

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- Matroids and Their Applications:

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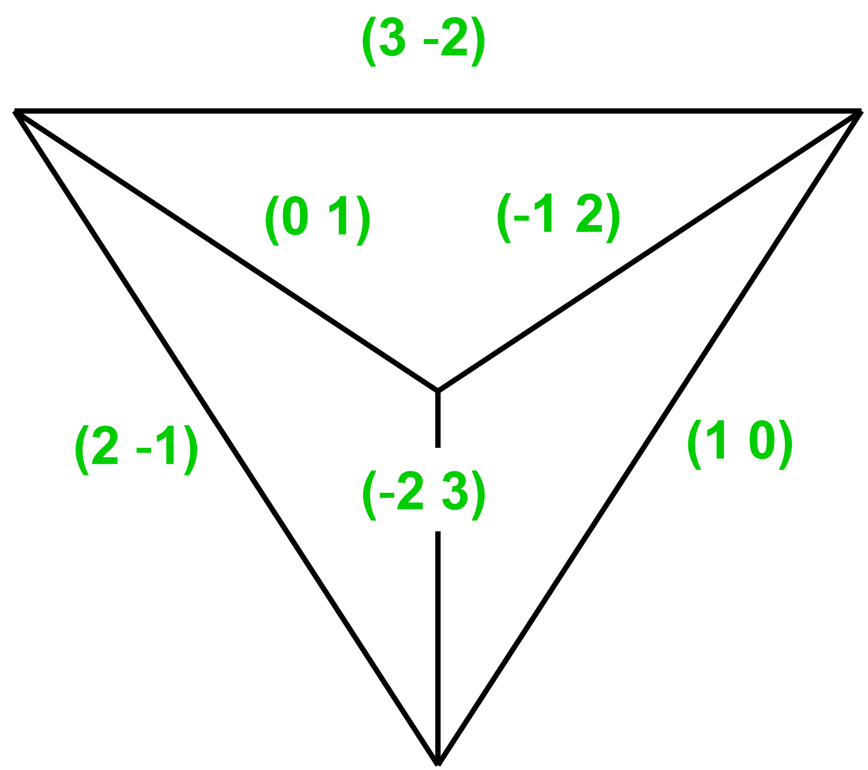
Example - spanning trees:

Consider $n=6$, the graph $G=K_4$, $d=2$, weights w, and the Euclidean norm (squared) $f(x) = |x|^2 = x_1^2 + x_2^2$



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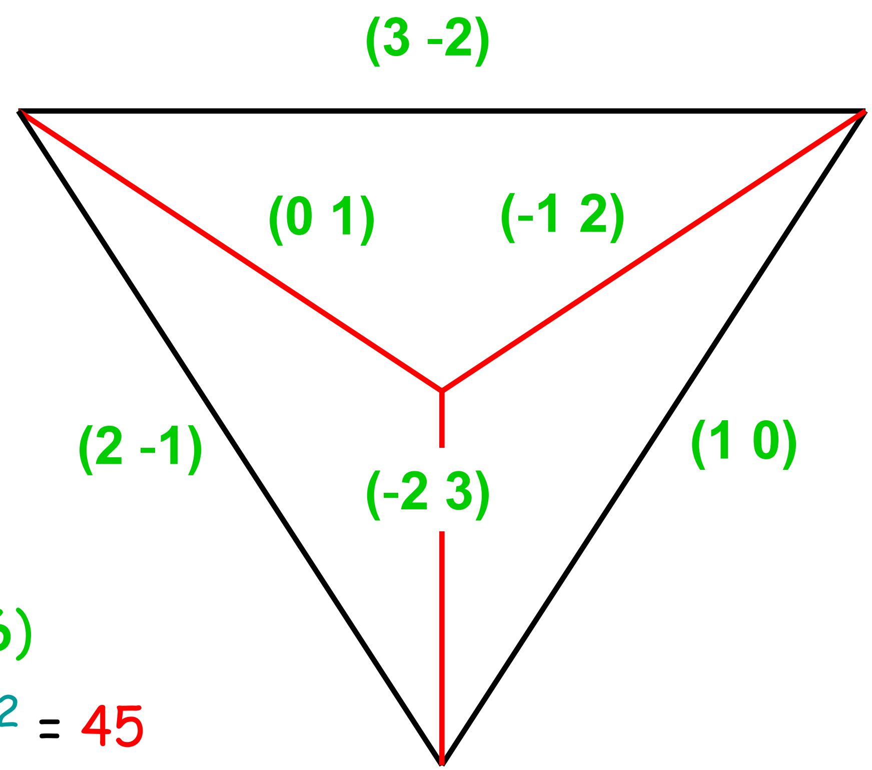

The optimal tree x has

$w(x) = (0\ \ 1) + (-1\ \ 2) + (-2\ \ 3) = (-3\ \ 6)$

and objective value $f(w(x)) = |-3\ \ 6|^2 = 45$

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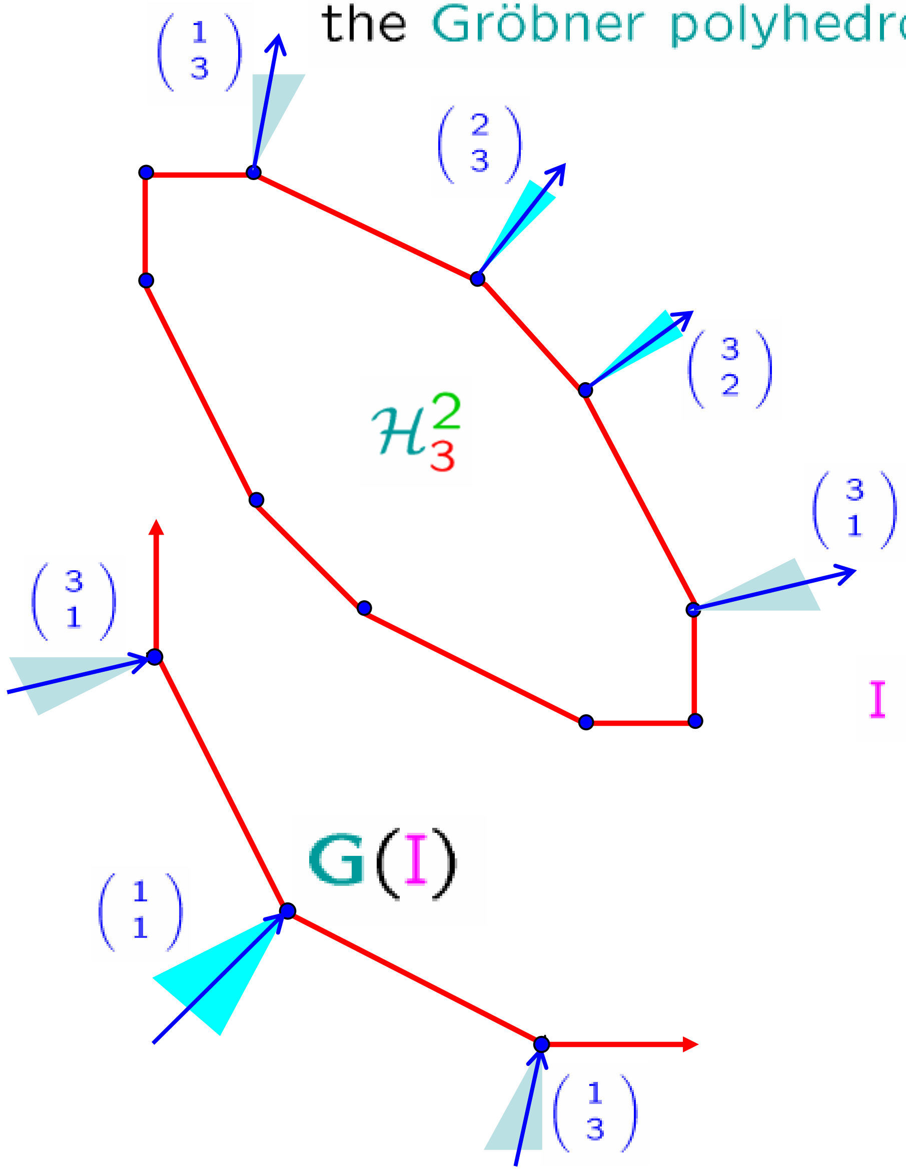
- Matroids and Their Applications:

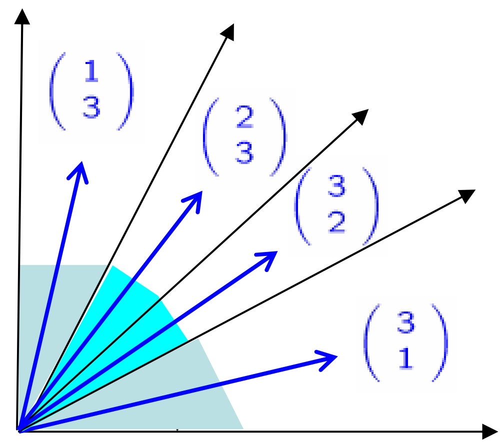
- Spanning trees, polymatroids, intersections of matroids:

- Systems of polynomial equations:
  simultaneous computation of universal Gröbner bases
  for all ideals on the Hilbert Scheme

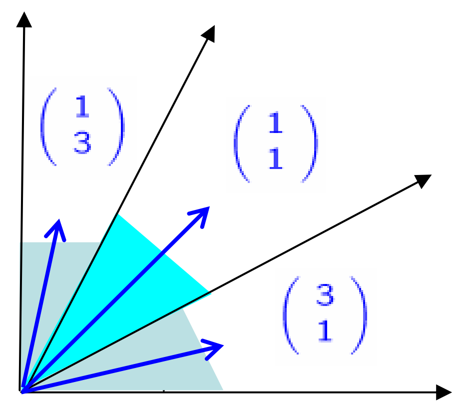
Example: The Hilbert Zonotope $\mathcal{H}_3^2$

It refines the Hilbert polyhedron $\mathbf{H}(\mathrm{Hilb}_3^2)$ and the Gröbner polyhedron $\mathbf{G}(\mathrm{I})$ of every $\mathrm{I} \in \mathrm{Hilb}_3^2$.





$$\mathrm{I} \;=\; \langle x^3 - 3x^2 + 2x,\, y - x^2 \rangle \;\in\; \mathrm{Hilb}_3^2$$

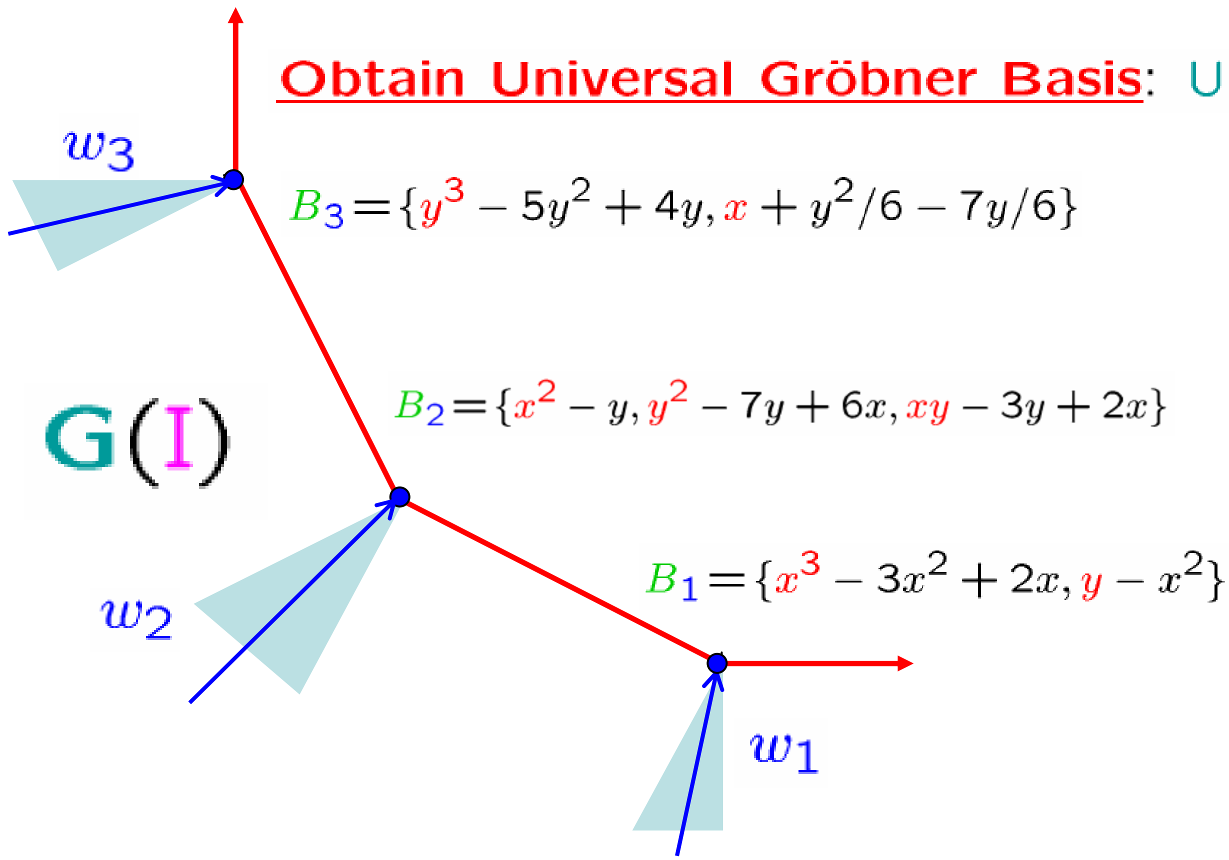


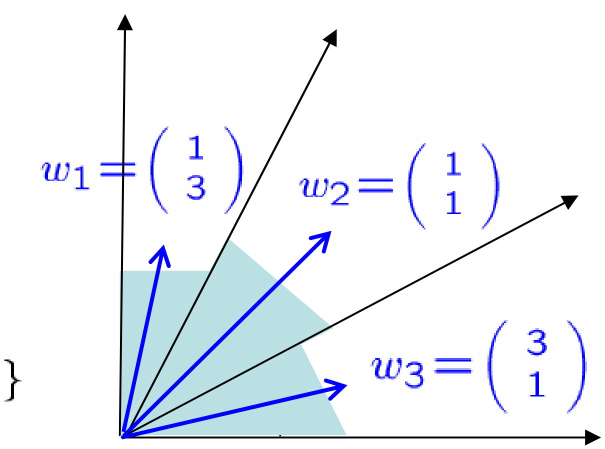
Gröbner Polyhedra

Each generic $w \in \mathbf{R}_+^d$ gives monomial order on $\mathbf{C}[x_1, \ldots, x_d]$ by the value $w \cdot v$ of the exponent $v \in \mathbf{N}^d$ of the monomial $\prod_i x_i^{v_i}$.

A Gröbner polyhedron $\mathbf{G}(\mathrm{I})$ of an ideal I is one with vertices in bijection with reduced Gröbner bases of I, where the RGB of v equals RGB_w whenever w is minimized over $\mathbf{G}(\mathrm{I})$ uniquely at v.

Obtain Universal Gröbner Basis: $\mathrm{UGB}(\mathrm{I}) = B_1 \cup B_2 \cup B_3$





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- Convex matroid optimization (SIAM Disc. Math.)
- Convex combinatorial optimization (Disc. Comp. Geom.)
- Cutting corners (Adv. App. Math.)
- The Hilbert zonotope and universal Gröbner bases (Adv. App. Math.)
- Nonlinear matroid optimization and experimental design (SIAM Disc. Math.)
- Nonlinear optimization over a weighted independence system (submitted)

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- The complexity of 3-way tables (SIAM Comp.)
- Markov bases of 3-way tables (J. Symb. Comp.)
- All linear and integer programs are slim 3-way programs (SIAM Opt.)
- N-fold integer programming (Disc. Opt. in memory of Dantzig)
- Graver complexity of integer programming (Annals Combin.)
- Nonlinear bipartite matching (Disc. Opt.)
- Convex integer maximization (J. Pure App. Algebra)
- Convex integer minimization (submitted)

Some Geometric Methods:

Convex Discrete Maximization

Convex Discrete Maximization

Consider the convex hull $P = \text{conv}\{S\}$ of the feasible set S in Z^n.
When we can control the edge-directions of P, we can reduce convex to linear maximization in strongly polynomial time.

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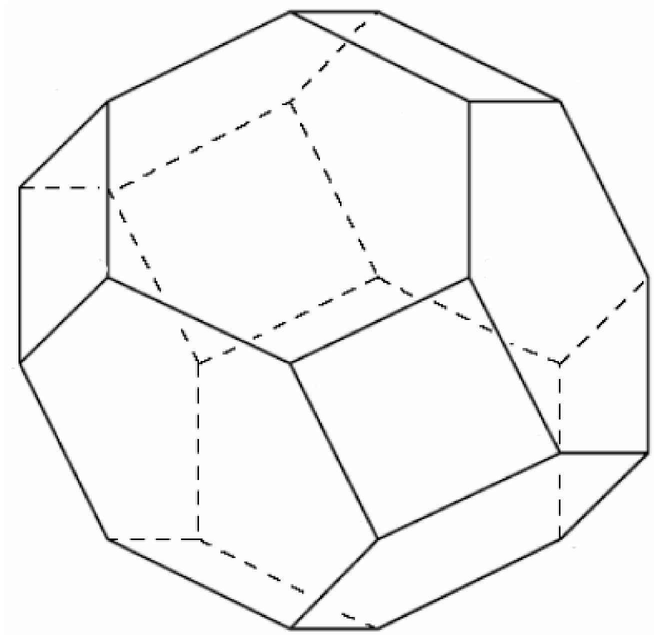

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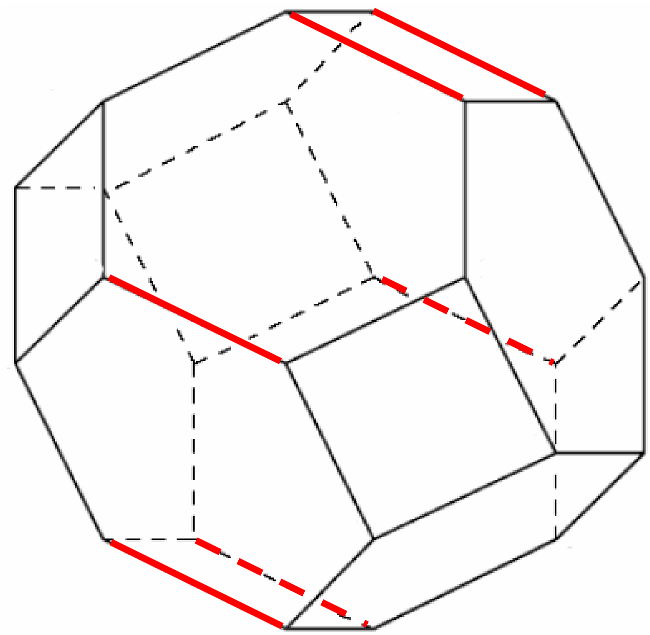

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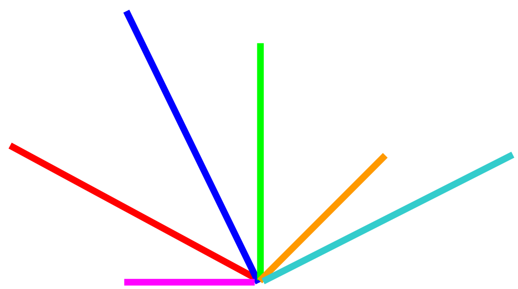

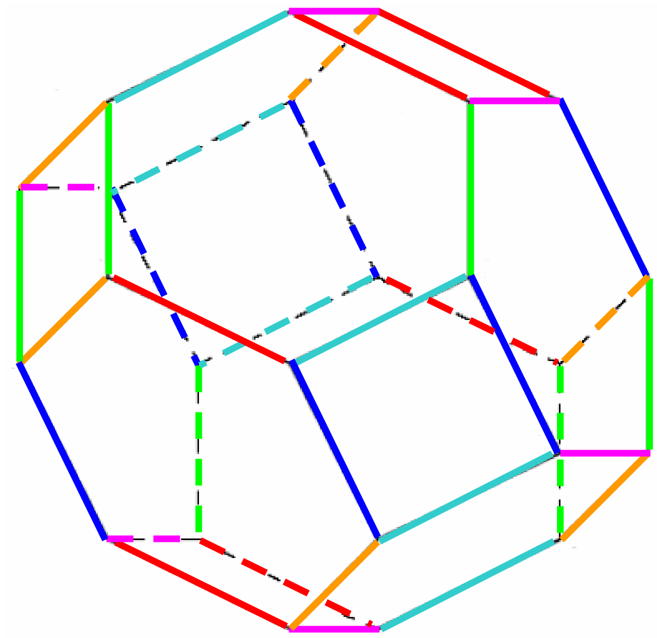

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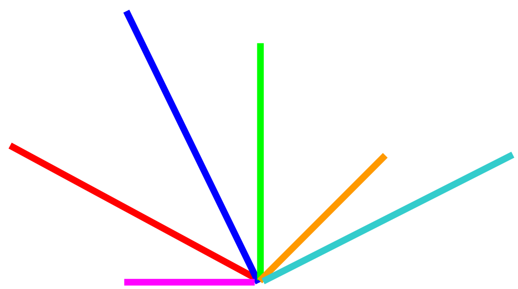

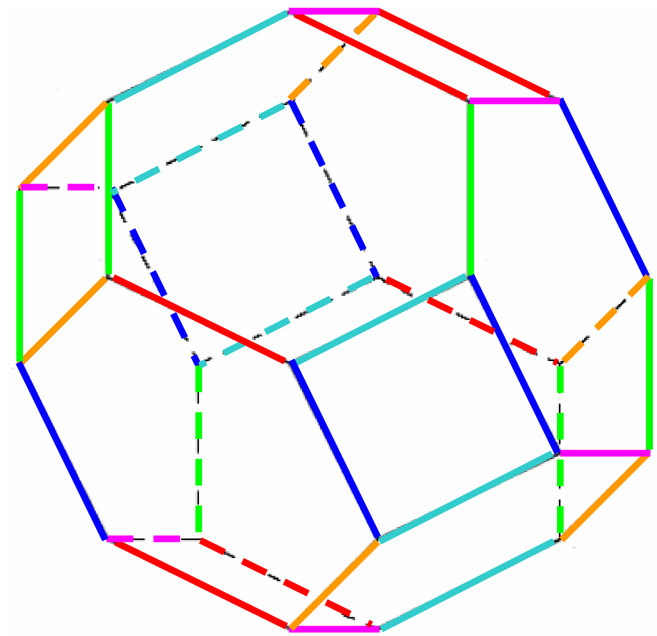

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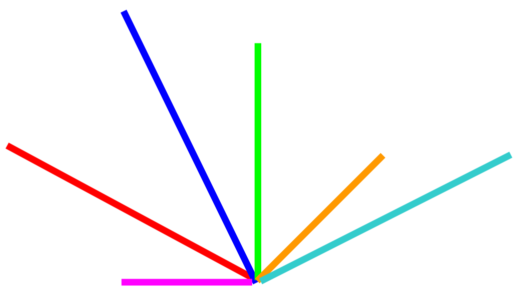

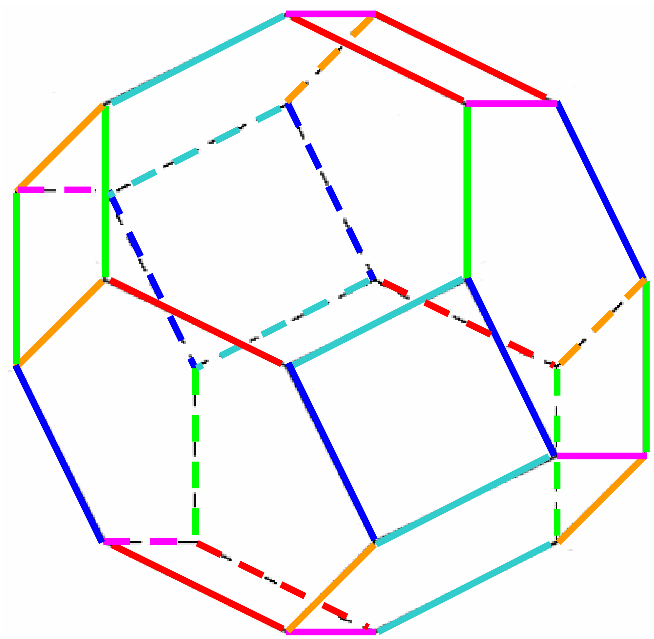

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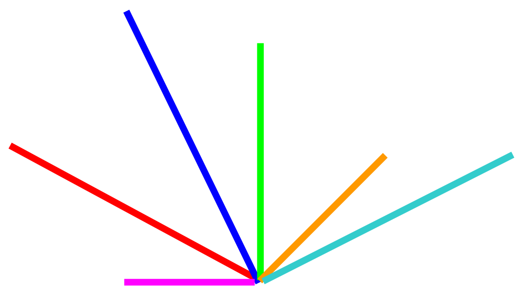

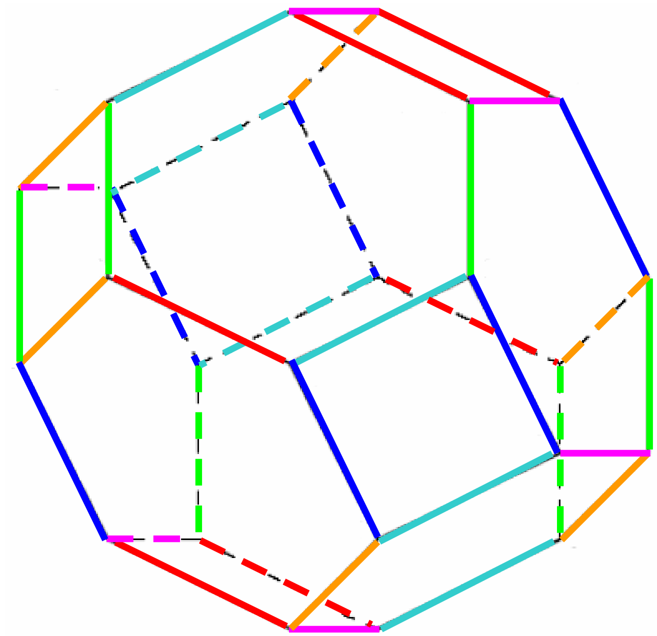

- Cubes: unit vectors 1_i (e.g. 0-1 quadratic programming)
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- Transportations: $n \times p$ circuit matrices (e.g. partitioning, clustering)

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Convex Discrete Maximization

Theorem: Fix any d. Then for any S in Z^n endowed with a set E that covers all edge-directions of conv{S}, and any convex f presented by a comparison oracle, the convex discrete maximization problem

$$\max \{ f(w_1 x, \ldots, w_d x) \ : \ x \text{ in } S \}$$

reduces to strongly polynomially linear counterparts over S.

Proof: preliminaries on zonotopes

Lemma: If $E = \{e^1, \ldots, e^m\}$ covers all edge-directions of a polytope P then the zonotope $Z = [-1, 1]\, e^1 + \ldots + [-1, 1]\, e^m$ is a refinement of P.

(Minkowsky, Grunbaum , ...,)

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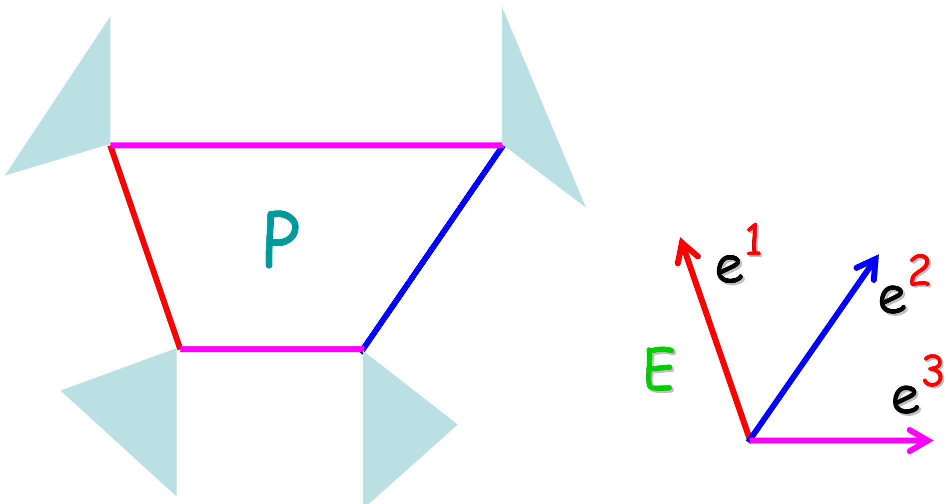

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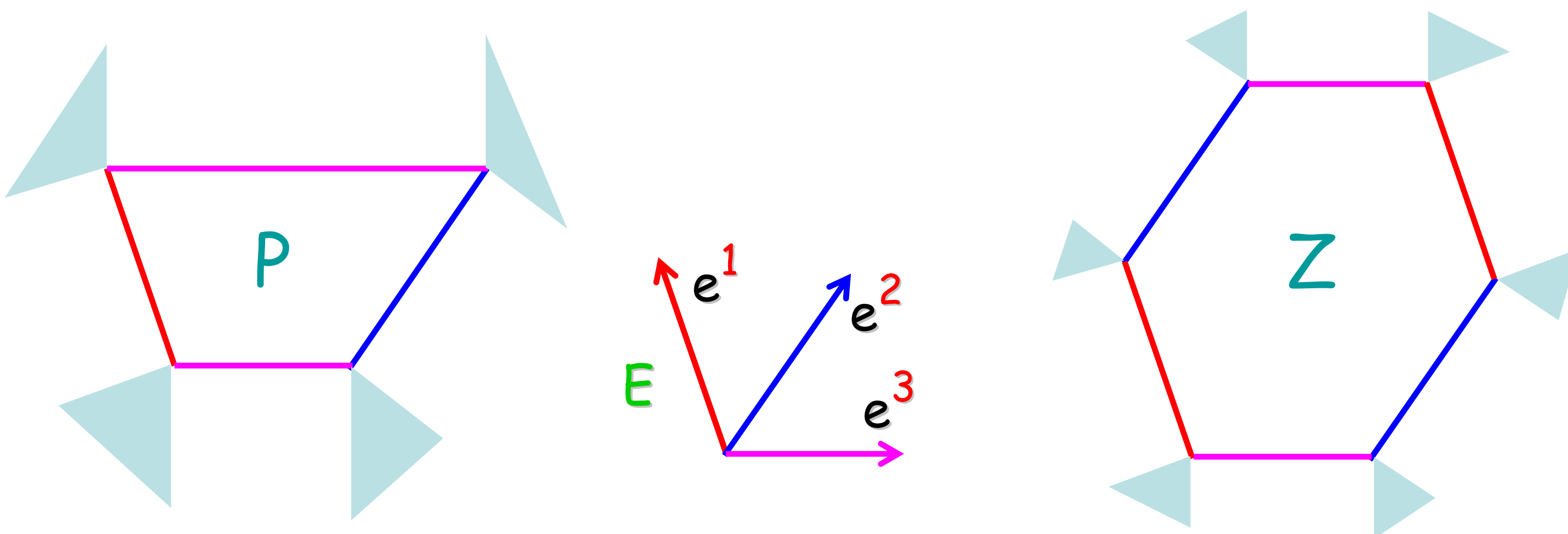

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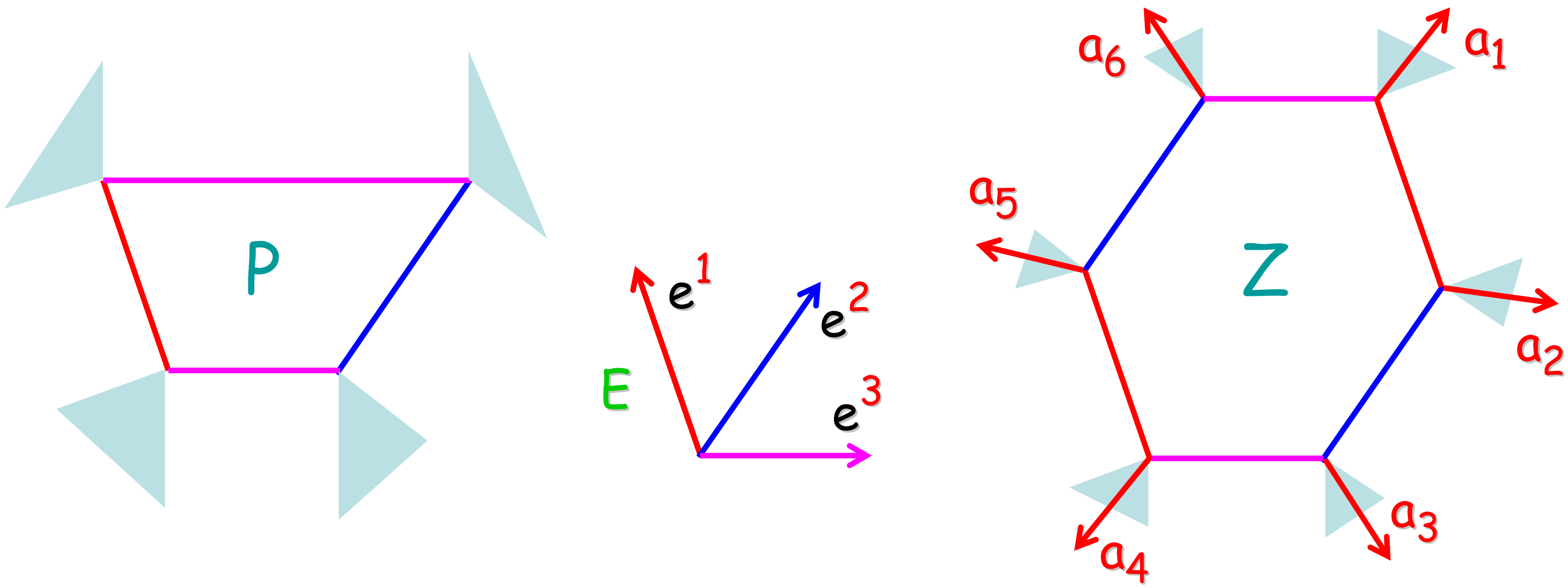

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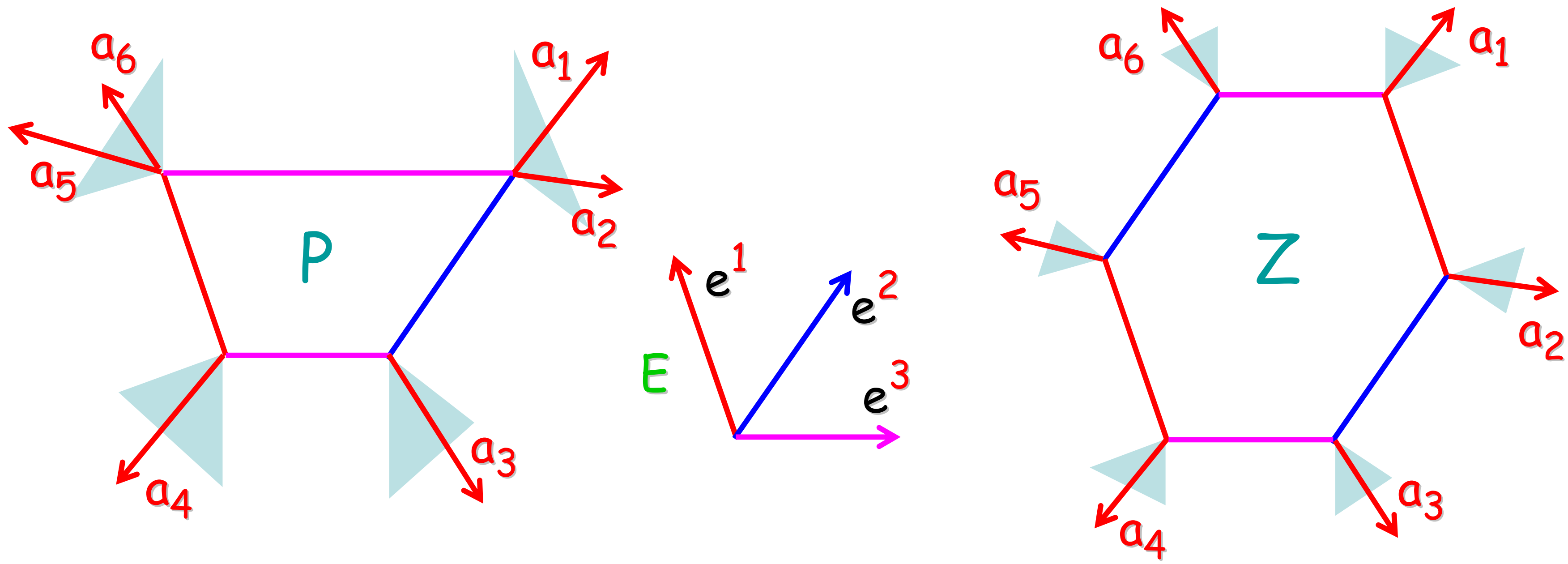

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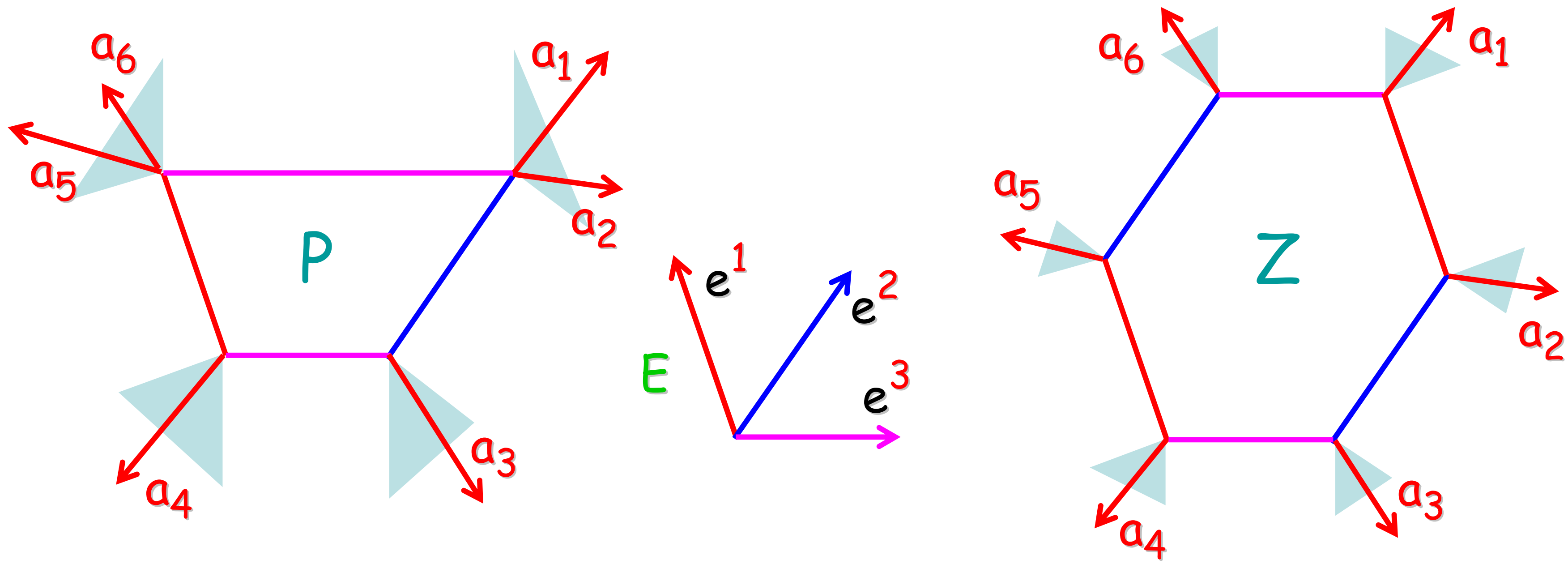

Lemma: In R^d, the zonotope Z can be constructed from $E = \{e^1, \ldots, e^m\}$ along with a vector a_i in the cone of every vertex in $O(m^{d-1})$ operations.

(Edelsbrunner, Gritzmann, Orourk, Seidel, Sharir, Sturmfels, …)

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Proof: the algorithm

Input: S in Z^n given by linear optimization oracle, set E of edge-directions of P=conv{S}, d x n matrix w, and convex f on R^d given by comparison oracle

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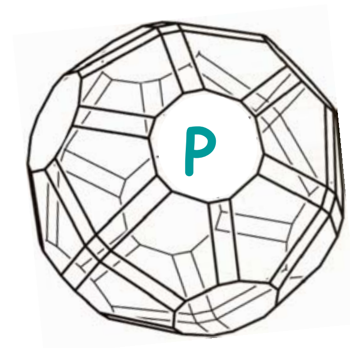
E



R^n

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Input: S in Z^n given by linear optimization oracle, set E of edge-directions of $P=conv\{S\}$, $d \times n$ matrix w, and convex f on R^d given by comparison oracle

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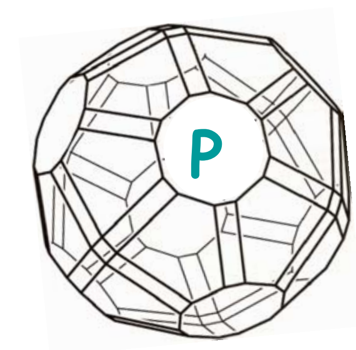

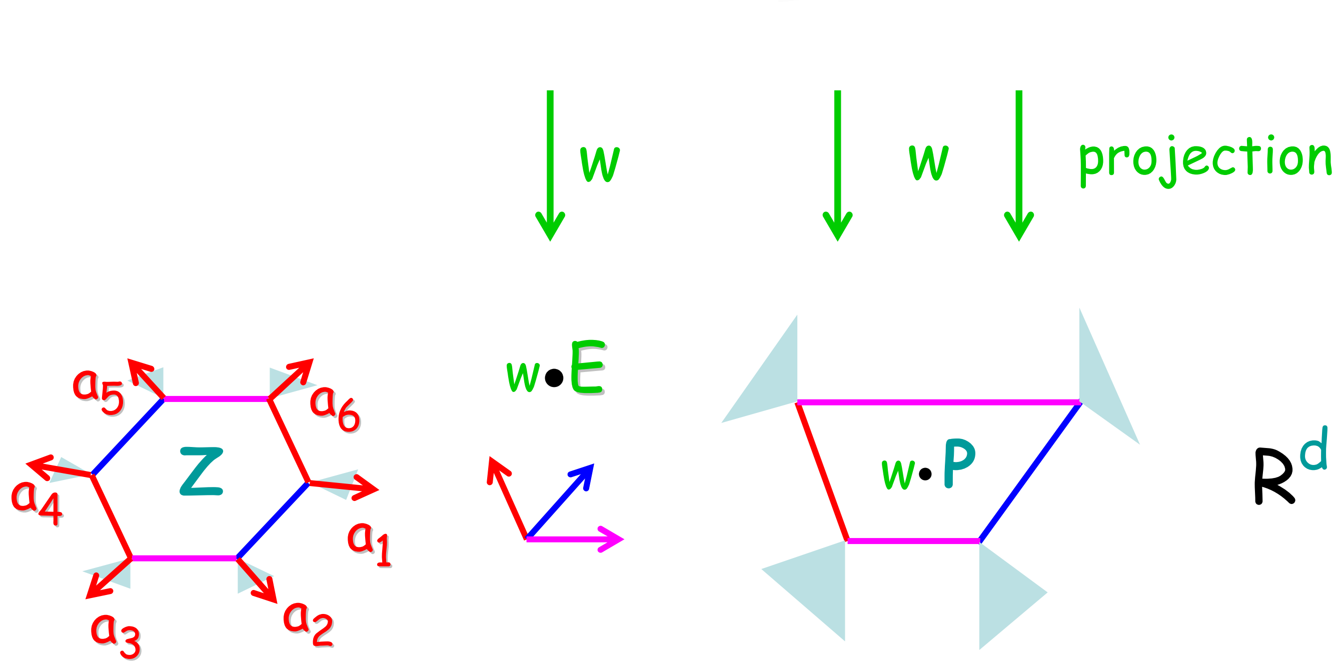

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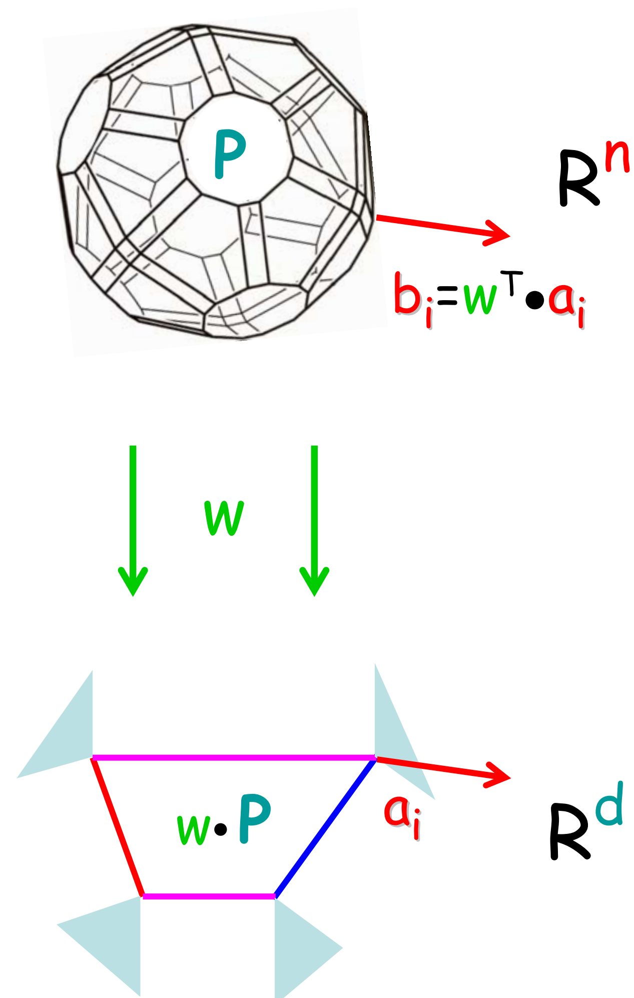
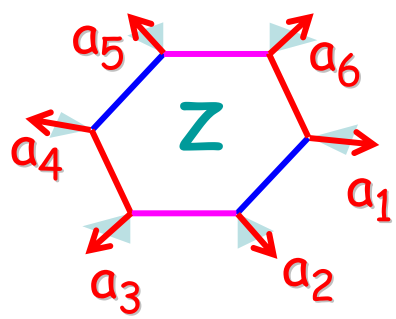
2. Lift each a_i in R^d to $b_i = w^T \bullet a_i$ in R^n and solve linear optimization with objective b_i over S





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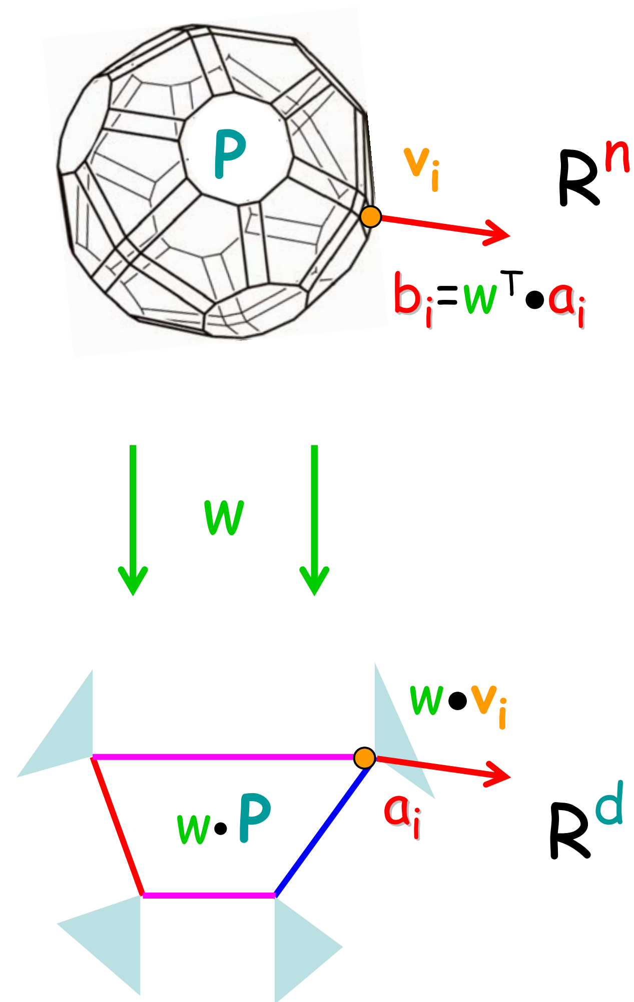
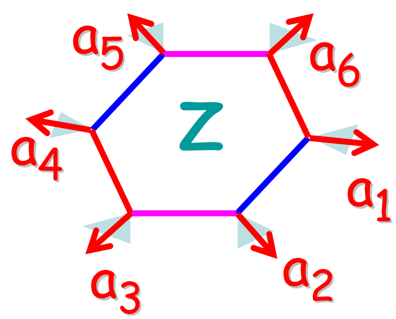
3. Obtain the vertex v_i of P and the vertex $w \bullet v_i$ of $w \bullet P$





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Proof: the algorithm

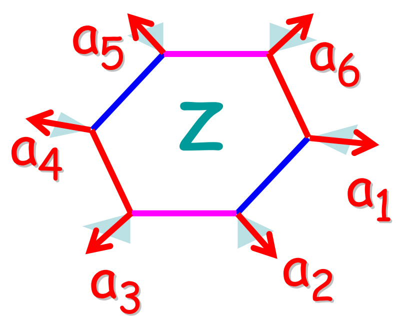
Input: S in Z^n given by linear optimization oracle, set E of edge-directions of $P=\text{conv}\{S\}$, $d \times n$ matrix w, and convex f on R^d given by comparison oracle

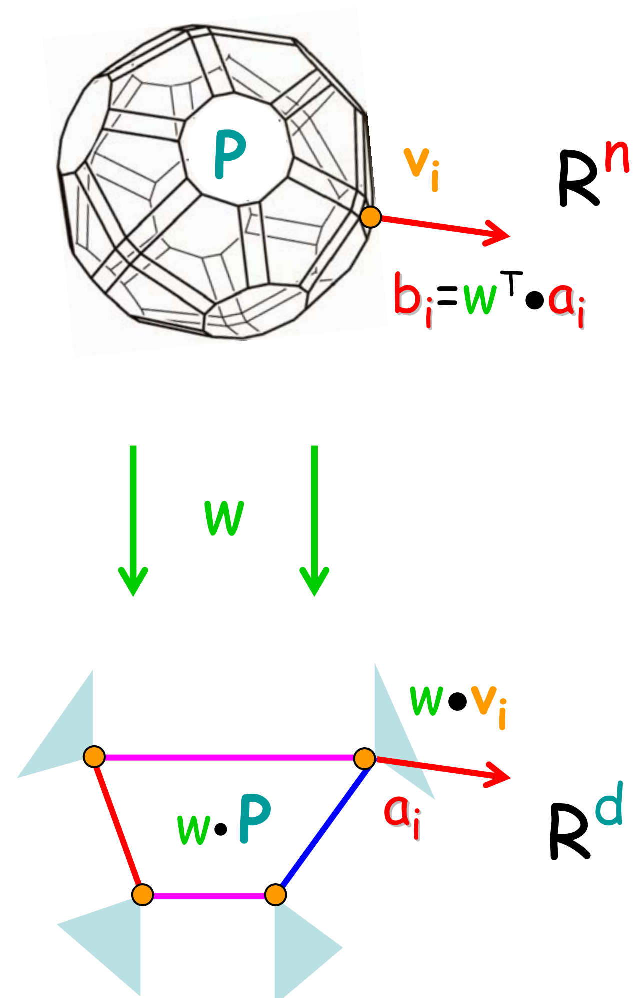
1. Construct the zonotope Z generated by the projection $w{\bullet}E$, and find a_i in each normal cone

2. Lift each a_i in R^d to $b_i = w^T{\bullet}\, a_i$ in R^n and solve linear optimization with objective b_i over S

3. Obtain the vertex v_i of P and the vertex $w{\bullet}v_i$ of $w{\bullet}P$

4. Output any v_i attaining maximum value $f(w{\bullet}\, v_i)$ using comparison oracle





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Strongly Polynomial
Convex Combinatorial Maximization

Theorem: For any fixed d, there is a strongly polynomial time algorithm that, given any S in $\{0,1\}^n$ presented by **membership oracle** and endowed with set E covering all edge-directions of conv$\{S\}$, and convex f, solves

$$\max \{ f(w_1 x, \ldots, w_d x) \ : \ x \text{ in } S \}$$

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- Convex combinatorial optimization (Disc. Comp. Geom.)

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Natural case with exponentially many edge-directions – permutation matrices and Birkhoff polytope - is treated in

- Nonlinear bipartite matching (Disc. Opt.)

Proof: membership → augmentation → linear optimization

Consider maximizing wx over S, with E all edge-directions of $\text{conv}\{S\}$:

Proof: membership → augmentation → linear optimization

Consider maximizing wx over S, with E all edge-directions of conv$\{S\}$:

Lemma: Membership → Augmentation

Proof: x in S can be improved if and only if there is an edge direction e in E such that $w \cdot e > 0$ and $x + e = y$ for some y in S.

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Lemma: Augmentation → linear Optimization

Proof: Schulz-Weismantel-Ziegler and Grötschel–Lovász using scaling ideas going back to Edmonds-Karp.

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Lemma: Augmentation → linear Optimization

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Lemma: Polynomial time → Strongly polynomial time

Proof: Frank-Tárdos show that using Diophantine approximation can replace w by w' of bit size depending polynomially only on n.

Some Algebraic Methods:

Nonlinear Integer Programming

Nonlinear Integer Programming

We now concentrate on the nonlinear integer programming problem:

$$\min \text{ or } \max \{f(w_1 x, \ldots, w_d x) : x \geq 0,\ Bx = b,\ x \text{ integer}\}$$

with $w_1 x, \ldots, w_d x$ linear forms on R^n and f real valued function on R^d.

N-Fold Systems

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N-Fold Systems

Let A be $(r+s) \times t$ matrix with submatrices A_1, A_2 of first r and last s rows.

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Define the n-fold product of A to be the following $(r+ns) \times nt$ matrix,

$$A^{(n)} = \underbrace{\begin{pmatrix} A_1 & A_1 & A_1 & \cdots & A_1 \\ A_2 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_2 \end{pmatrix}}_{n} .$$

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We have the following four theorems on n-fold integer programming:

Theorem 1: For any fixed matrix A, **linear integer programming** over n-fold products of A can be done in polynomial time:

$$\max \{ wx : \ A^{(n)}x = a,\ x \geq 0,\ x \text{ integer} \}$$

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We have the following four theorems on n-fold integer programming:

Theorem 2: For any fixed d and matrix A, **convex integer maximization** over n-fold products of A can be done in polynomial time:

$$\max \{ f(w_1 x, \ldots, w_d x) \ : \ A^{(n)} x = a,\ x \geq 0,\ x \text{ integer} \}$$

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Theorem 3: For fixed matrix A, **separable convex integer minimization** over n-fold products of A can be done in polynomial time:

$$\min \{ f_1(x_1) + \ldots + f_{nt}(x_{nt}) \ : \ A^{(n)}x = a,\ x \geq 0,\ x \text{ integer} \}$$

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We have the following four theorems on n-fold integer programming:

Theorem 4: For fixed matrix A, **integer point l_p-nearest to a given x** over n-fold products of A can be determined in polynomial time:

$$\min \{ \, |x - \mathbf{x}|_p \; : \; A^{(n)}x = a, \; x \geq 0, \; x \text{ integer} \}$$

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- N-fold integer programming (Disc. Opt. in memory of Dantzig)
- Convex integer maximization via Graver bases (J. Pure App. Algebra)
- Convex integer minimization (submitted)

Universality of N-Fold Systems

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Universality of N-Fold Systems

Define the n-product of A as the following variant of the n-fold operator:

$$A^{[n]} = \underbrace{\begin{pmatrix} I & I & I & \cdots & I \\ A & 0 & 0 & \cdots & 0 \\ 0 & A & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A \end{pmatrix}}_{n} \quad .$$

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Consider m-products of the 1×3 matrix $(1 \;\; 1 \;\; 1)$. For instance,

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$$(1\ \ 1\ \ 1)^{[3]} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} .$$

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Universality Theorem: Any bounded set $\{x \text{ integer} : x \geq 0,\ Bx = b\}$ is in polynomial-time-computable coordinate-embedding-bijection with some

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- All linear and integer programs are slim 3-way programs (SIAM Opt.)

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Scheme for Nonlinear Integer Programming:

any program: $\max \{f(w_1 x, \ldots, w_d x) : x \geq 0,\ Bx = b,\ x \text{ integer}\}$

can be lifted to

n-fold program: $\max \{f(w'_1 x, \ldots, w'_d x) : x \geq 0,\ (1\ 1\ 1)^{[m][n]}x = a,\ x \text{ integer}\}$

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The Computational Complexity of Nonlinear Integer Programming:

The Graver Complexity of $K_{3,m}$

Graver Bases

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Vector x conforms to y if lie in same orthant and $|x_i| \leq |y_i|$ and for all i.

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non-circuits: ±(1 -1 1)

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Graver Bases of N-Products

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Lemma: Every integer matrix A has a finite Graver complexity $g(A)$ such that any element in the Graver basis $G(A^{[n]})$ for any n has type $\leq g(A)$.

(Aoki-Takemura, Santos-Sturmfels, Hosten-Sullivant)

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Practicality: the complexity is high and dominated by $n^{g(A)}$, but promising scheme is to consider Graver elements of gradually increasing type.

The Graver Complexity of a Graph

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Definition: The Graver complexity of a graph or a digraph G is the Graver complexity $g(G):=g(A)$ of its vertex-edge incidence matrix A.

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For instance,

$$g(3) = g\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} = 9$$

The Graver Complexity of a Graph

Recall: $g(m) := g(K_{3,m}) = g((1\ 1\ 1)^{[m]})$

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So if P≠NP then $g(m)$ must grow with m. How fast ?

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The Graver Complexity of a Graph

Recall: $g(m) := g(K_{3,m}) = g((1\ 1\ 1)^{[m]})$

Theorem: We have $\Omega(2^m) = g(m) = O(m^4 6^m)$.

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Also: $g(2)=3$, $g(3)=9$, $g(4) \geq 27$, $g(5) \geq 61$.

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- Graver complexity of integer programming (Annals Combin.)

Proofs of Theorems 1-4

about

Nonlinear N-Fold Integer Programming

Proof of Theorem 1 - Linear Optimization

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- Compute the Graver basis of $A^{(n)}$ efficiently.

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- Use an auxiliary n-fold program to find an initial feasible point.

Proof of Theorem 2 - Convex Maximization

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- Simulate linear optimization oracle using Theorem 1.

Proof of Theorem 2 – Convex Maximization

- Compute the Graver basis of $A^{(n)}$ efficiently.

- Simulate linear optimization oracle using Theorem 1.

- Reduce convex to linear maximization using the Geometric Theorem with the Graver basis $G(A^{(n)})$ providing a set of edge-directions of

$$\text{conv}\{x : A^{(n)}x = a,\ x \geq 0,\ x \text{ integer}\}$$

Proof of Theorem 3 - Separable Convex Minimization

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- Compute Graver basis of n-fold product of auxiliary larger matrix.

Proof of Theorem 3 – Separable Convex Minimization

- Compute Graver basis of n-fold product of auxiliary larger matrix.

- Use integer caratheodory, convergence property, and superadditivity, and use the Graver basis to augment feasible to optimal point.

Proof of Theorem 3 – Separable Convex Minimization

- Compute Graver basis of n-fold product of auxiliary larger matrix.

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Proof of Theorem 4 - l_p-Nearest Point

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Proof of Theorem 4 - l_p-Nearest Point

- For positive integer p this is the special case of Theorem 3 with

$$(|x - \mathbf{x}|_p)^p = |x_1 - \mathbf{x}_1|^p + \ldots + |x_{nt} - \mathbf{x}_{nt}|^p$$

Proof of Theorem 4 - l_p-Nearest Point

- For positive integer p this is the special case of Theorem 3 with

$$(|x - \mathbf{x}|_p)^p = |x_1 - \mathbf{x}_1|^p + \ldots + |x_{nt} - \mathbf{x}_{nt}|^p$$

- For p infinity this reduces to the case of finite q for suitably large q.

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Multiway Tables and Applications

Multiway Tables and Margins

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Multiway Tables and Margins

A k-way table is an $m_1 \times \cdots \times m_k$ array of nonnegative integers.

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Multiway Tables and Margins

A k-way table is an $m_1 \times \cdots \times m_k$ array of nonnegative integers.

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2	3	2	

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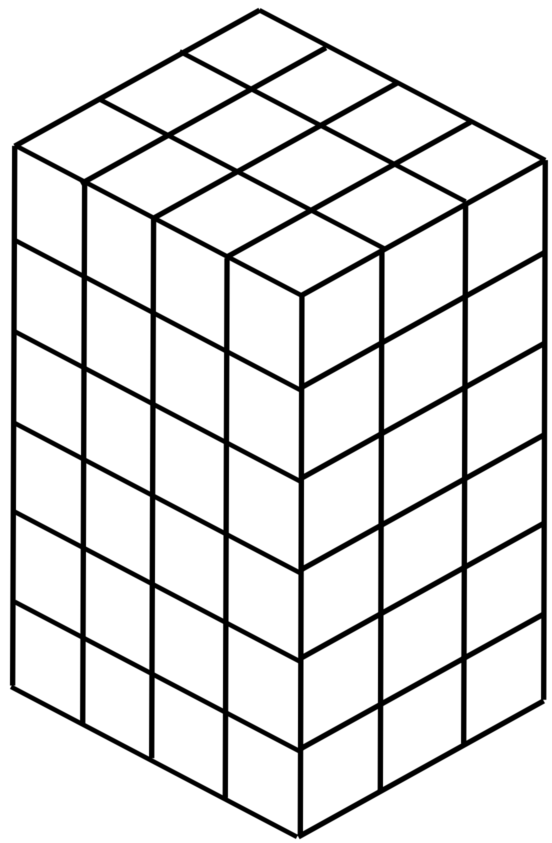
Example: 3-way table of size $6 \times 4 \times 3$:



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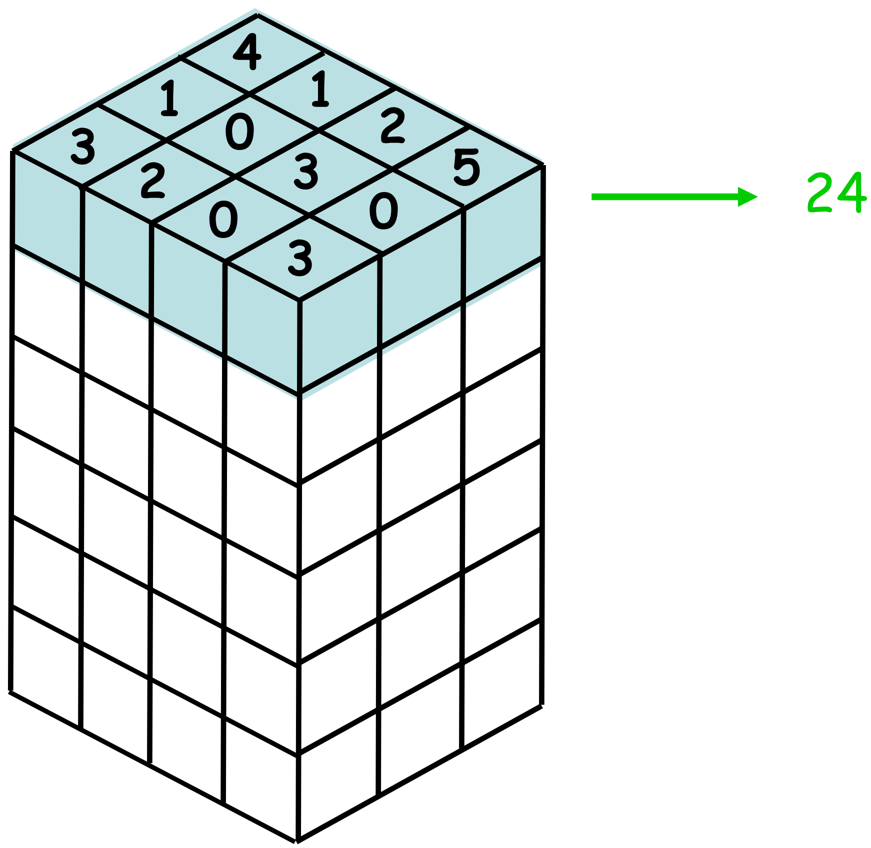
Example: 3-way table of size $6 \times 4 \times 3$ with a plane-sum:



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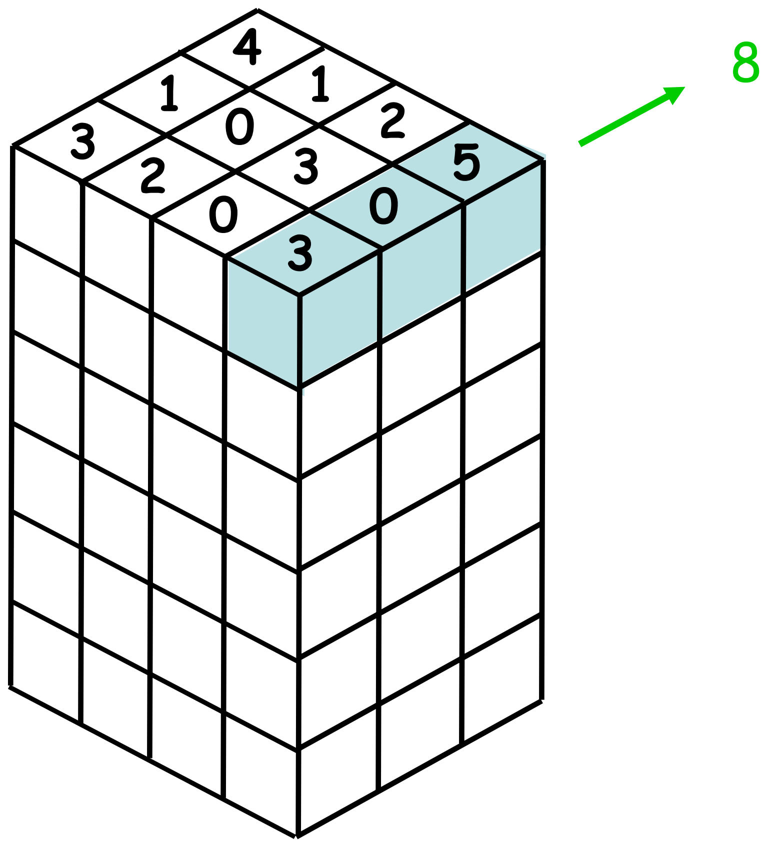
Example: 3-way table of size $6 \times 4 \times 3$ with a line-sum:



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Casting "Long" Multiway Tables as N-fold Programs

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Casting "Long" Multiway Tables as N-fold Programs

The margin equations for $m_1 \times \cdots \times m_k \times n$ tables form an n-fold system defined by a suitable $(r+s) \times t$ matrix A, where A_1 controls the equations of margins involving summation over layers, whereas A_2 controls the equations of margins involving summation within a single layer at a time.

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$$A^{(n)} = \underbrace{\begin{pmatrix} A_1 & A_1 & A_1 & \cdots & A_1 \\ A_2 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_2 \end{pmatrix}}_{n} .$$

$$A^{(n)} x = a, \qquad x = (x^1, \ldots, x^n), \qquad a = (a^0, a^1, \ldots, a^n)$$

$$A_1(x^1 + \ldots + x^n) = a^0, \qquad A_2 x^k = a^k, \quad k = 1, \ldots n$$

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Example:

The line-sum equations for $3 \times 3 \times n$ tables are defined by the $(9+6) \times 9$ matrix A where A_1 is the 9×9 identity matrix and

$$A_2 = (1 \;\; 1 \;\; 1)^{[3]} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} .$$

Long Tables are Efficiently Treatable

Corollary: nonlinear optimization over $m_1 \times \cdots \times m_k \times n$ tables with hierarchical margin constraints can be done in polynomial time.

Privacy and Confidentiality in Statistical Data Bases: Entry Uniqueness Problem

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Corollary: For $m_1 \times \cdots \times m_k \times n$ tables can check entry uniqueness in polynomial time and refrain from margin disclosure if vulnerable.

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Congestion Avoiding Transportation

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In the high dimensional extension, wish to find an optimal multiway table satisfying margin constraints and upper bounds on the entries.

Corollary: We can find congestion avoiding transportation over $m_1 \times \cdots \times m_k \times n$ tables in polynomial time.

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Error Correcting Codes

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Corollary: We can error-correct messages of format $m_1 \times \cdots \times m_k \times n$ in polynomial time.

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- Nonlinear optimization over a weighted independence system (submitted)