Quasisymmetric Schur functions

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Compositions and partitions

A composition $\alpha_1 \ldots \alpha_k$ of n is a list of positive integers whose sum is n: $2213 \vDash 8$.

A composition is a partition if $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k > 0$: $3221 \vdash 8$.

Any composition determines a partition: $\lambda(2213) = 3221$.

$\alpha = \alpha_1 \ldots \alpha_k$ is a coarsening of $\beta = \beta_1 \ldots \beta_l$ (β is a refinement of α) if

$$\underbrace{\beta_1 + \ldots + \beta_i}_{\alpha_1} \underbrace{\beta_{i+1} + \ldots + \beta_j}_{\alpha_2} \ldots \underbrace{\beta_m + \ldots + \beta_l}_{\alpha_k}$$

is true: $53 \geq 2213$.

Macdonald polynomials ...

1988: (Macdonald) Introduced $\{P_\lambda(X;q,t)\}_\lambda$

$$P_\lambda(X;q,t) = \prod_{u \in \mathsf{dg}'(\lambda^\circ)} (1 - q^{l(u)+1} t^{a(u)}) \sum_{\lambda(\mu)=\lambda} \frac{E_\gamma(X;q^{-1},t^{-1})}{\prod(1 - q^{l(u)+1} t^{a(u)})}$$

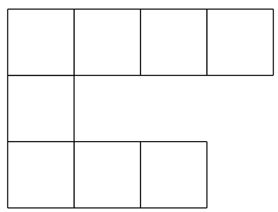
letting $q, t \to 0$

$$s_\lambda(X) = \sum_{\lambda(\mu)=\lambda} E_\gamma(X;\infty,\infty).$$

(Lascoux-Schützenberger (90); Reiner-Shimozono (95); Haglund, Haiman, Loehr (05); Mason (06))

Composition diagrams and tableaux

The composition diagram $\alpha = \alpha_1 \ldots \alpha_k > 0$ is the array of boxes with α_i boxes in row i from the top.



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A (standard) composition tableau of shape α is a filling of α with (each first n) $1, 2, 3, \ldots$ such that

Rules for composition tableaux I

- First column entries strictly increase top to bottom.
- Rows weakly decrease left to right.

Example

5	4	3	1
6			
8	7	2	

Rules for composition tableaux II

- When the filling is extended for $1 \le i < j, 2 \le k$

$$(j,k) \neq 0, (j,k) \geq (i,k) \Rightarrow (j,k) > (i,k-1).$$

Example

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Example

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Weights and weakness

Given a composition tableau T we have

$$x^T := x_1^{\#1s} x_2^{\#2s} x_3^{\#3s} \ldots$$

Example

$$\mathrm{T} = \begin{array}{|c|c|c|c|}\hline 5 & 4 & 3 & 1 \\ \hline 6 \\ \cline{1-1} 8 & 7 & 2 \\ \cline{1-3}\end{array} \qquad x^T = x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8$$

A weak composition $\gamma_1 \ldots \gamma_k$ of n is a list of non-negative integers whose sum is n: $2021003 \vDash 8$.

Demazure atoms

The collapse of a weak composition γ, $\alpha(\gamma)$, is γ with 0's removed: $\alpha(2021003) = 2213$.

The foundation of a weak composition γ, $\mathcal{F}o(\gamma)$, is all i so $\gamma_i > 0$: $\mathcal{F}o(2021003) = \{1, 3, 4, 7\}$.

Then

$$NS_\gamma(X) = E_{\gamma_n,\ldots,\gamma_1}(x_n, \ldots, x_1; \infty, \infty) = \sum_\gamma x^T$$

over composition tableaux shape $\alpha(\gamma)$, entries $1, 2, 3, \ldots, n$, first column $\mathcal{F}o(\gamma)$.

NS_γ for $\gamma = 102$

1	
3	3

1	
3	2

1	
3	1

$$NS_\gamma = x_1 x_3^2 + x_1 x_2 x_3 + x_1^2 x_3.$$

NS_γ for $\gamma = 102$

1	0
3	3

1	
3	2

1	
3	1

$$NS_\gamma = x_1 x_3^2 + x_1 x_2 x_3 + x_1^2 x_3.$$

NS_γ for $\gamma = 102$

1	
3	3

1	
3	2

1	0
3	1

$$NS_\gamma = x_1 x_3^2 + x_1 x_2 x_3 + x_1^2 x_3.$$

NS_γ for $\gamma = 102$

1	
3	3

1	
3	2

$$NS_\gamma = x_1 x_3^2 + x_1 x_2 x_3 + \qquad\qquad .$$

Why quasisymmetric functions?

1. P-partitions (Stanley 74; Gessel 83).

2. Combinatorial Hopf algebras (Ehrenborg 96; Aguiar/Begeron/Sottile 06). Dual to the cd-index (Billera/Hsaio/vW 03).

3. Equality of ribbon Schur functions (Billera/Thomas/vW 06).

4. Coloured, type B (Hsaio/Petersen 06; Ehrenborg/Readdy 06).

5. KL-polynomials (Billera/Brenti 07).

Quasisymmetric functions I

Let $\mathcal{Q} \subset \mathbb{Q}[[x_1, x_2, \ldots]]$ be the algebra of all quasisymmetric functions

$$\mathcal{Q} := \mathcal{Q}_0 \oplus \mathcal{Q}_1 \oplus \cdots$$

where

$$\mathcal{Q}_n := \text{span}_{\mathbb{Q}}\{M_\alpha \mid \alpha = \alpha_1 \ldots \alpha_k \models n\}$$

$$M_\alpha := \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.$$

Example $M_{121} = \sum_{i_1 < i_2 < i_3} x_{i_1}^1 x_{i_2}^2 x_{i_3}^1 = x_1 x_2^2 x_3 + \ldots$

Quasisymmetric functions II

For $\alpha \models n$ define

$$F_\alpha = \sum_{\beta \preceq \alpha} M_\beta.$$

The F_α are the fundamental quasisymmetric functions.

Example

$$F_{121} = M_{121} + M_{1111}$$

Symmetric functions I

Let $\Lambda \subset \mathbb{Q}[[x_1, x_2, \ldots]]$ be the algebra of all symmetric functions

$$\Lambda := \Lambda_0 \oplus \Lambda_1 \oplus \cdots$$

where

$$\Lambda_n := \text{span}_{\mathbb{Q}}\{m_\lambda \mid \lambda \vdash n\}$$

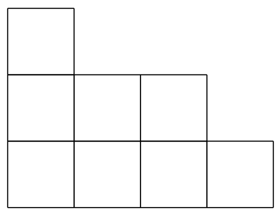
$$m_\lambda := \sum_{\lambda(\alpha)=\lambda} M_\alpha.$$

Example

$$m_{211} = M_{211} + M_{121} + M_{112}$$

Contretableaux

The diagram $\lambda = \lambda_1 \geq \ldots \geq \lambda_k > 0$ is the array of boxes with λ_i boxes in row i from the bottom.



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A (standard) contretableau T of shape λ is a filling of λ with (each first n) $1, 2, 3, \ldots$ so rows weakly decrease and columns strictly increase.

Contretableaux

The diagram $\lambda = \lambda_1 \geq \ldots \geq \lambda_k > 0$ is the array of boxes with λ_i boxes in row i from the bottom.

5			
6	4	2	
8	7	3	1

431

A (standard) contretableau T of shape λ is a filling of λ with (each first n) $1, 2, 3, \ldots$ so rows weakly decrease and columns strictly increase.

Descents and subsets

$D(T)$ is i where $i+1$ is not strictly west:

5			
6	4	2	
8	7	3	1

composition $\alpha_1 \dots \alpha_k \vDash n \leftrightarrow$ subset $\{i_1, \dots, i_{k-1}\} \subseteq [n-1]$

$\beta \qquad 2312 \vDash 8 \leftrightarrow \{2,5,6\} \subseteq [7] \qquad S(\beta)$

Symmetric functions II

For $\lambda \vdash n$ define

$$s_\lambda = \sum_\beta d_{\lambda\beta} F_\beta$$

where $d_{\lambda\beta}$ = number of standard contretableaux T of shape λ and $D(T) = S(\beta)$.

The s_λ are the Schur functions.

Example

$$s_{21} = F_{21} + F_{12}$$

from

$$\begin{array}{|c|c} \hline 2 & \\ \hline 3 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \qquad \begin{array}{|c|c} \hline 1 & \\ \hline 3 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

Quasisymmetric Schur functions

If $m_\lambda = \sum_{\lambda(\alpha)=\lambda} M_\alpha$ then $s_\lambda = \sum_{\lambda(\alpha)=\lambda} \mathcal{S}_\alpha$

but what is $\mathcal{S}_\alpha$?

The quasisymmetric Schur function is given by . . .

Quasisymmetric Schur functions

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but what is $\mathcal{S}_\alpha$?

The quasisymmetric Schur function is given by

$$\mathcal{S}_\alpha = \sum_{\alpha(\gamma)=\alpha} NS_\gamma .$$

$\mathcal{S}_\alpha$ for $\alpha = 12$

$$\begin{array}{|c|}\hline 1 \\ \hline \end{array} \qquad \begin{array}{|c|} \hline 1 \\ \hline \end{array} \qquad \begin{array}{|c|} \hline 1 \\ \hline \end{array} \qquad \begin{array}{|c|} \hline 2 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline \end{array}$$

$$\begin{aligned} \mathcal{S}_{12} &= NS_{120} + NS_{102} + NS_{012} \\ &= x_1x_2^2 + x_1x_2x_3 + x_1x_3^2 + x_2x_3^2 \\ &= M_{12} + M_{111} \end{aligned}$$

and

$$s_{21} = F_{21} + F_{12} = M_{21} + M_{12} + 2M_{111} = \mathcal{S}_{21} + \mathcal{S}_{12}.$$

What properties should $\mathcal{S}_\alpha$ have?

- $\mathbb{Z}$-basis for $\mathcal{Q}$.

- Expression in F_β.

- Quasisymmetric Kostka numbers.

- Quasisymmetric Pieri rules.

- Quasisymmetric LR rule.

What properties should $\mathcal{S}_\alpha$ have?

- $\mathbb{Z}$-basis for $\mathcal{Q}$. ✔
- Expression in F_β. ✔
- Quasisymmetric Kostka numbers. ✔
- Quasisymmetric Pieri rules. ✔
- Quasisymmetric LR rule...

Expression in F_β

$\mathcal{D}(T)$ is i where $i+1$ is not strictly west:

4	3	2
5	1	

For $\lambda \vdash n$

$$s_\lambda = \sum_\beta d_{\lambda\beta} F_\beta$$

where $d_{\lambda\beta}$ = number of standard tableaux T of shape λ and $D(T) = S(\beta)$.

Expression in F_β

$\mathcal{D}(T)$ is i where $i+1$ is not strictly west:

4	3	2
5	1	

For $\alpha \vDash n$

$$\mathcal{S}_\alpha = \sum_\beta d_{\alpha\beta} F_\beta$$

where $d_{\alpha\beta}$ = number of standard tableaux T of shape α and $\mathcal{D}(T) = S(\beta)$.

$\mathcal{S}_\alpha$ for $\alpha = 32$

3	2	1
5	4	

4	3	1
5	2	

4	3	2
5	1	

and

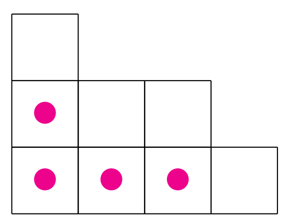
$$\mathcal{S}_{32} = F_{32} + F_{221} + F_{131}.$$

Skew diagrams and strips

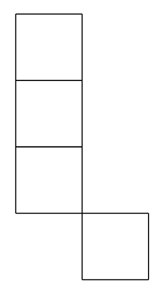
For λ, μ the skew diagram λ/μ of size $|\lambda/\mu|$ is the array of boxes contained in λ but not in μ.

A skew diagram λ/μ is a row (column) strip if

no 2 boxes in same column (row).



row strip



col strip

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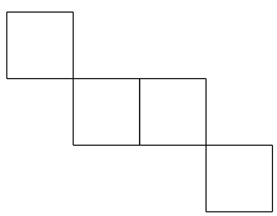
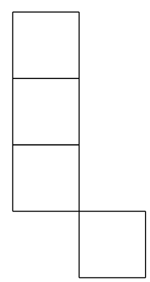


row strip



col strip

Pieri rules for partitions λ, μ

$$s_n s_\lambda = \sum_\mu s_\mu \qquad s_{1^n} s_\lambda = \sum_\mu s_\mu$$

- $\delta = \mu/\lambda$ row strip, $|\delta| = n$.
- $\varepsilon = \mu/\lambda$ column strip, $|\varepsilon| = n$.

Removal from compositions

Let $\alpha = \alpha_1 \ldots \alpha_k$ have largest part m, and $s \in [m]$

$$rem_s(\alpha) = \alpha_1 \ldots \alpha_{i-1}(s-1)\alpha_{i+1} \ldots \alpha_k$$

for largest i.

If $S = \{s_1 < \cdots < s_j\}$ then

$$row_S(\alpha) = rem_{s_1}(\ldots(rem_{s_{j-1}}(rem_{s_j}(\alpha)))\ldots).$$

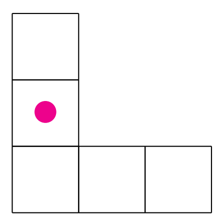
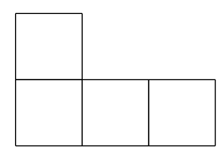
If $M = \{m_1 \leq \cdots \leq m_j\}$ then

$$col_M(\alpha) = rem_{m_j}(\ldots(rem_{m_2}(rem_{m_1}(\alpha)))\ldots).$$

rem_s in action

rem_s removes rightmost box from lowest row length s.

Example

$rem_1(113) =$  $=$  $= 13$

Quasisymmetric Pieri rules for compositions α, β

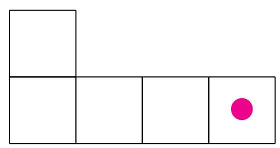
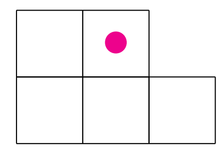
$$\mathcal{S}_n \mathcal{S}_\alpha = \sum_\beta \mathcal{S}_\beta \qquad \mathcal{S}_{1^n} \mathcal{S}_\alpha = \sum_\beta \mathcal{S}_\beta$$

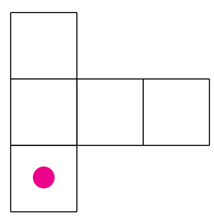
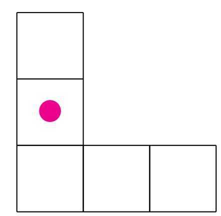
- $\delta = \lambda(\beta)/\lambda(\alpha)$ row strip, $|\delta| = n$.
- $row_{S(\delta)}(\beta) = \alpha$.
- $\varepsilon = \lambda(\beta)/\lambda(\alpha)$ column strip, $|\varepsilon| = n$.
- $col_{M(\varepsilon)}(\beta) = \alpha$.

$\mathcal{S}_1\mathcal{S}_{13}$ in action

$$41/31, \; 32/31, \; 311/31, \; 311/31$$

with row strips containing a box in column $4, 2, 1, 1$ respectively.

$row_{\{4\}}(14) =$  $row_{\{2\}}(23) =$ 

$row_{\{1\}}(131) =$  $row_{\{1\}}(113) =$ 

and

$$\mathcal{S}_1\mathcal{S}_{13} = \mathcal{S}_{14} + \mathcal{S}_{23} + \mathcal{S}_{131} + \mathcal{S}_{113}.$$