# An Introduction to Graph Coloring Problems

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WAM Workshop
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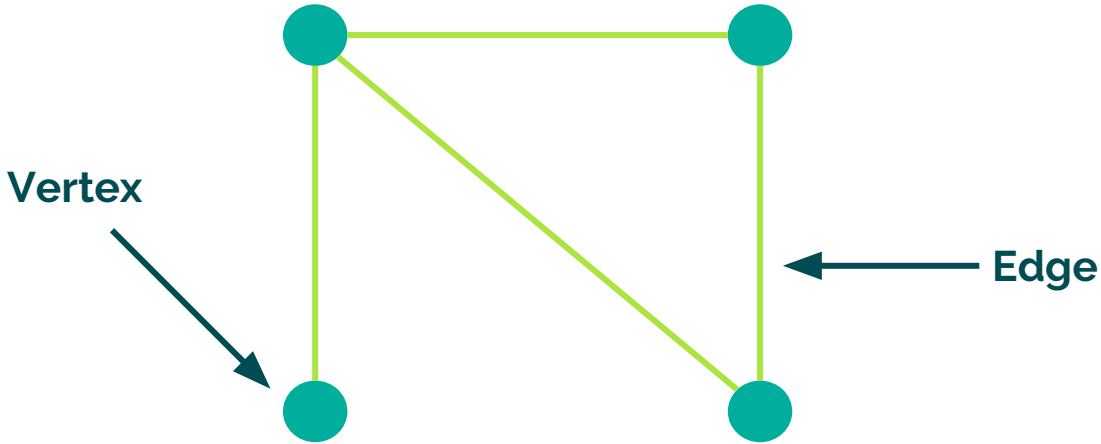
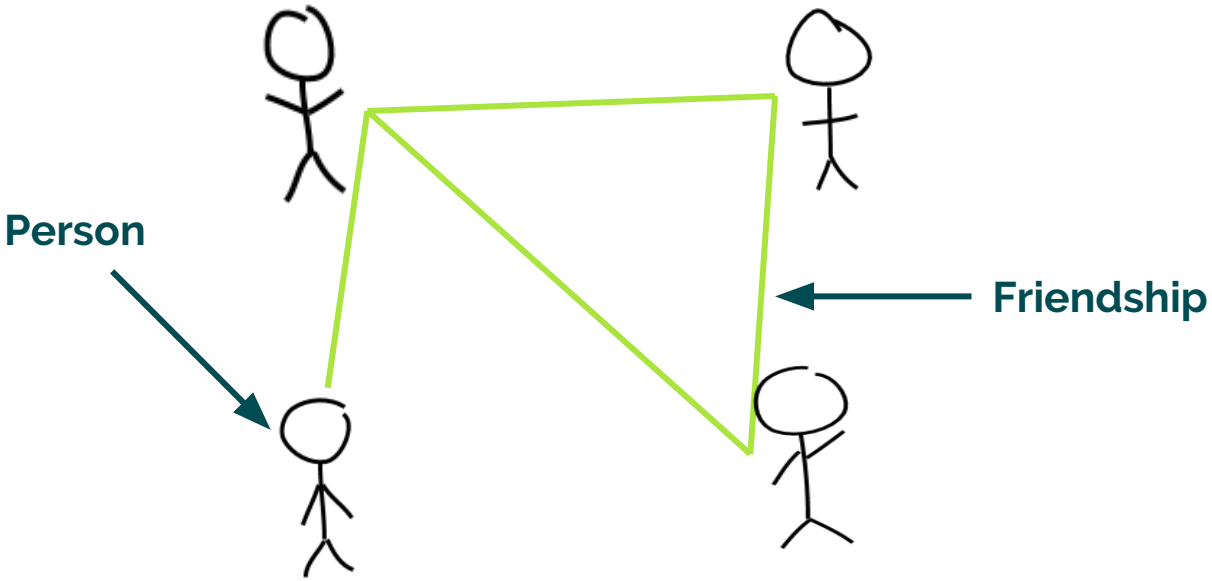
# What is a graph?

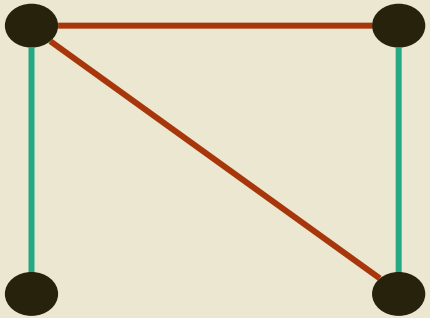


# What is a graph?

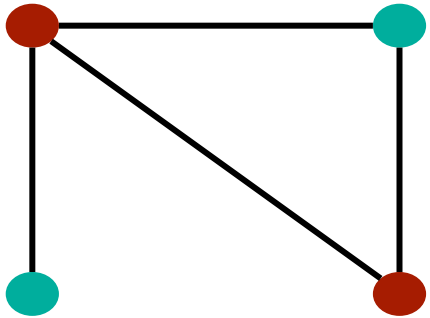


# What are graph colorings?

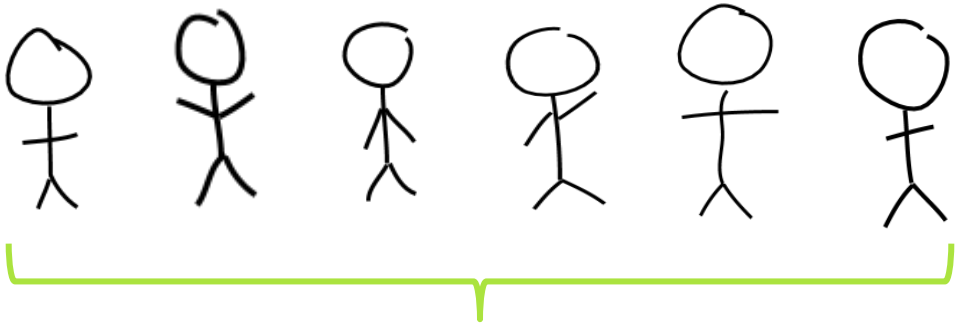
## Edge Colorings



## Vertex Colorings



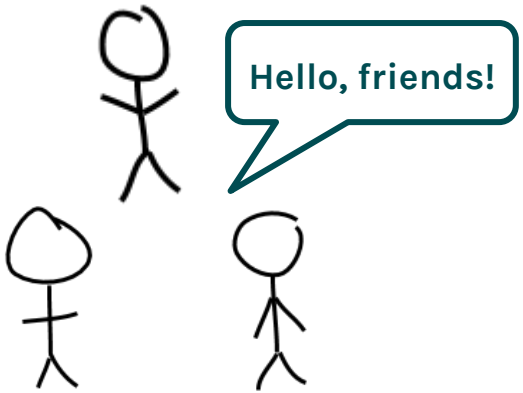
# The party problem



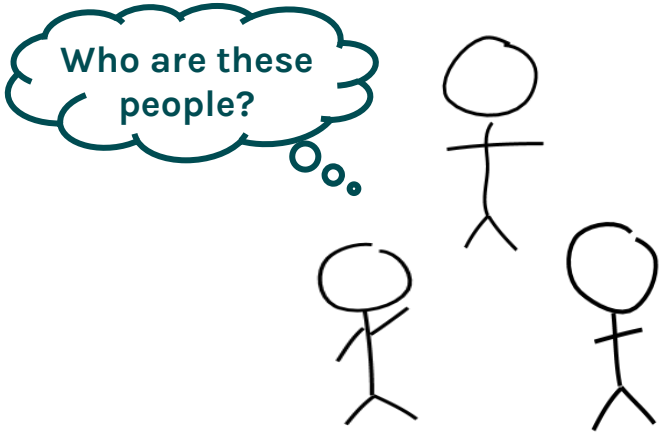
6 people

# The party problem

**3 mutual friends**




**3 mutual non-friends**



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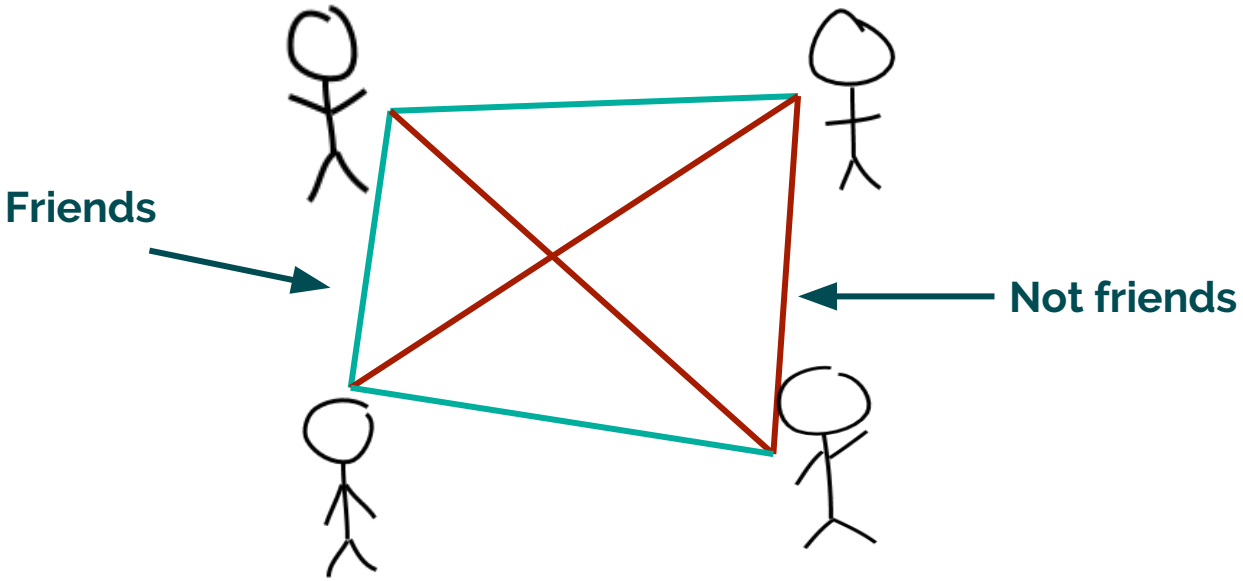

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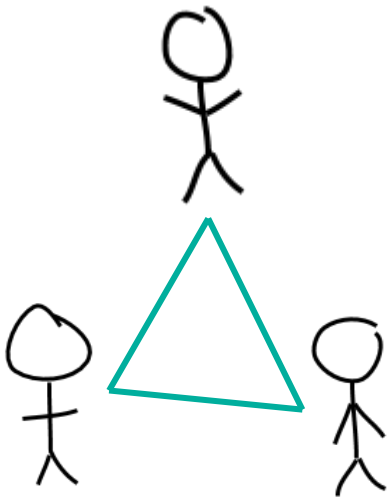

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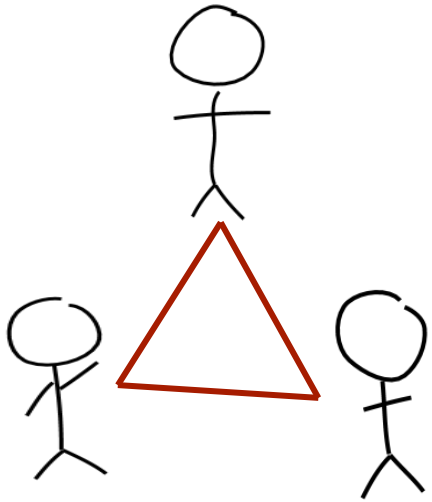

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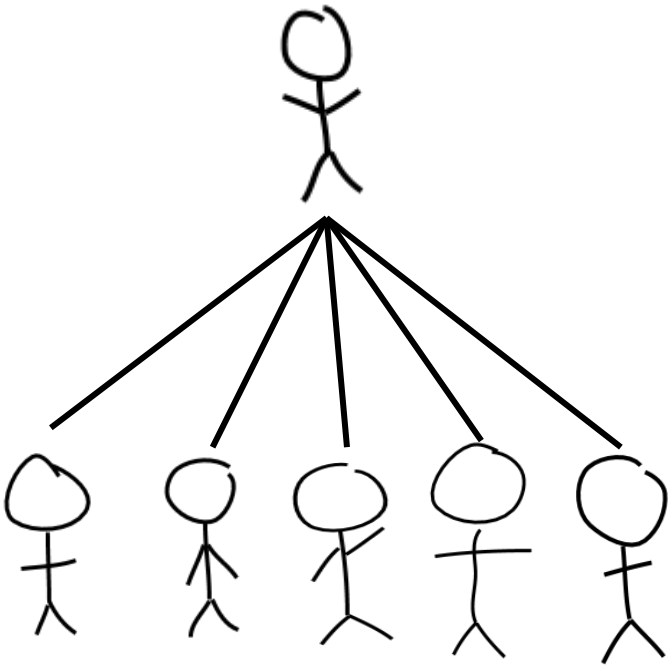

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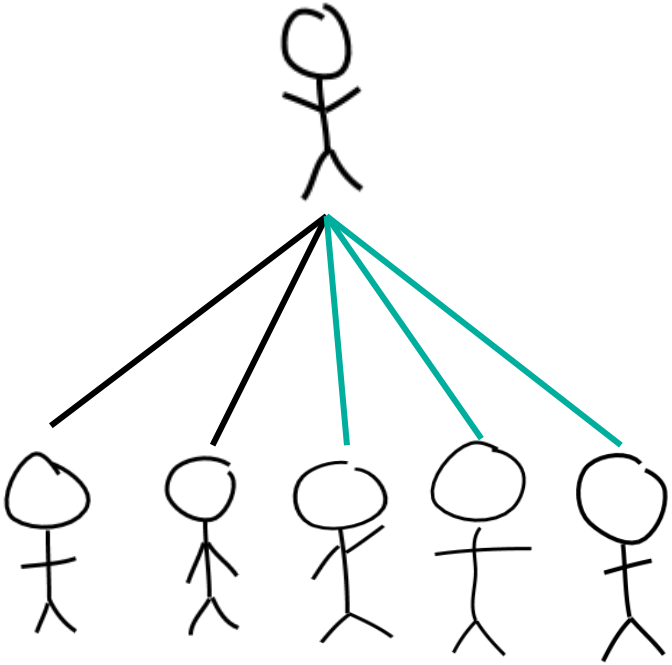

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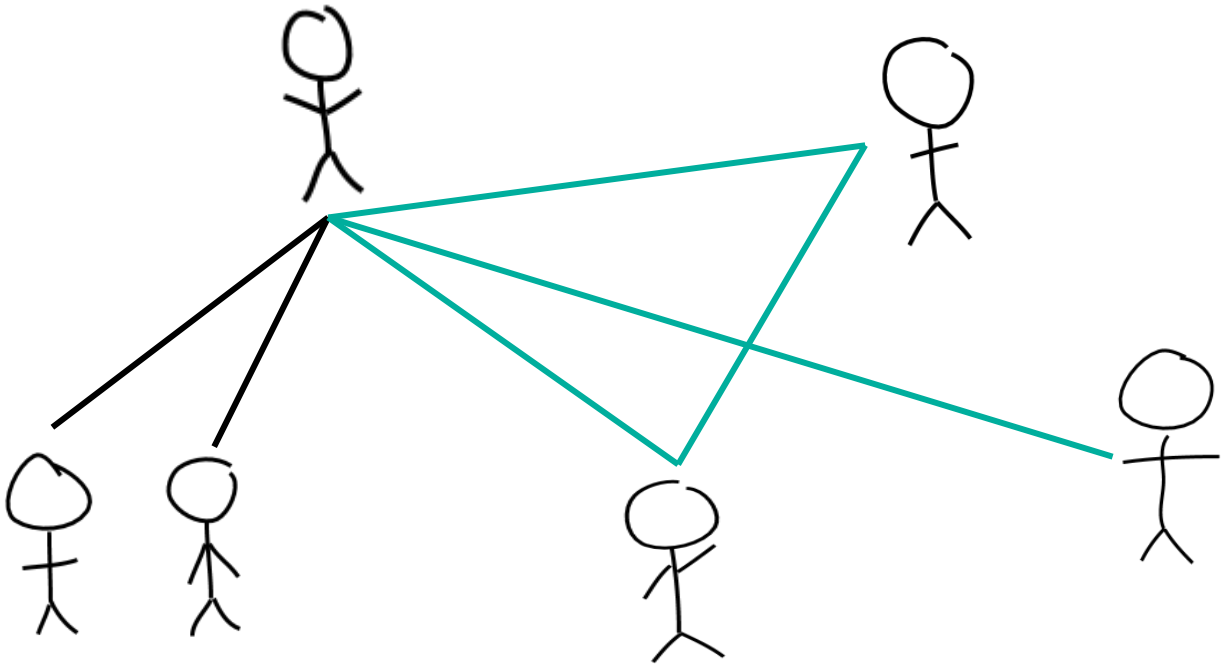

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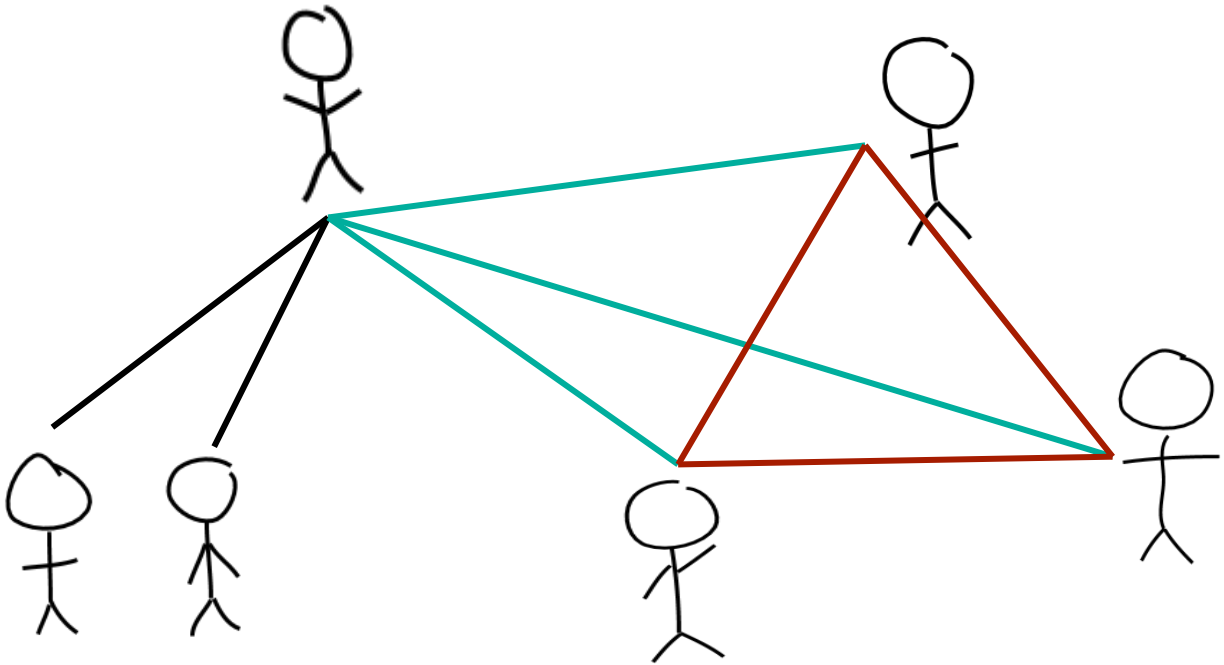

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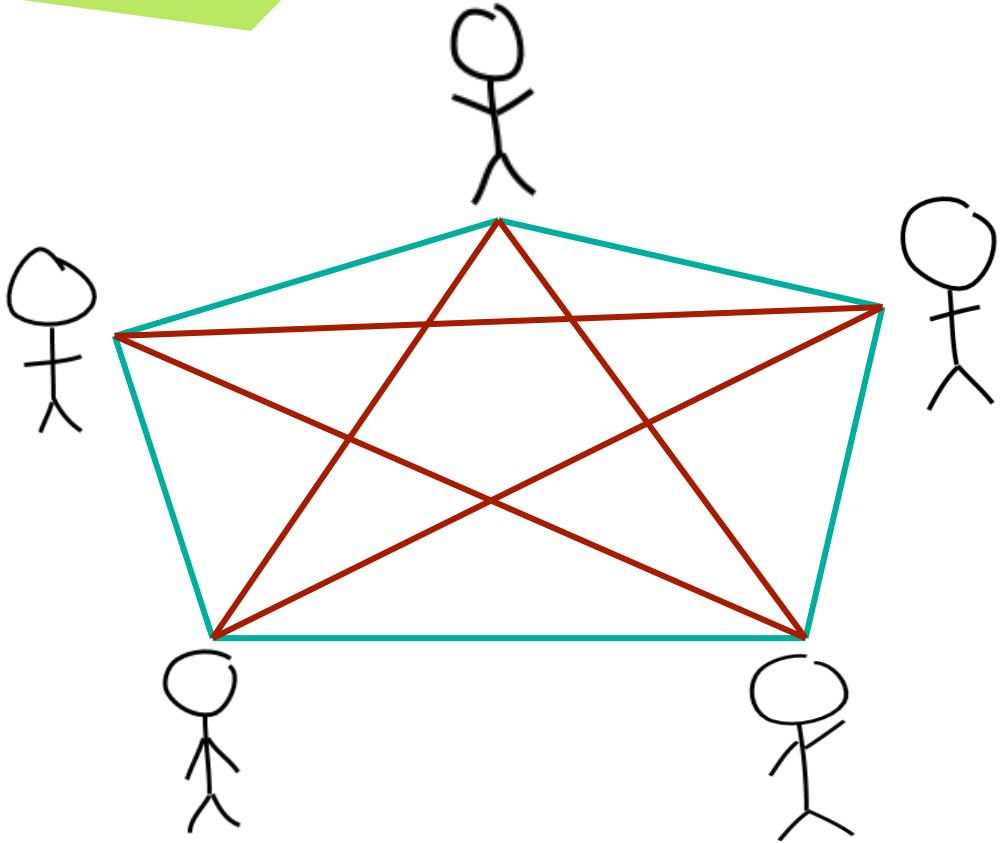
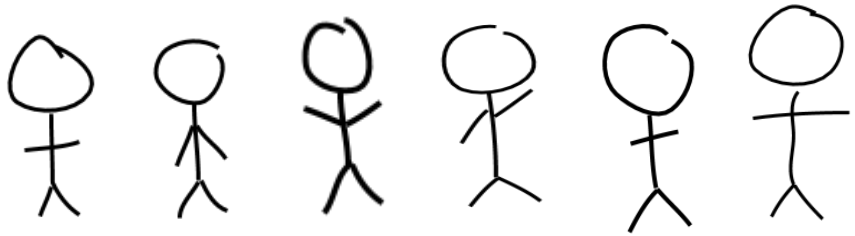


# Ramsey theory

$$R(m,n) = min\{k \mid K_k \text{ with edges colored red/blue has either a red } K_n \text{ or a blue } K_m\}$$

$$R(3,3) = 6$$



# Ramsey theory

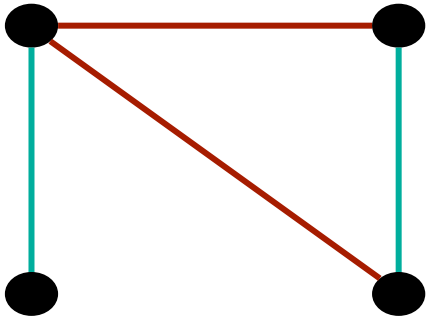
- Proven:
  - R(2,n) = n
  - R(3,4) = 9
  - R(4,4) = 17
- Proven through computation:
  - R(3,8) = 28
  - R(3,9) = 36
  - R(4,5) = 24

# Ramsey theory

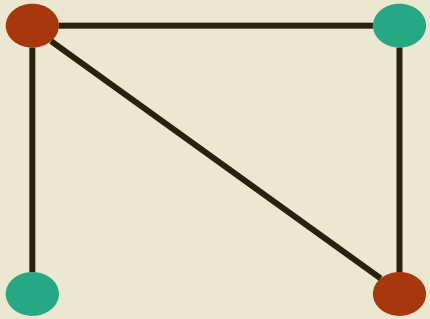
- General bounds:
  - $R(n, n) \leq 4^n$
  - $R(m, n) \leq R(m-1, n) + R(m, n-1)$

# What are graph colorings?

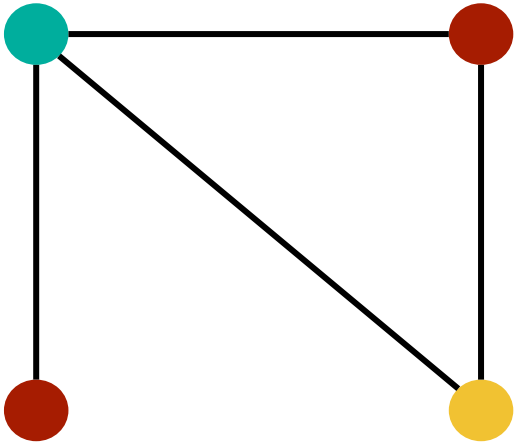
## Edge Colorings



## Vertex Colorings



# Vertex coloring

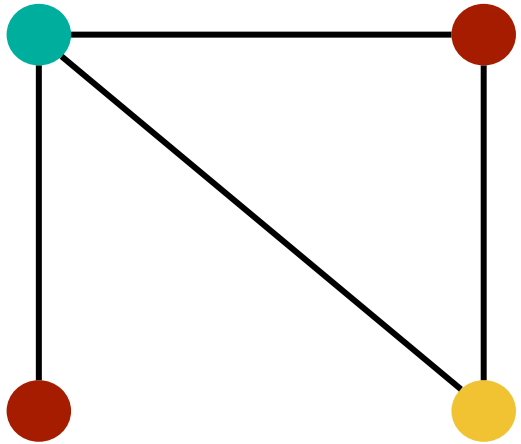


# Application: map coloring

**Chromatic number:**
**Minimum number of colors needed to properly color the graph**

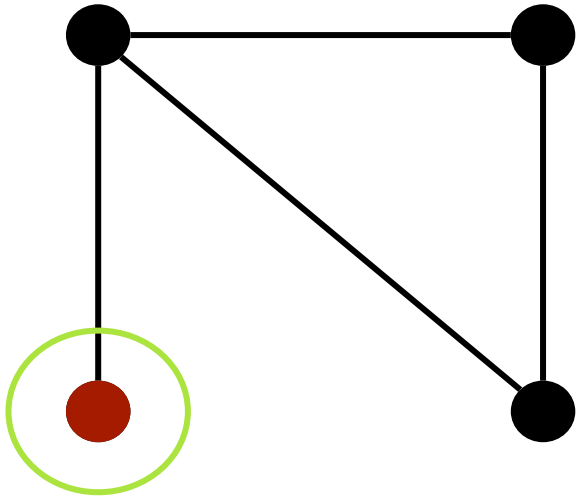
# Vertex coloring

**Problem: find the chromatic number of a graph**



# Greedy vertex coloring

Color 1
Color 2
Color 3
Color 4
. . .



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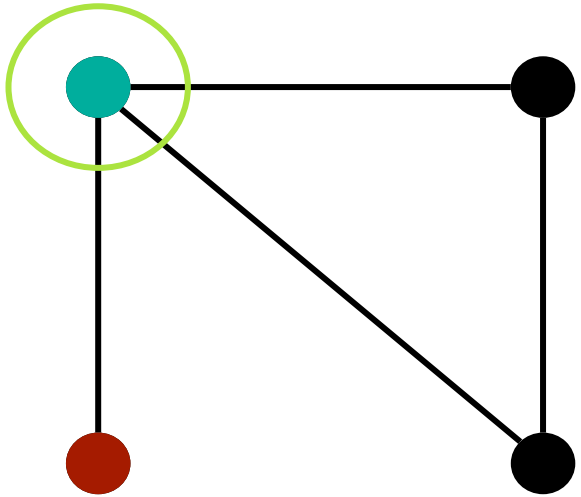

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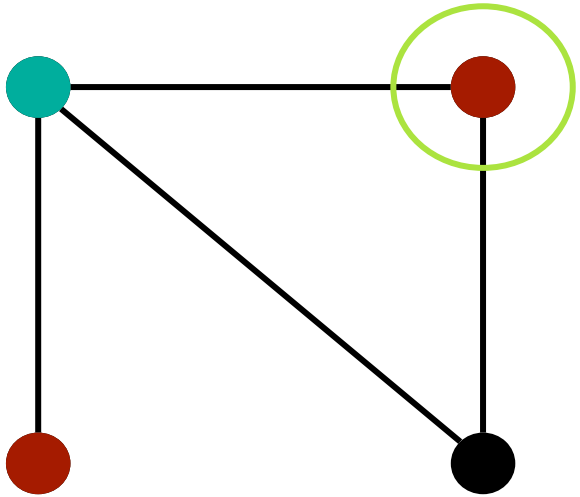

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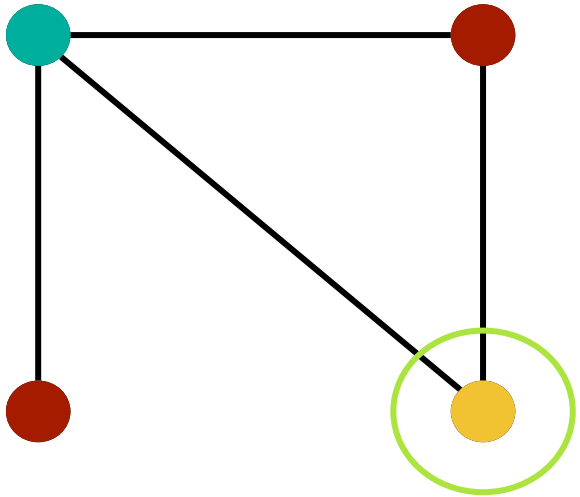
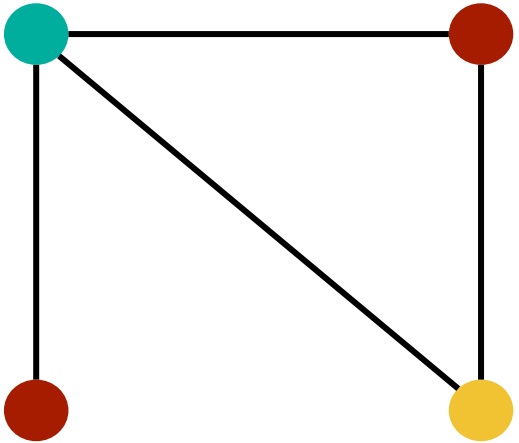


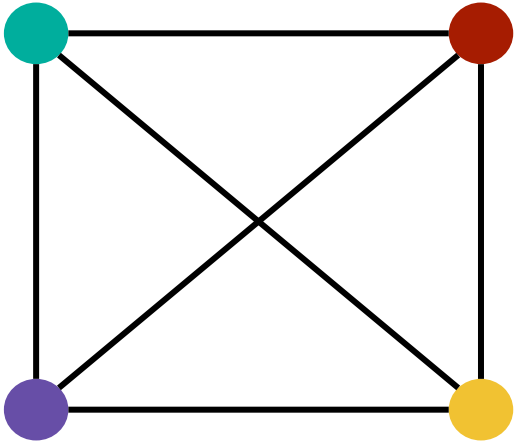
# Greedy vertex coloring

**Shows chromatic number is no more than max degree + 1**

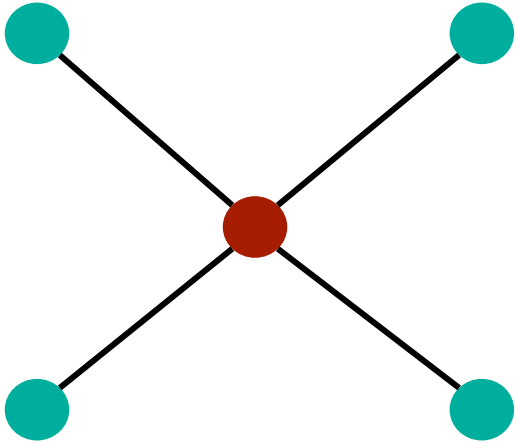


# Greedy vertex coloring

A tight example:



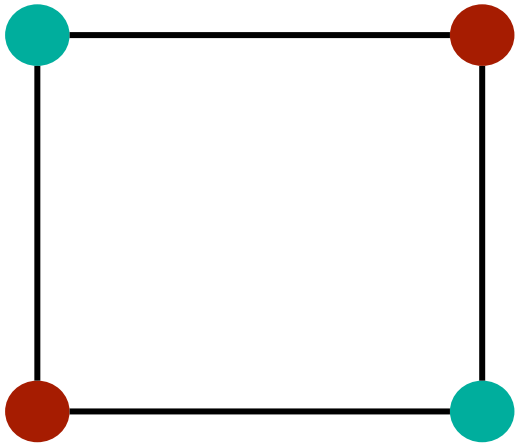
A loose example:



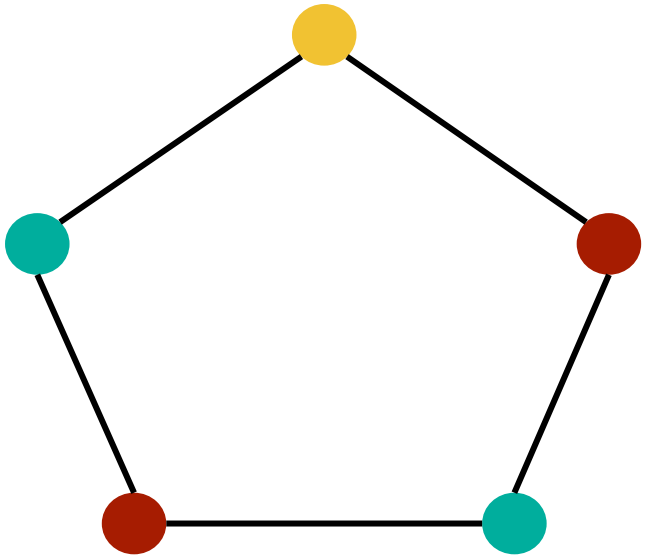
# Chromatic number

Are there any scalable graphs with low chromatic number?

Idea: limit graph structure to no triangles



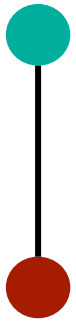
# Chromatic number: no triangles



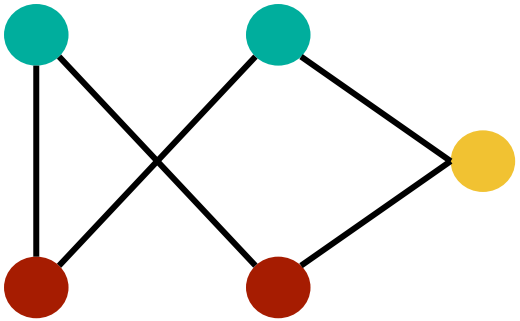
# Mycielski graphs

$M_2$

# Mycielski graphs

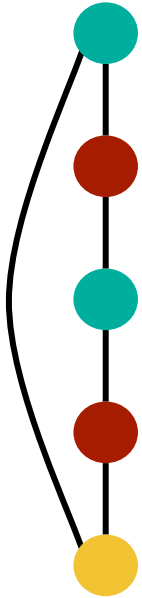


$M_2$



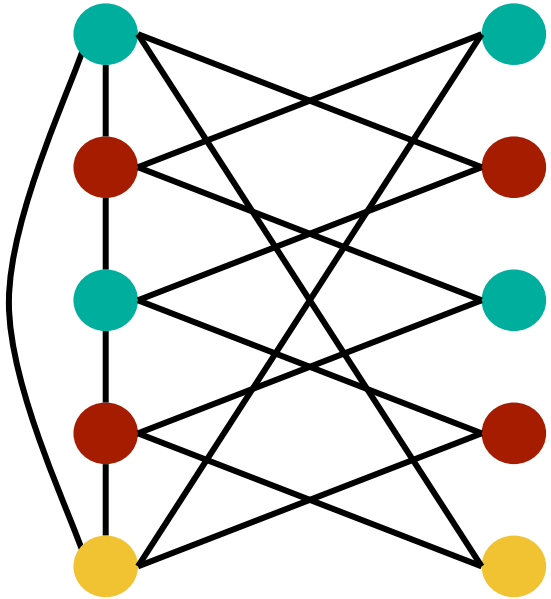
$M_3$

# Mycielski graphs



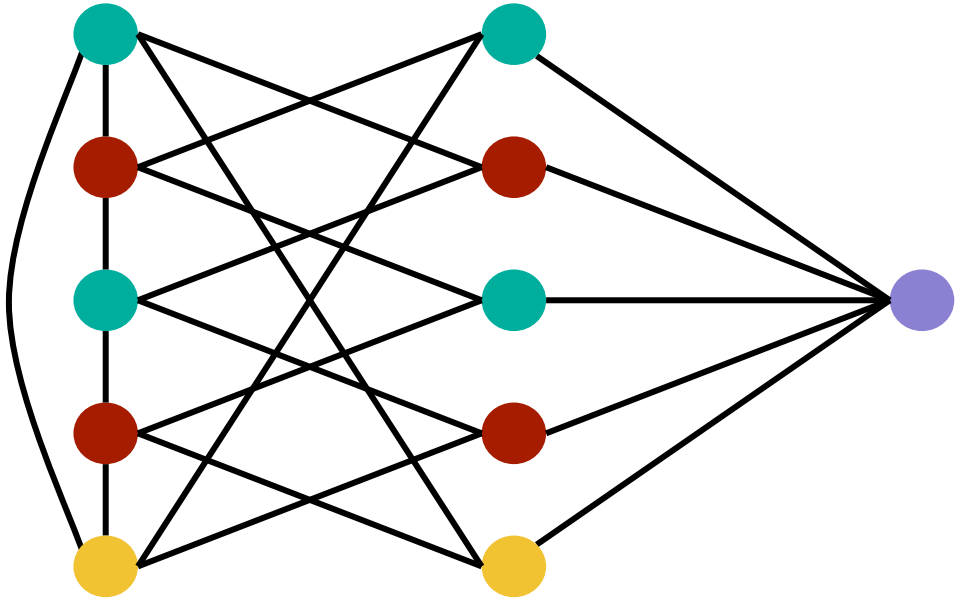
$M_4$

# Mycielski graphs



$M_4$

# Mycielski graphs



$M_4$

# Vertex colorings

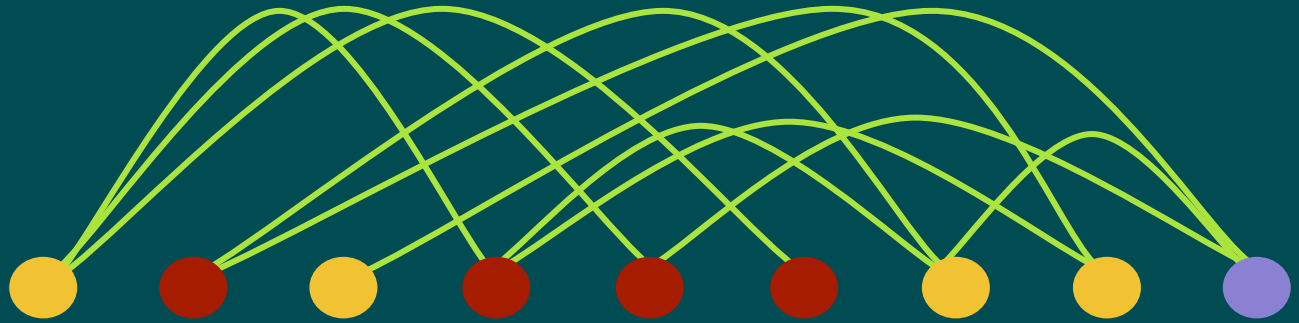
- Other sparse constructions of high chromatic number graphs
- Vertex colorings on hypergraphs
- Better algorithms to color vertices?
- Better bounds on chromatic number?

# Resources

- Material: Budapest Semesters in Mathematics “Advanced Combinatorics Handout,” *https://pdfs.semanticscholar.org/03f5/e4c50ceda0303e46608939c3a6a48f488728.pdf*
- South America map coloring: *https://mapchart.net/world.html*
- Slide template: *https://www.slidescarnival.com/escalus-free-presentation-template/963*

# Thank you!

## ...Questions?



The shift graph $SH_5$