Line Shelling Zonotopes of Polytopes

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Motivation:

One nice way to show a quantity is nonnegative for all polytopes is through shellings: Show that at each step of a shelling, the addition of the new facet changes the quantity by a nonnegative amount.

Some times this doesn't work. Say the addition of a certain type of facet actually decreases the quantity.

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One nice way to show a quantity is nonnegative for all polytopes is through shellings: Show that at each step of a shelling, the addition of the new facet changes the quantity by a nonnegative amount.

Some times this doesn't work. Say the addition of a certain type of facet actually decreases the quantity.

Idea: Consider the reverse shelling.

If that doesn't work, consider more shellings.

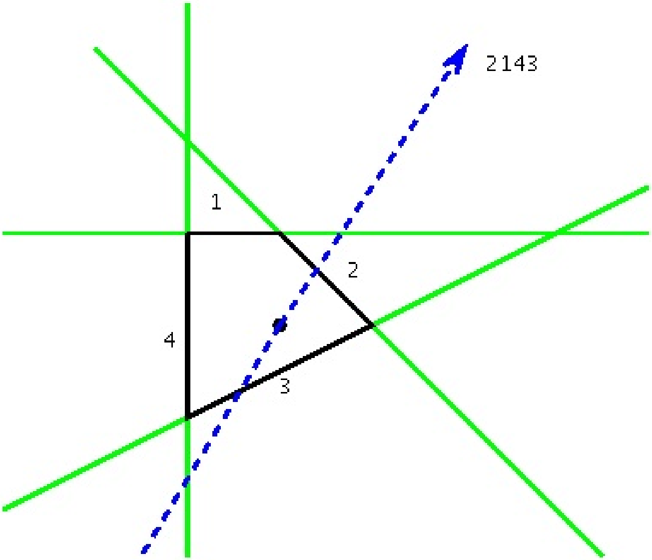
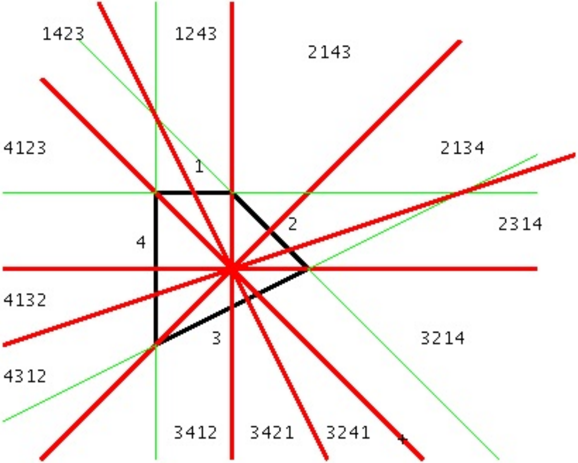
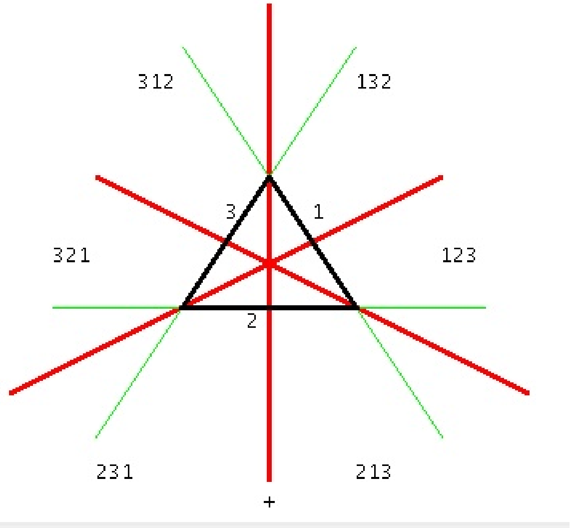


Figure: A line shelling of a quadrilateral.



We get the line shelling hyperplane arrangement by taking the linear spans of the pairwise intersections of the facet-defining hyperplanes. The line shelling zonotope is dual to this.



The line shelling hyperplane arrangement of a triangle.

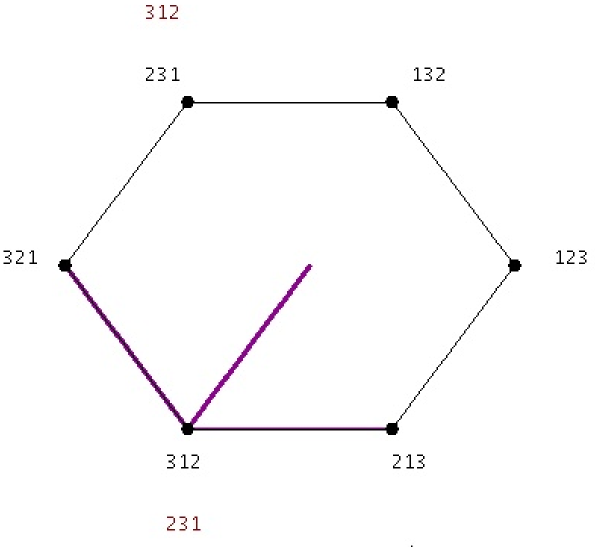


Figure: The permutahedron P_3.

Theorem

The line shelling hyperplane arrangement of the standard simplex is the braid arrangement, and the corresponding line shelling zonotope is a permutahedron.

Well-known fact (Grünbaum, p. 71): Every d-polytope P with $n+1$ facets is the intersection of an n-simplex with a d-dimensional flat.

Corollary

The line shelling hyperplane arrangement of P is the intersection of this d-dimensional flat with the braid arrangement.

Dual version:

Well-known fact (Grunbaum, p. 71): Every d-polytope P with $n+1$ vertices is the projection of the standard n-simplex onto a d-dimensional flat given by the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ v_1 & v_2 & \cdots & v_{n+1} \end{bmatrix}$$

where v_i is the column vector of the coordinates of vertex i.

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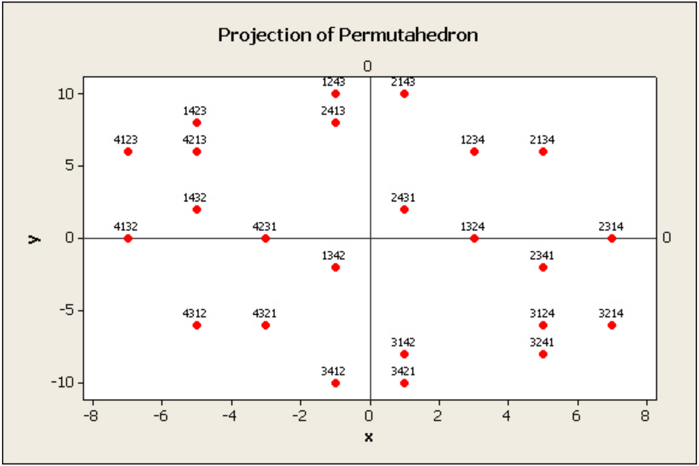
Corollary

Let P be a d-polytope with $n+1$ facets. The line shelling zonotope of P is the projection of the (negative of the) permutahedron P_{n+1} by the standard projection of the n-simplex to the dual of P.



The projection of the negative of the permutahedron to the dual of our original quadrilateral (with appropriate relabeling).

Question: What are the faces of a line shelling zonotope?

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Answer: Projections of faces of the permutahedron.

Billera and Sarangarajan described the faces of the permutahedron in terms of ordered partitions of $\{1, 2, \ldots, n\}$.

Faces of permutahedra are products of lower-dimensional permutahedra.

Faces of the line shelling zonotope are

- projections of products of permutahedra
- products of projections of permutahedra
- products of other line shelling zonotopes.

Up to scaling and a shift, the zones of the permutahedron are given by the vectors $[0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0] = e_i - e_j$. Since our projection matrix is $\begin{bmatrix} 1 & 1 & \cdots & 1 \\ v_1 & v_2 & \cdots & v_{n+1} \end{bmatrix}$, the zones of our line shelling zonotope are given by $v_i - v_j$.

Question: Does it matter where you put the origin?

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Answer: Yes

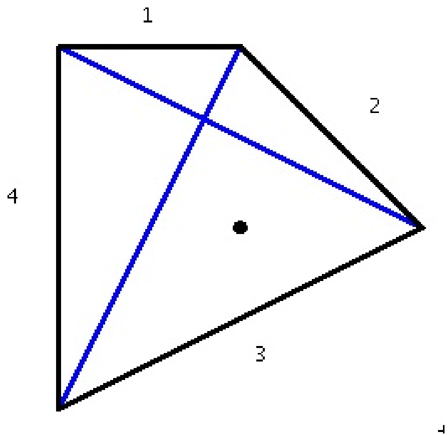

Figure: Placing the origin: different regions give different line shelling zonotopes.

Question: Does the line shelling zonotope of P depend only on the combinatorial type of P?

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Answer: No. Geometry matters.

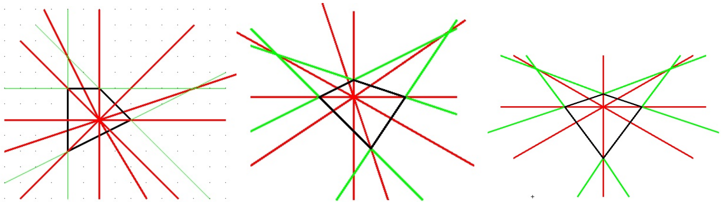


Figure: Three quadrilaterals with different line shelling zonotopes.

Projection of the permutahedron in a sufficiently general direction corresponds to truncation of its underlying matroid.

Good news (Zaslavsky; Bayer and Sturmfels; Billera, Ehrenborg and Readdy): Anything you want to count about the faces of the line shelling zonotope depends only on the underlying matroid.

Whenever the origin of the original polytope is in a sufficiently general position, the line shelling zonotope should have the same number of vertices.

Allowable sequences (Goodman and Pollack)

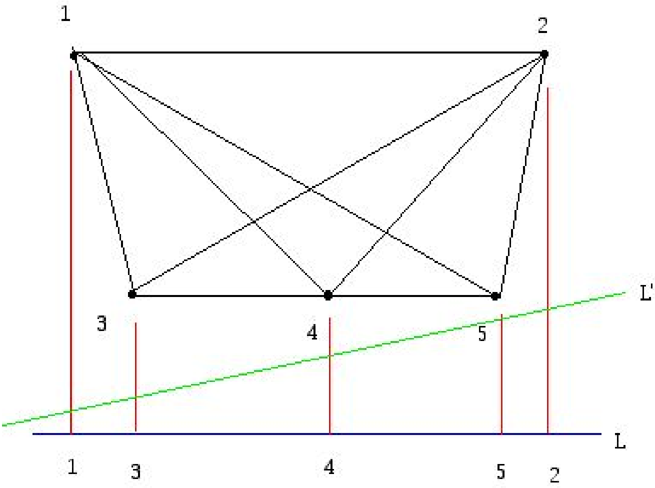


Figure: Malkevitch

Definition

(Goodman and Pollack) An *allowable sequence of permutations* is a periodic sequence of permutations with the following two properties:

1. The move from each permutation to the next consists of reversing one or more (non-overlapping)substrings.
2. If a move results in the reversal of a pair ij, then every other pair is reversed subsequently by the time i and j are again.

Remark Some allowable sequences of permutations come from point configurations in the plane, and some do not.

In the plane, the line shelling zonotope of a polytope P gives the allowable sequence of the vertices of the polar polytope P^*.

Higher dimensions: Goodman and Pollack (1986, 1993) say that the appropriate way to generalize allowable sequences to points configurations in R^d is to orthogonally project the points to a line and then to rotate this line through all possible directions on the sphere. "One gets, in this way, a much richer structure than in the case of $d = 2$, one which has not yet fully been investigated."

Theorem

This structure is given by the projection of the permutahedron by

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ v_1 & v_2 & \cdots & v_{n+1} \end{bmatrix}$$

where v_i is the column vector of the coordinates of point i (even if the points are not the vertices of a polytope).

Two point configurations can have the same oriented matroid but different allowable sequences.

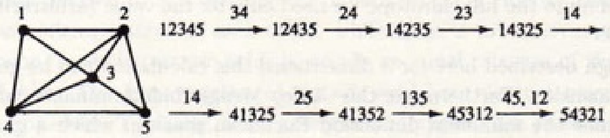


Figure 1.10.1: An allowable sequence of permutations.

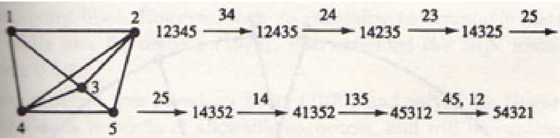
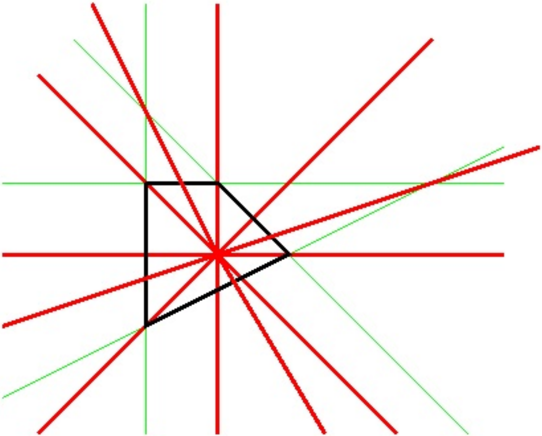


Figure 1.10.2: Different allowable sequence, same oriented matroid.

Figure: Bjorner et al.

If you add certain points and lines to a point configuration, you get the "big" oriented matroid which determines the allowable sequence. We already have the hyperplane version of this, and it should work for higher dimensions.



Theorem

(Stanley; Edelman and Greene) Simple allowable sequences containing the identity permutation $12\ldots n$ *are counted by maximal chains in the weak Bruhat order on* S_n.

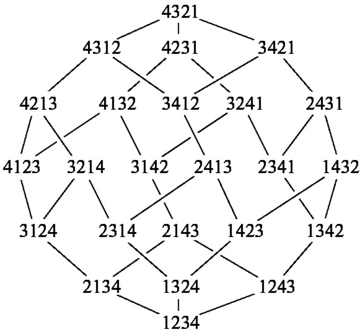


Figure: Zabroski

Questions:

Can we say something similar using this?

Palacios and Ronco, Weak Bruhat order on the set of faces of the permutohedron and the associahedron. *Journal of Algebra* 2006.

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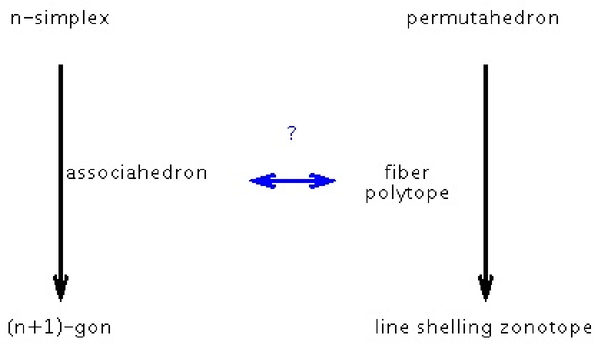
What do we do to the permutahedron to get the allowable sequences that don't come from point configurations?

What's the right set of rules to define the higher-dimensional version of allowable sequences that don't come from point configurations?

What about fiber polytopes?



Projections of permutahedra are interesting.

Back to my original question:
Can we prove certain quantities are non-negative for polytopes by averaging them over all line shellings?

For Adin's problem: How often is a facet first, second, kth in a shelling? How often does it appear after one, two, k of the facets adjacent to it have already been added?

Euler: In a shelling of a polytope with n facets, the first facet contributes 1 to the quantity $V - E + F$. The next $n - 2$ facets contribute 0, and the final facet contributes 1.

$$V - E + F =$$

$$\frac{1}{\#\text{ of shellings}} \sum_{\text{shellings}} \sum_{\text{facets}} \text{contribution of facet}$$

$$= \frac{1}{\#\text{ of shellings}} \sum_{\text{facets}} \sum_{\text{shellings}} \text{contribution of facet}$$

$$= \frac{1}{\#\text{ of shellings}} \sum_{\text{facets}} \#\text{ times facet is first}$$

$$+\#\text{ times facet is last}$$

$$= \frac{1}{\#\text{ of shellings}} \sum_{\text{facets}} 2 * \#\text{ times facet is first}$$

$$= 2$$