The Role of Mathematics in Understanding the Earth's Climate

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CORNELL UNIVERSITY FOUNDED A.D. 1865

MCRN Mathematics and Climate Research Network

Outline

- What is climate (change)?
- History of mathematics in climate science
- How do we study the climate?
- Dynamical systems
- Large-scale (Atlantic) ocean circulation
- Ice ages and the mid-Pleistocene transition
- Winter is coming?

Weather vs. Climate

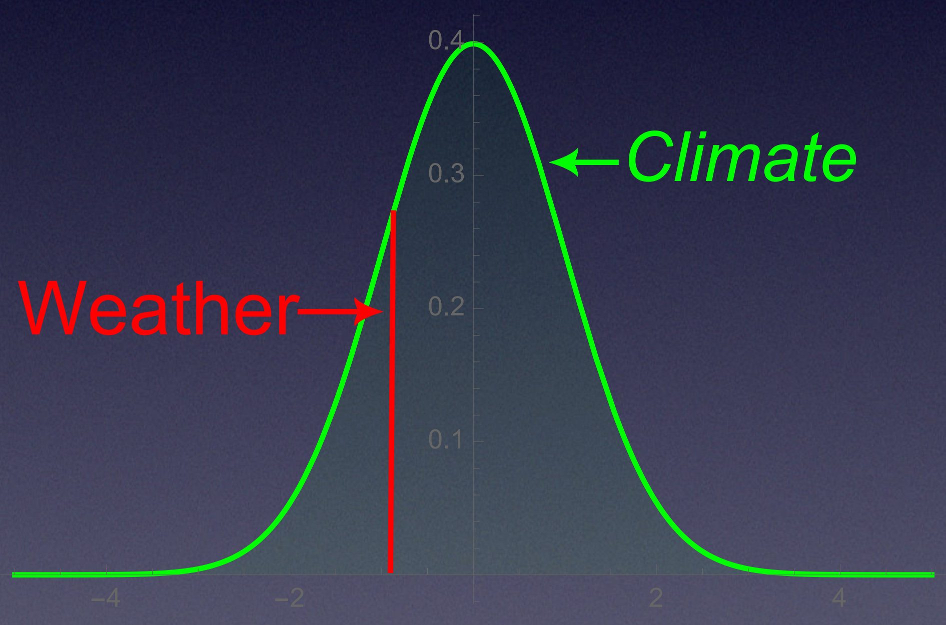
- Conditions of the atmosphere over a short period of time (minutes - months)
- Temp, humidity, precip, cloud coverage (today)
- Snowfall on November 14, 2014
- Heat wave in 2010
- Hurricane

- How the atmosphere “behaves” over a long period of time
- Average of weather over time and space (usually 30-yr avg)
- Historical average November precipitation
- Record high temperature
- Average number and strength of tropical cyclones, annually

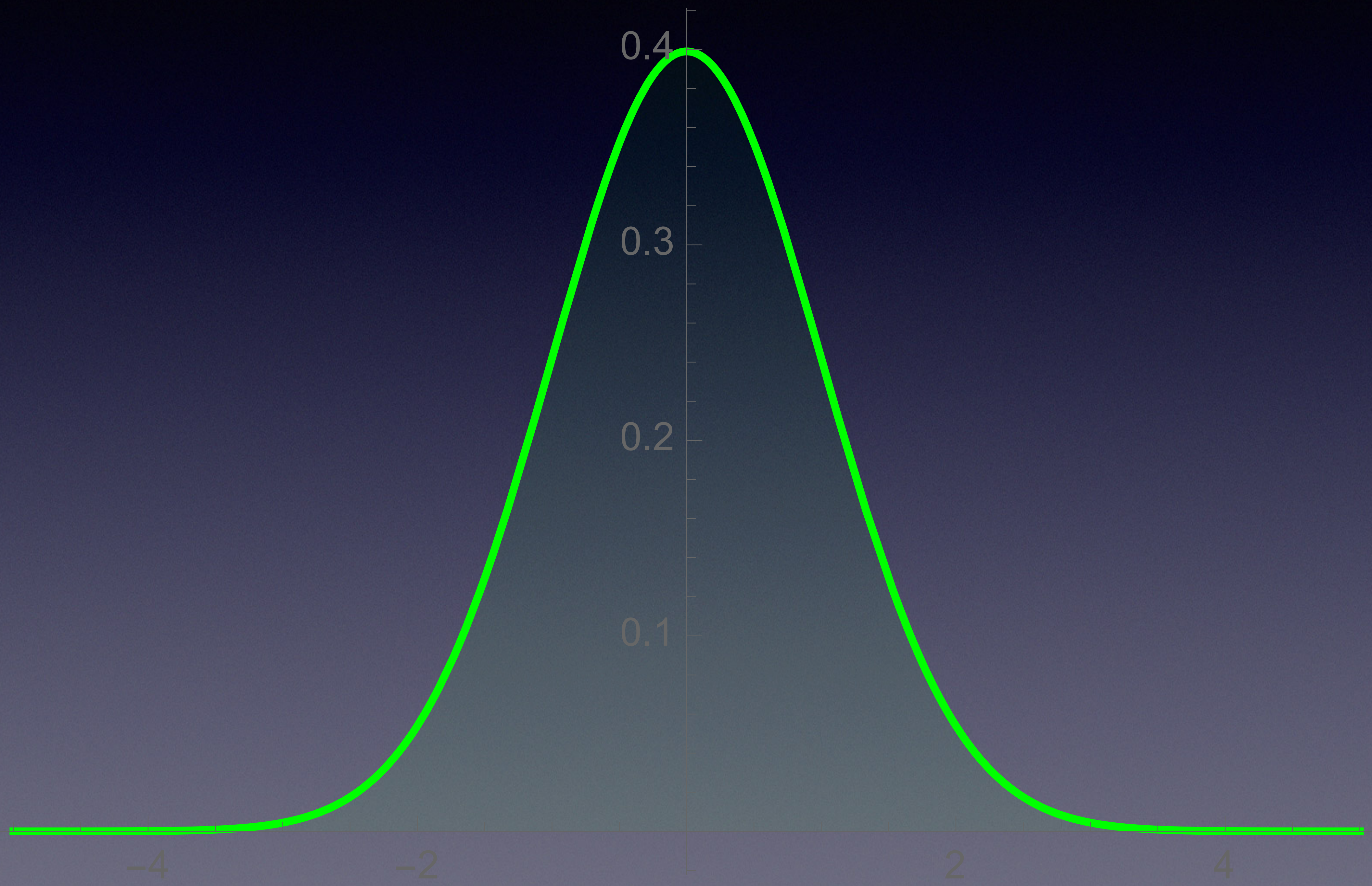
Weather vs. Climate

- Climate is what you expect, weather is what you get

- Can view climate as a probability distribution of possible weather



What is Climate Change?



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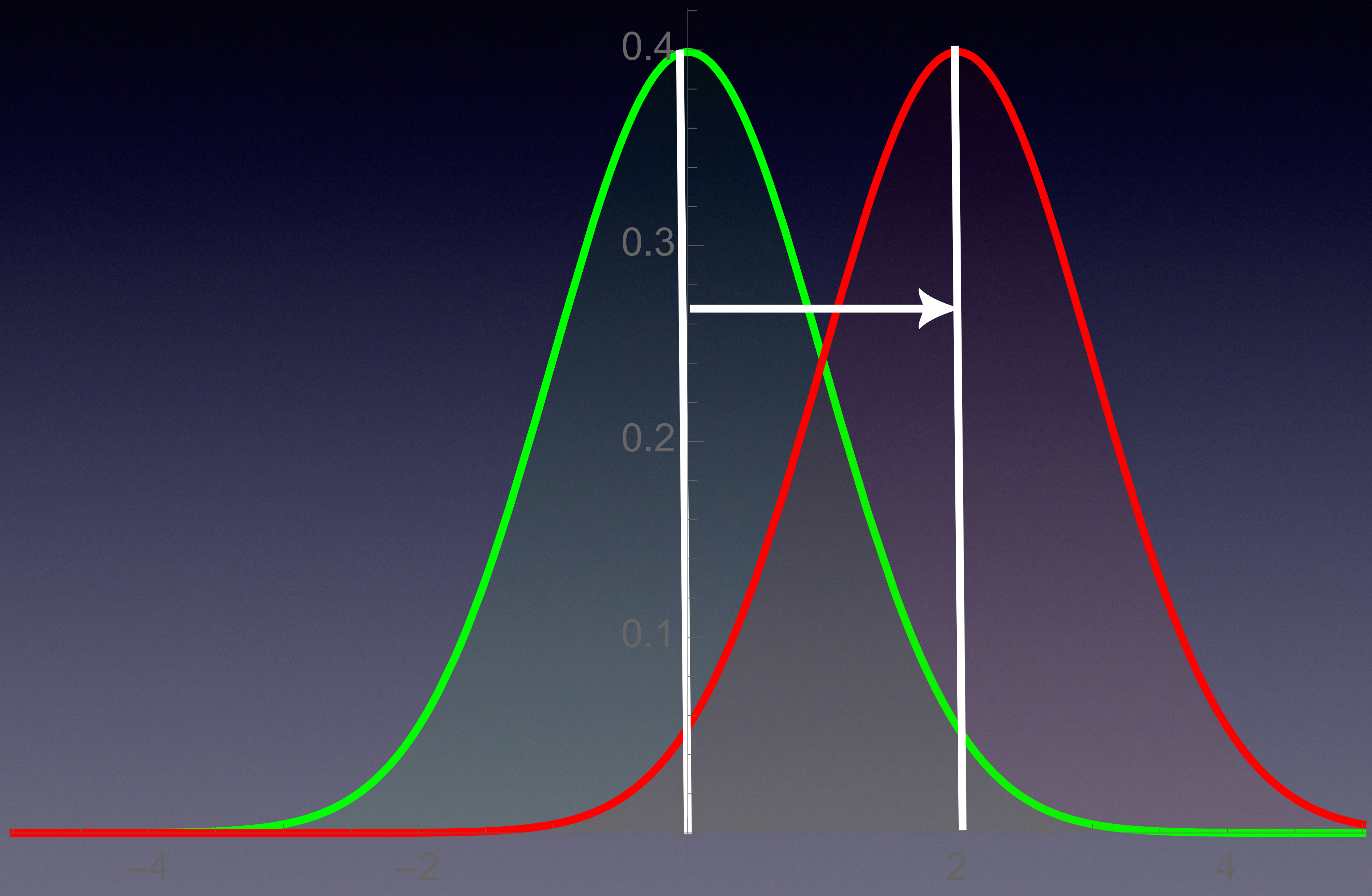

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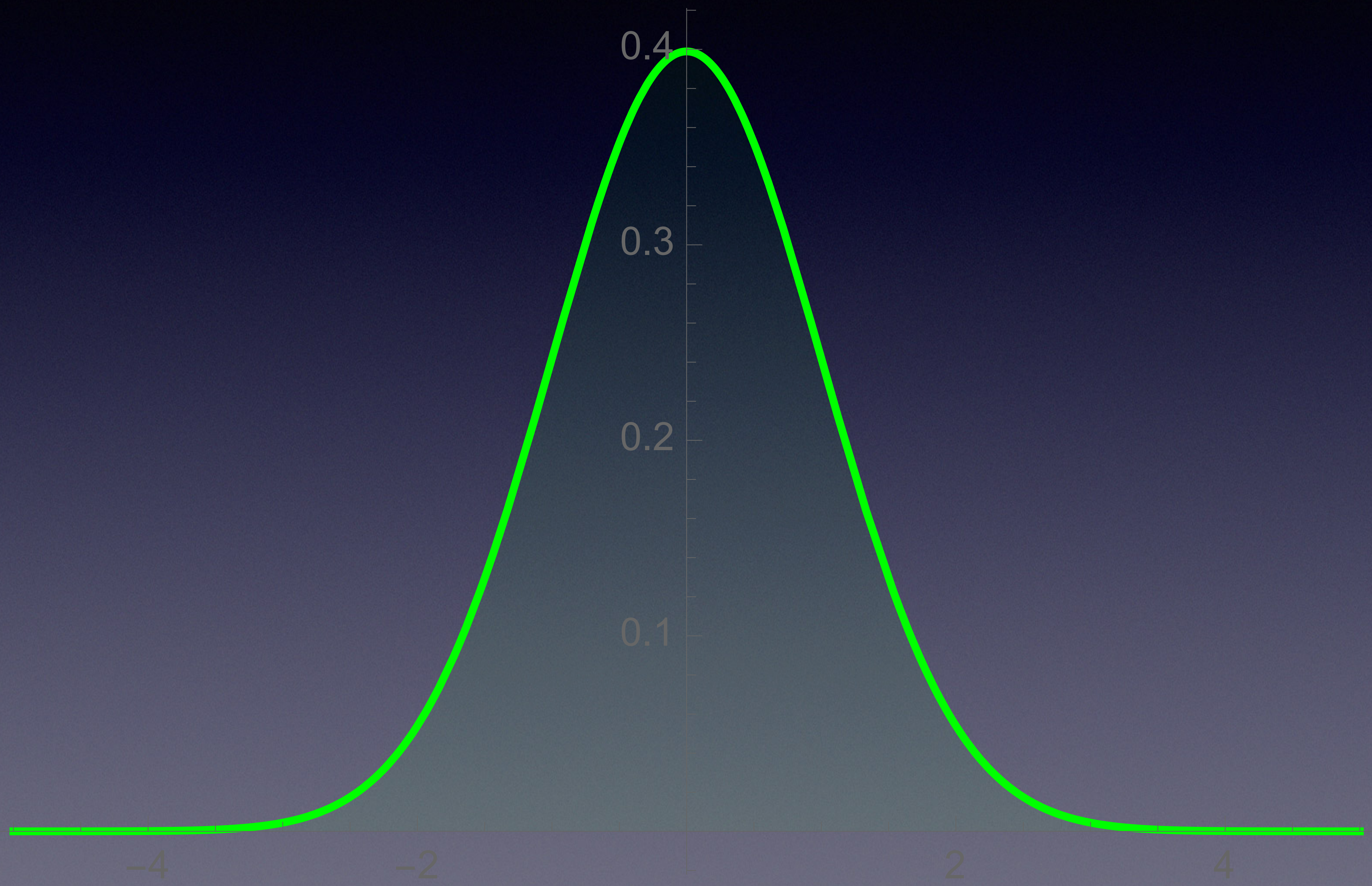

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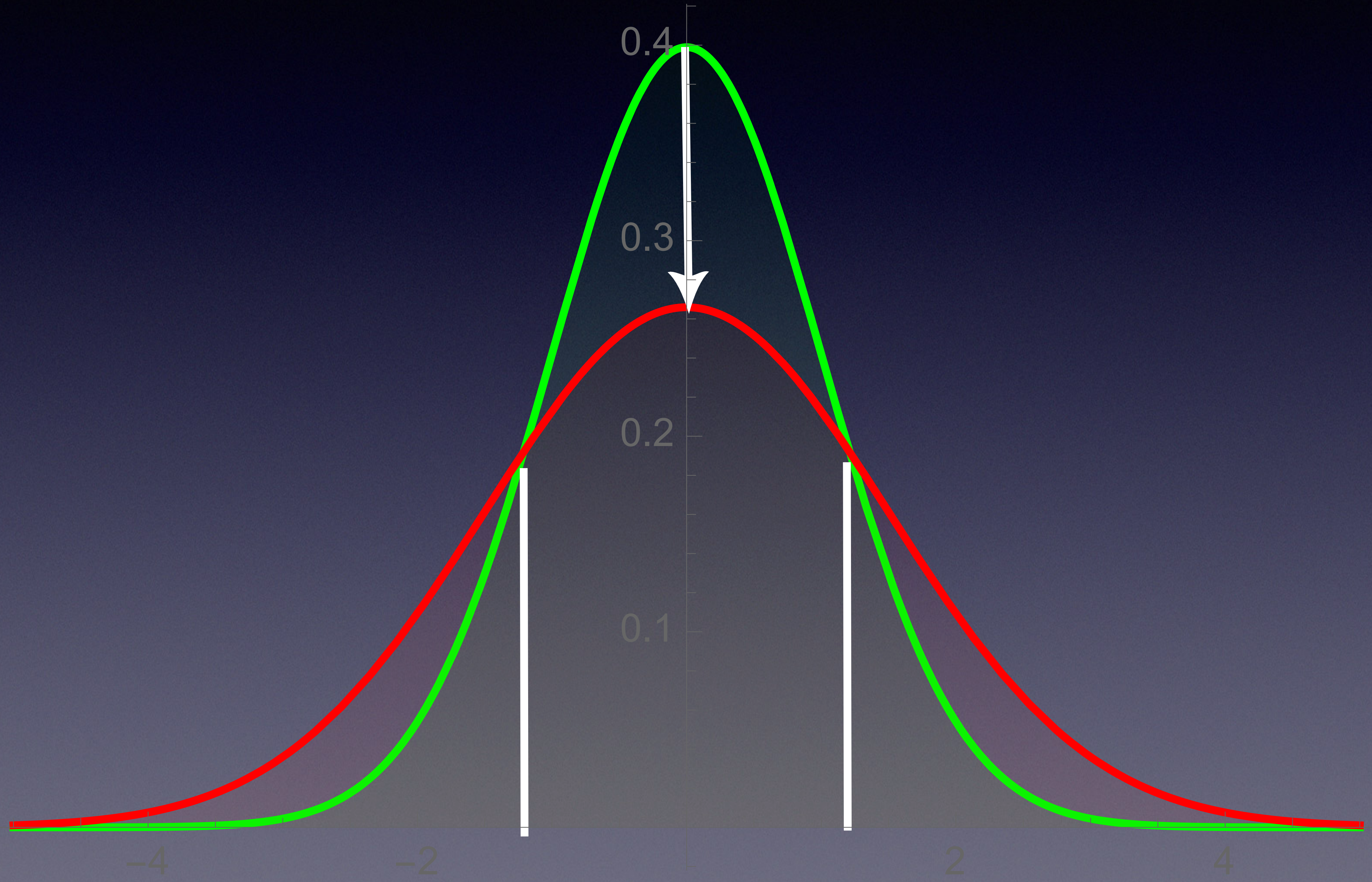
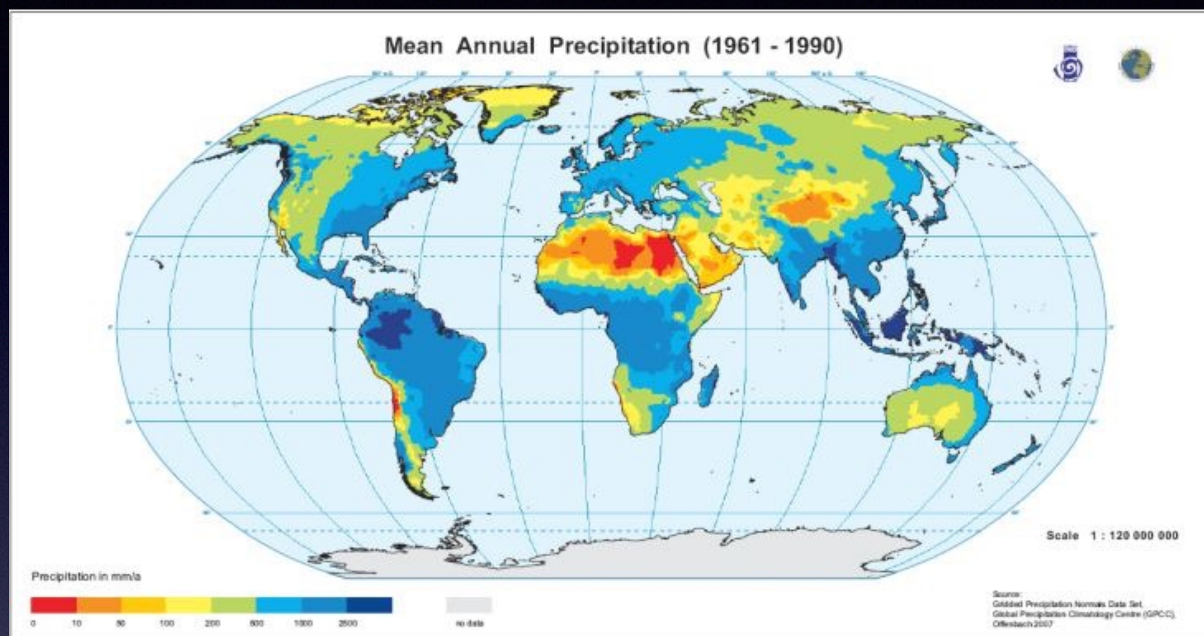
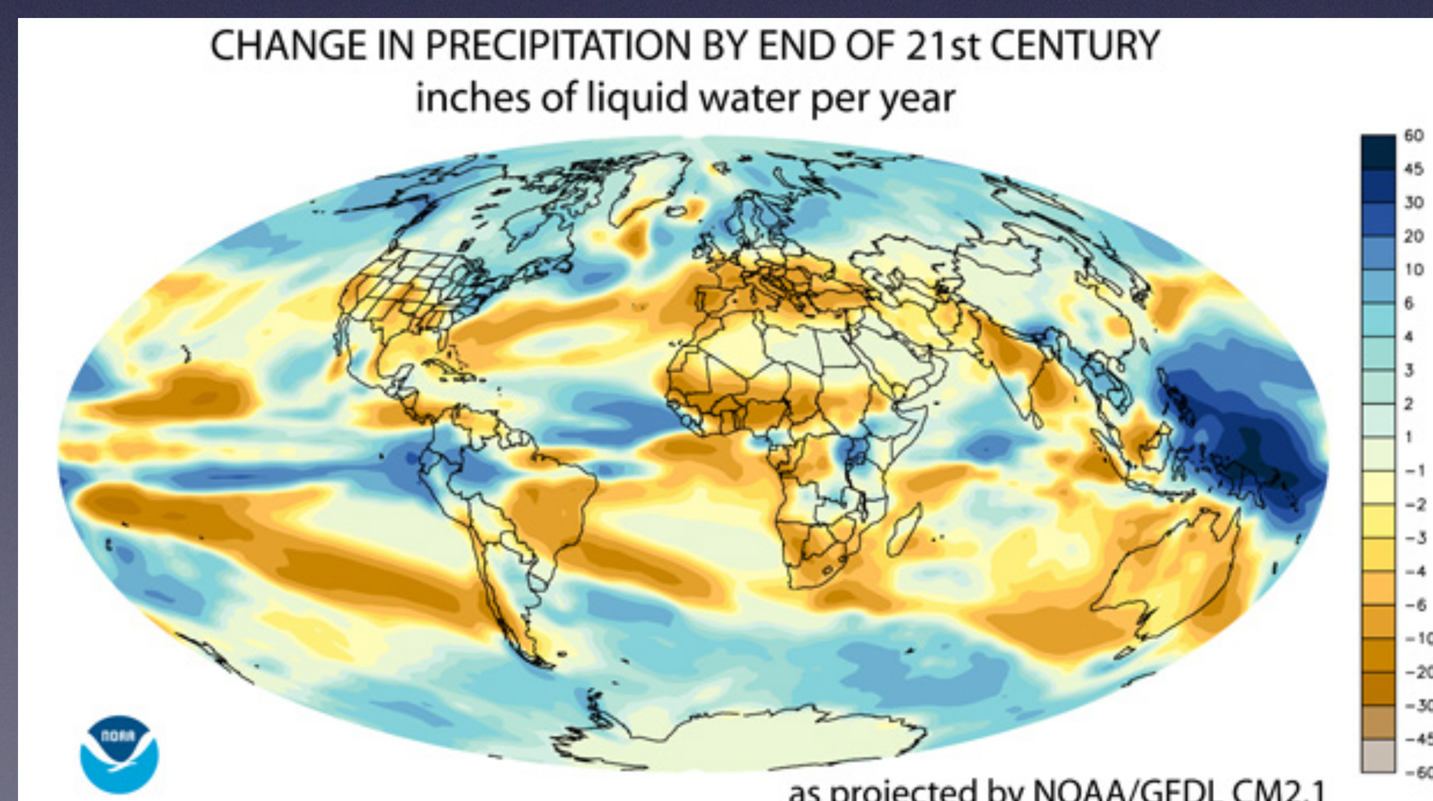


Precipitation



Dry areas
get dryer,
wet areas
get wetter

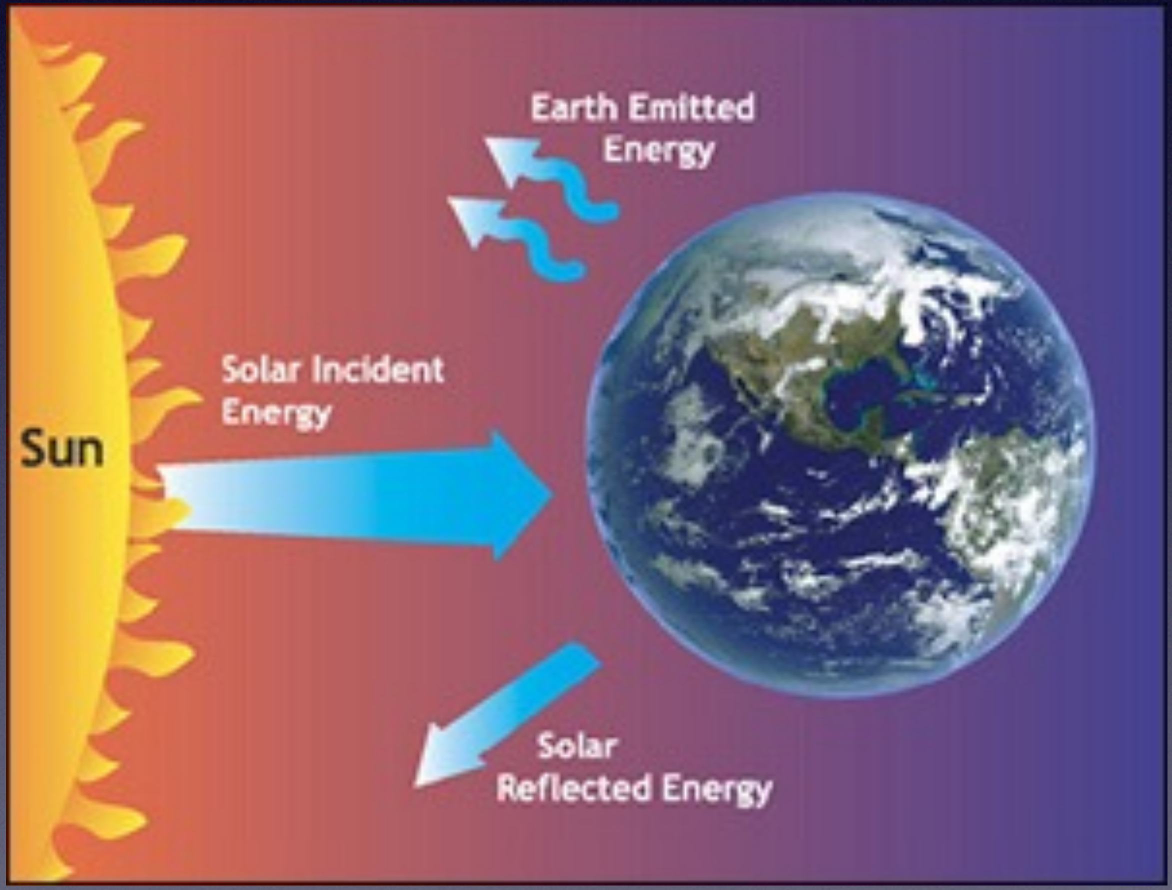
Climate scientists
predict more
floods and
more droughts!



Mathematics and Climate Change

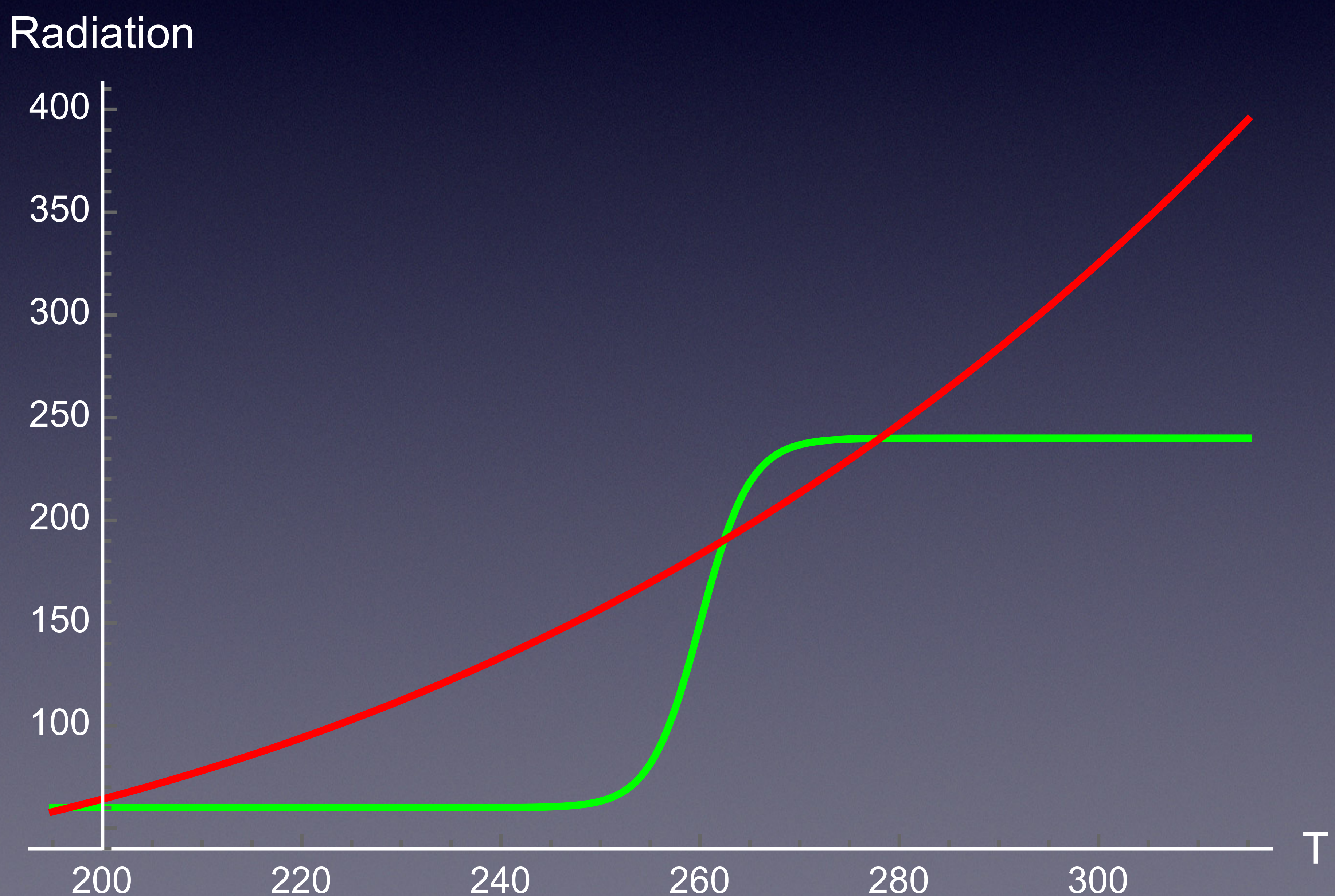
$$\Delta T = Q(1 - \alpha(T)) - \sigma T^4$$

- Energy balance equation
- Q: incoming solar radiation
- $(1 - \alpha(T))$: proportion absorbed by the Earth
- σT^4 :heat re-radiated back to space



Energy Balance

$$\Delta T = 0 \Rightarrow Q(1-\alpha(T)) = \sigma T^4$$

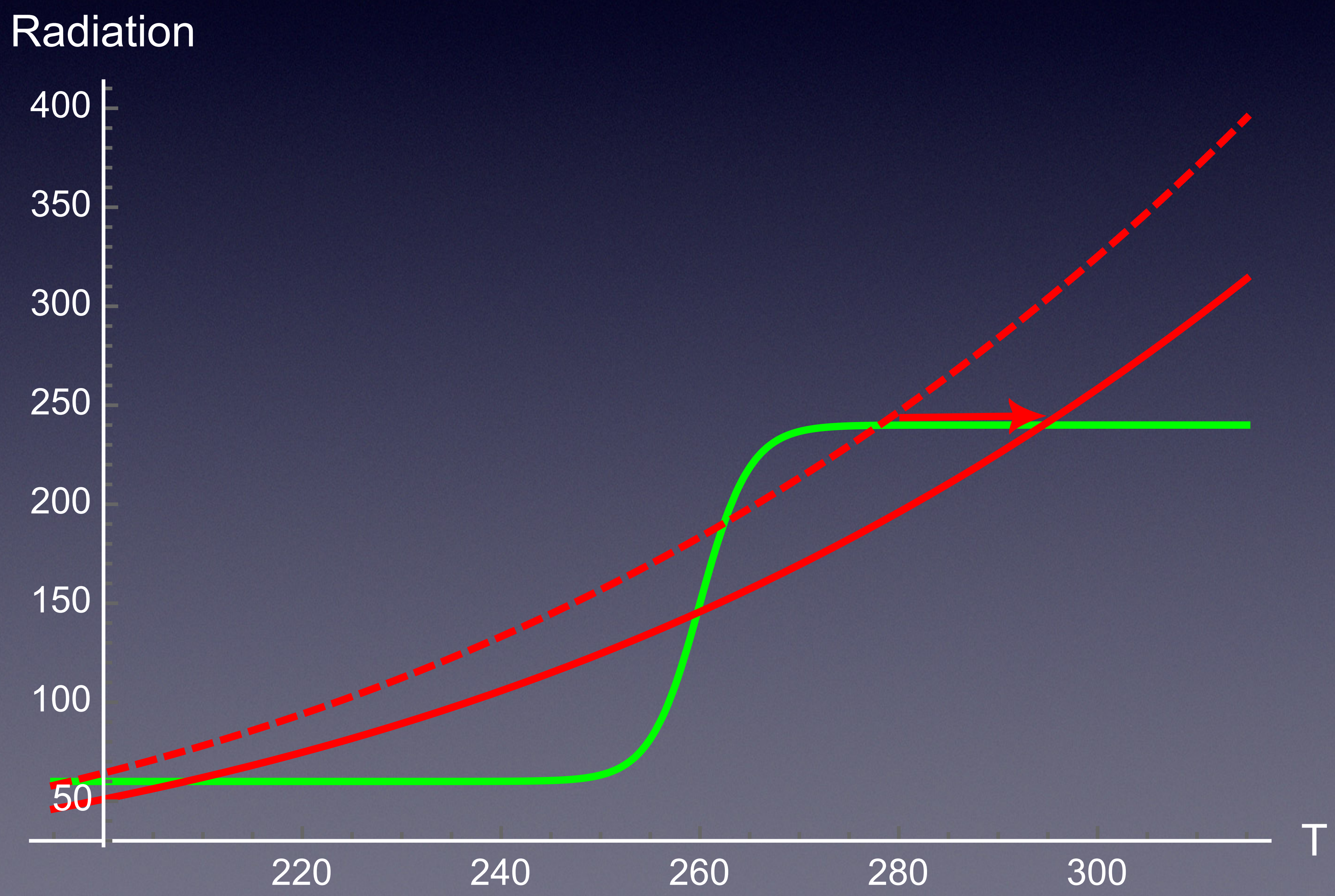


Greenhouse Effect

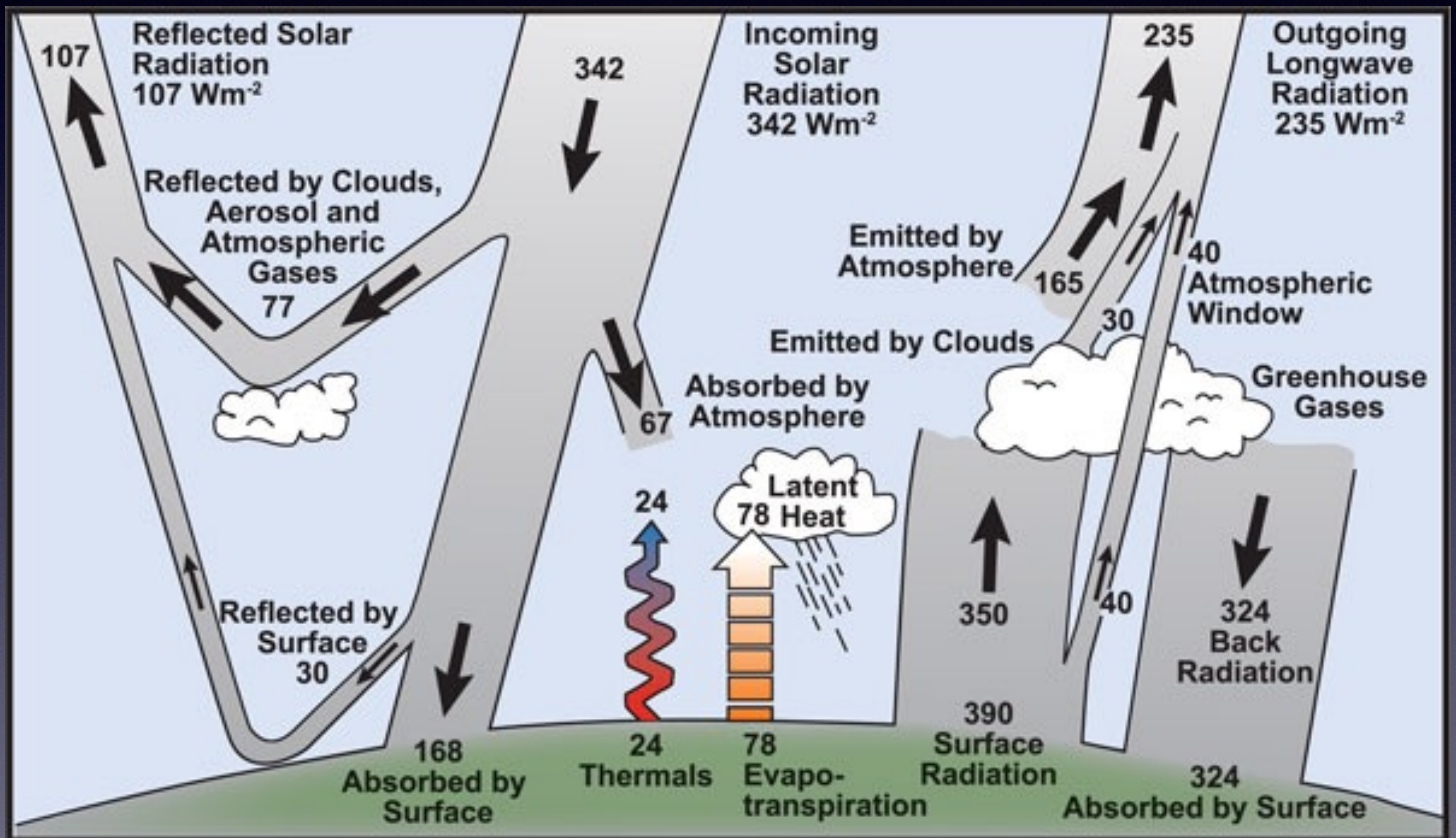
- Joseph Fourier attempted to calculate the average temperature of the Earth (c. 1820)
- Hypothesized what has come to be known as the “greenhouse effect” — something is trapping heat in the Earth’s atmosphere
- 50 years before Stefan-Boltzmann energy balance equation
- 75 years before Arrhenius quantified how much colder the Earth “should” be

Greenhouse Effect

$$\Delta T = Q(1 - \alpha(T)) - \varepsilon\sigma T^4$$

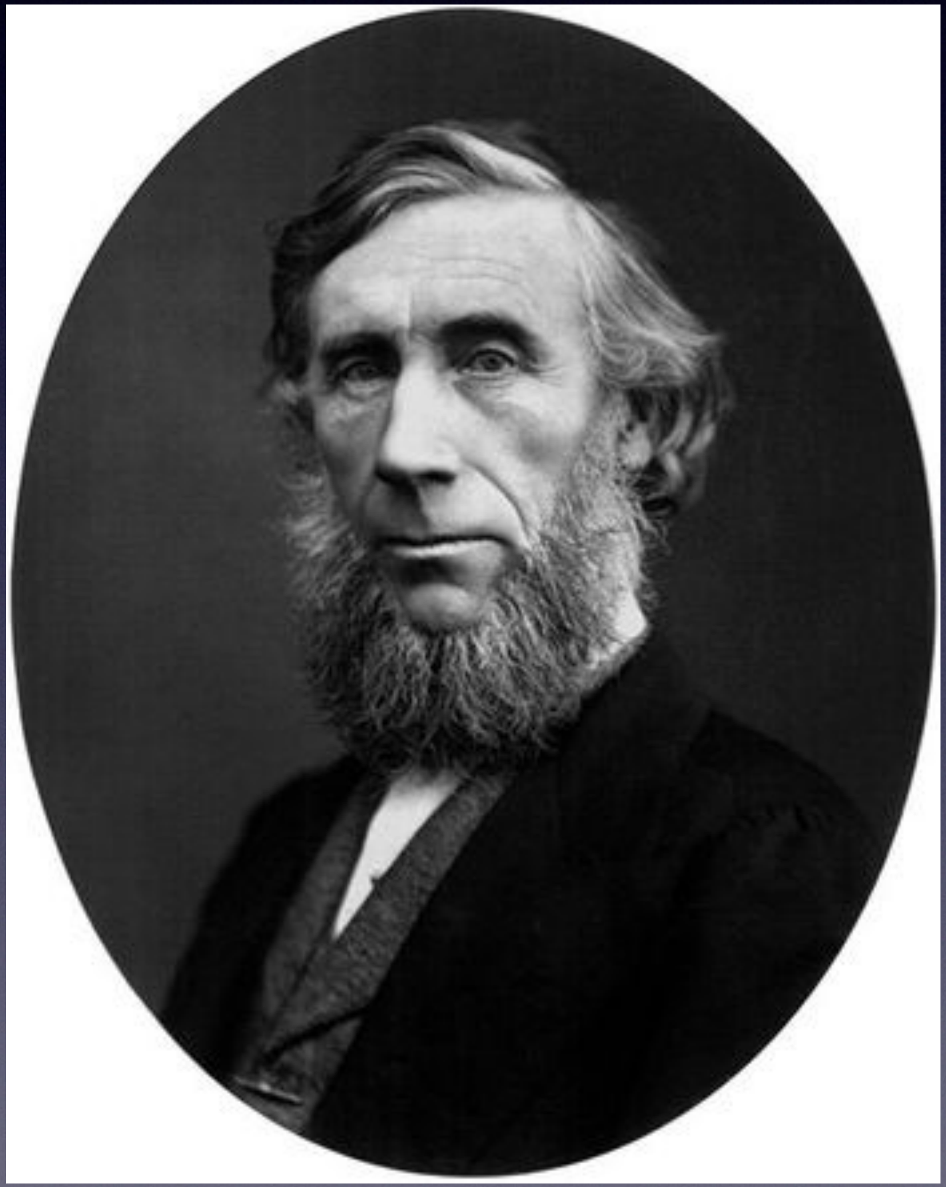


Energy Balance Cartoon



Ice Ages

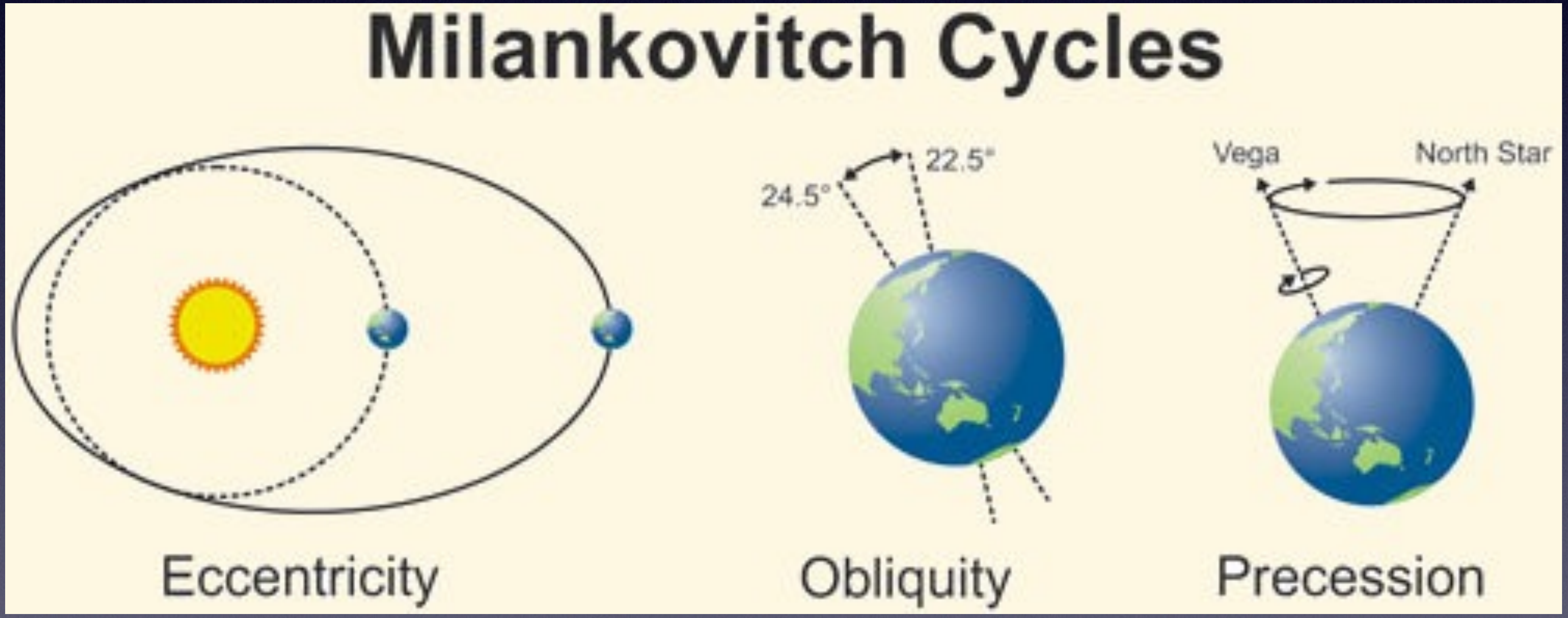
- Mid-1700s: speculation that ice ages exists
- 1830s: A few geologists claim ice-ages happend, ideas rejected
- 1842: Joseph Adhémar (mathematician) is first to propose ice-ages caused by variation in solar radiation



Ice Ages

- 1870s: Geologists reach consensus that ice-ages occurred (James Croll)
- 1912-1924: Milutin Milankovic
  - Eccentriciy (100 kyr) — Kepler 1609
  - Obliquity/Axial tilt (41 kyr)— Milankovic 1912
  - Precession (23 kyr)— Hipparchus 130 B.C.

Milankovic Cycles

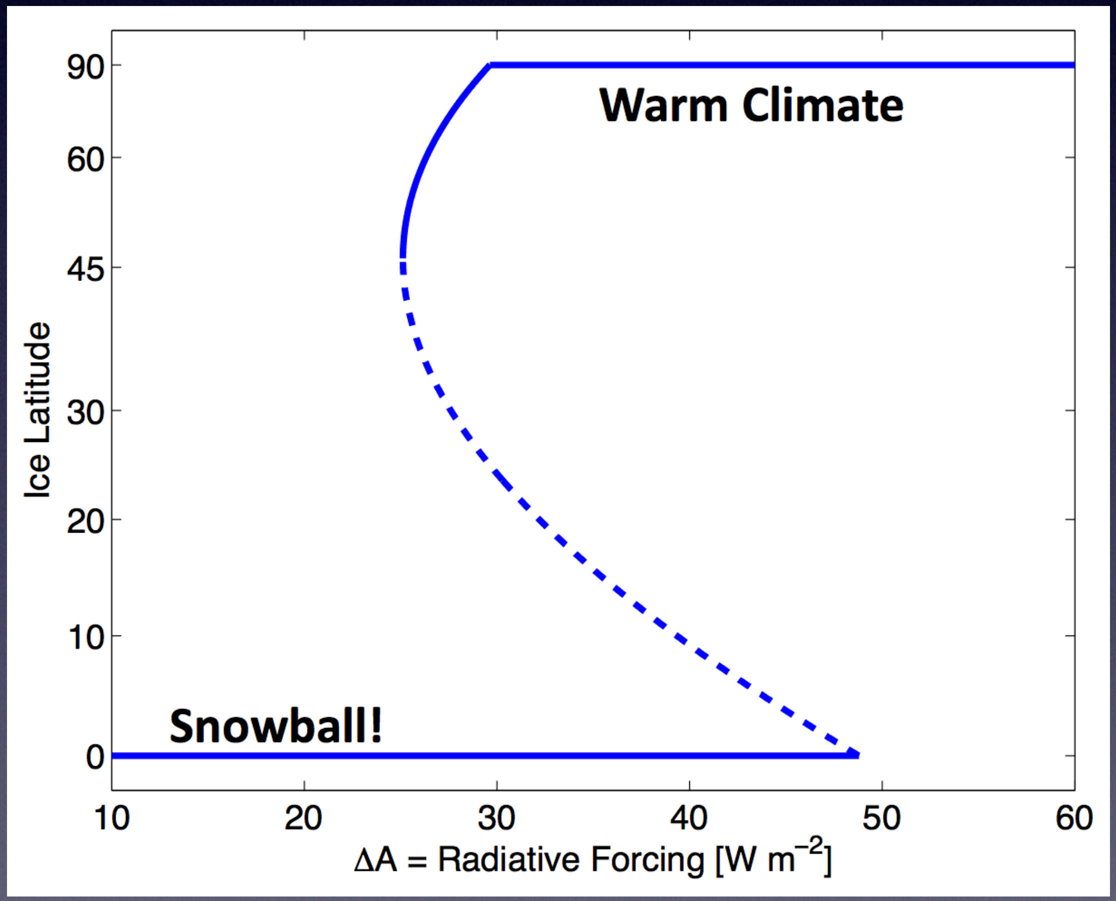


Snowball Earth

$$\frac{\partial T(y)}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT(y)) - C(T(y)-\bar{T}(\eta))$$

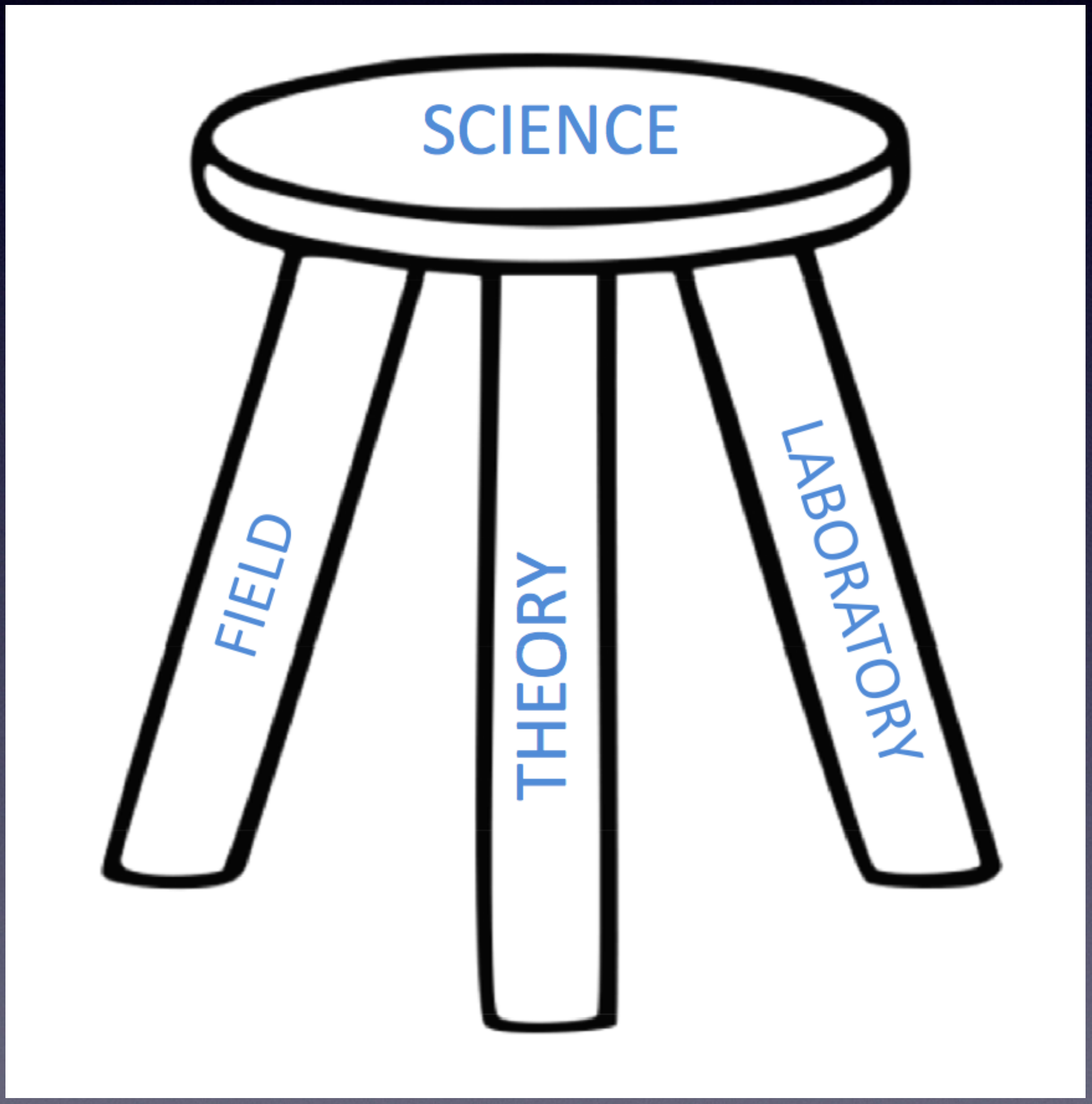
- Budyko and Sellers (1969) describe spatially dependent energy balance model
- Assume Northern and Southern hemisphere symmetric
- Assume temperature is the same for fixed latitude (y)
- Includes energy transport term

Snowball Earth

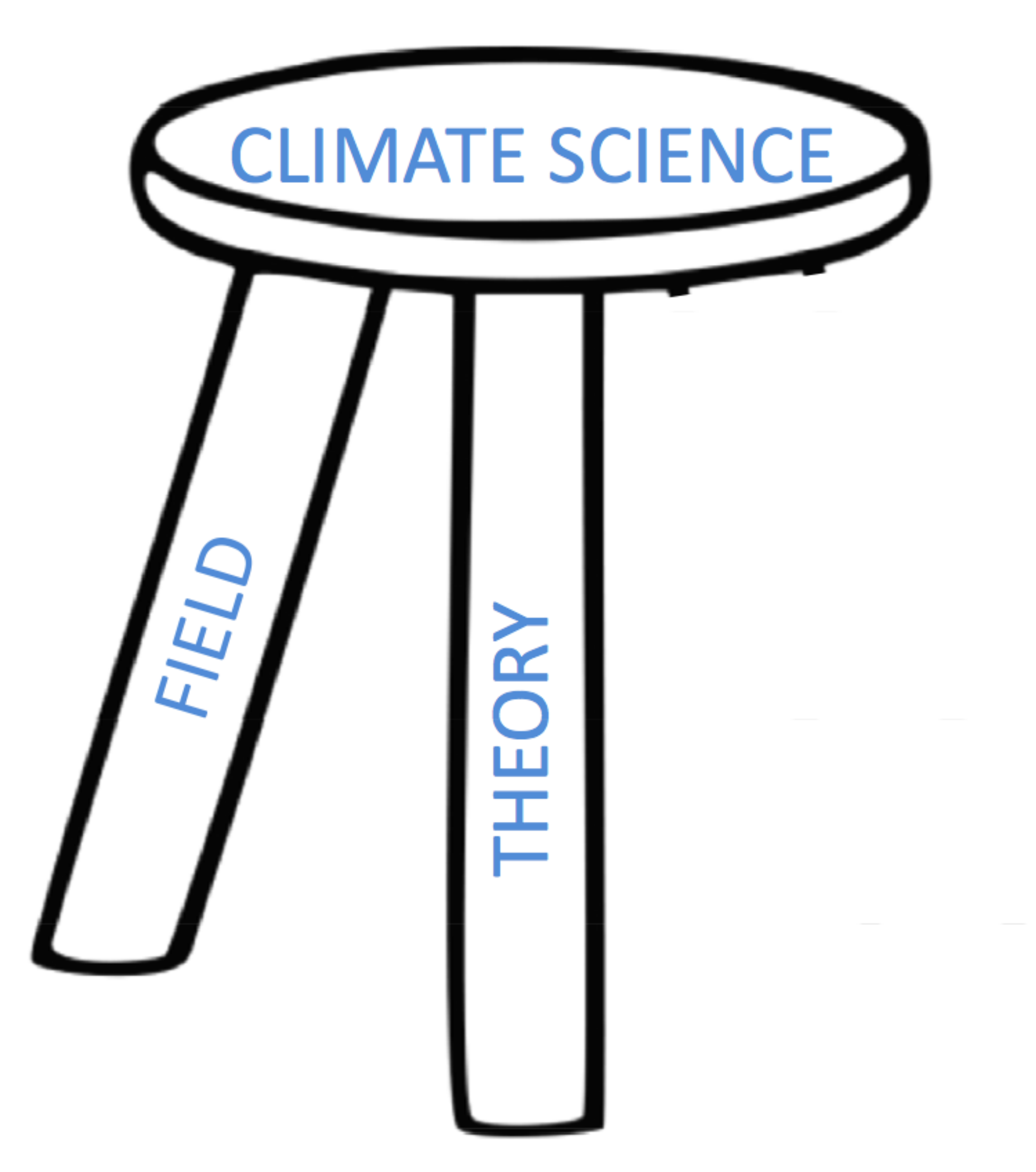


- 2 stable states of ice coverage:
  - Warm climate (like now - and even ice ages)
  - Snowball climate (entire Earth covered in ice)
- Dismissed as “mathematical artifact” until 1990s
  - New consensus: 3 snowball events (all over 600 myr ago)

How do we study the climate?



How do we study the climate?



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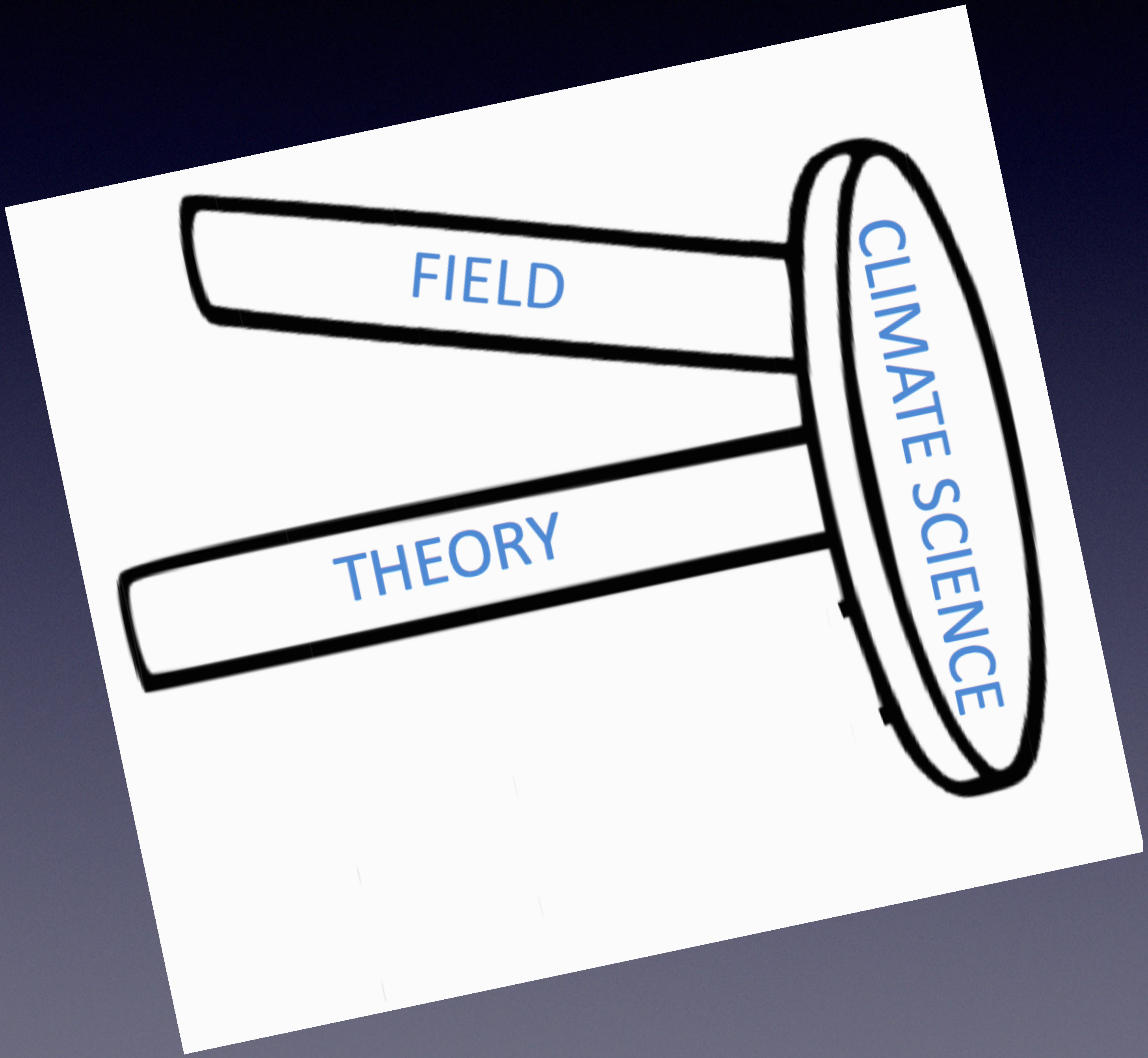
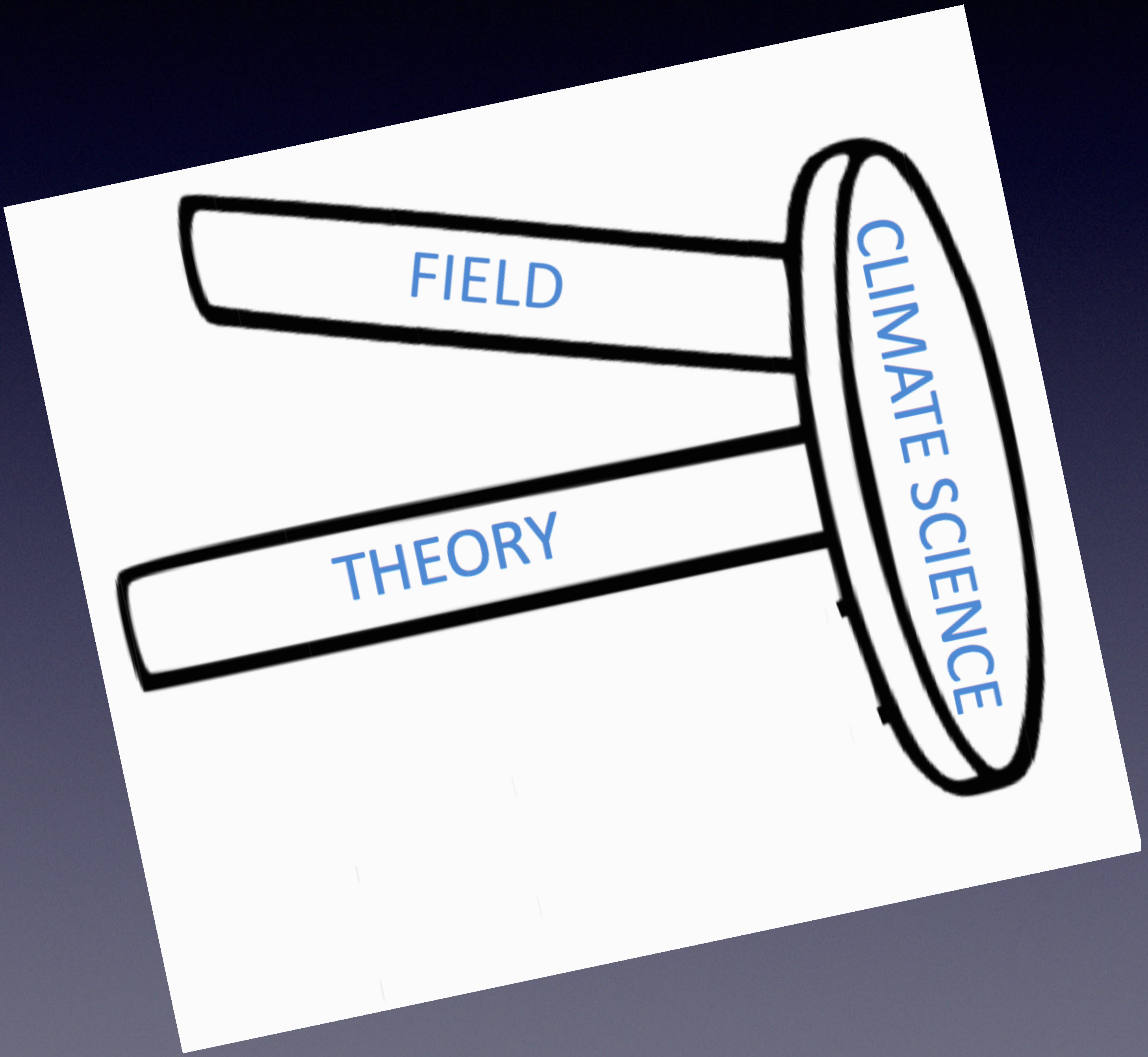
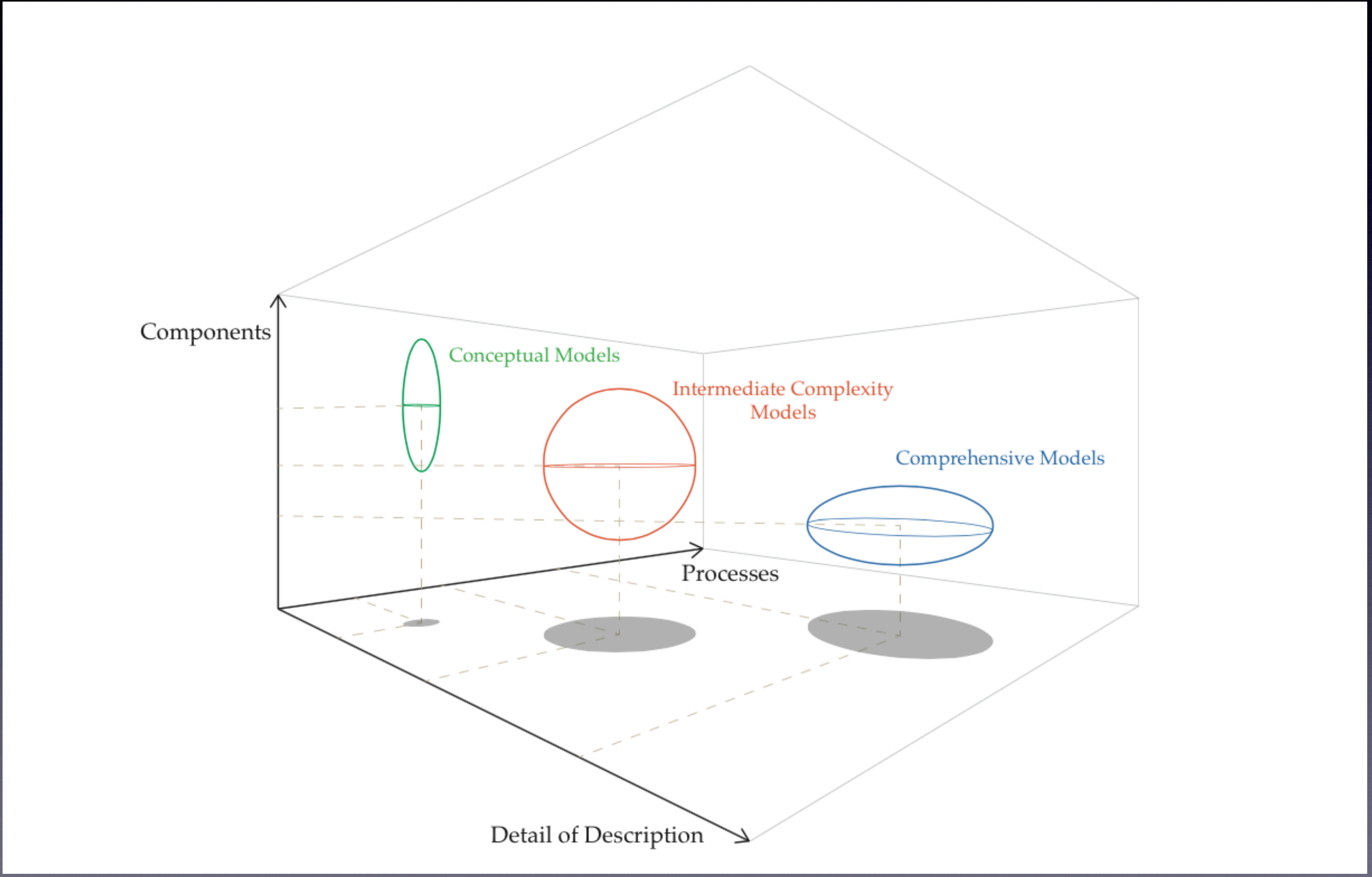


How do we study the climate?

Models!



Model Hierarchies



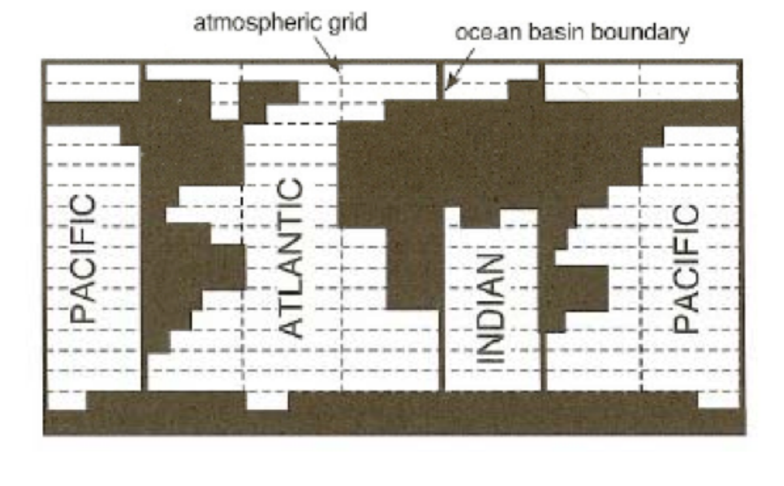
Conceptual Models

- Examples: Energy balance models
- Typically model 1 or 2 processes/phenomena
- Large-scale average behavior
- Help explain climate to non-experts
- Motivate large experiments

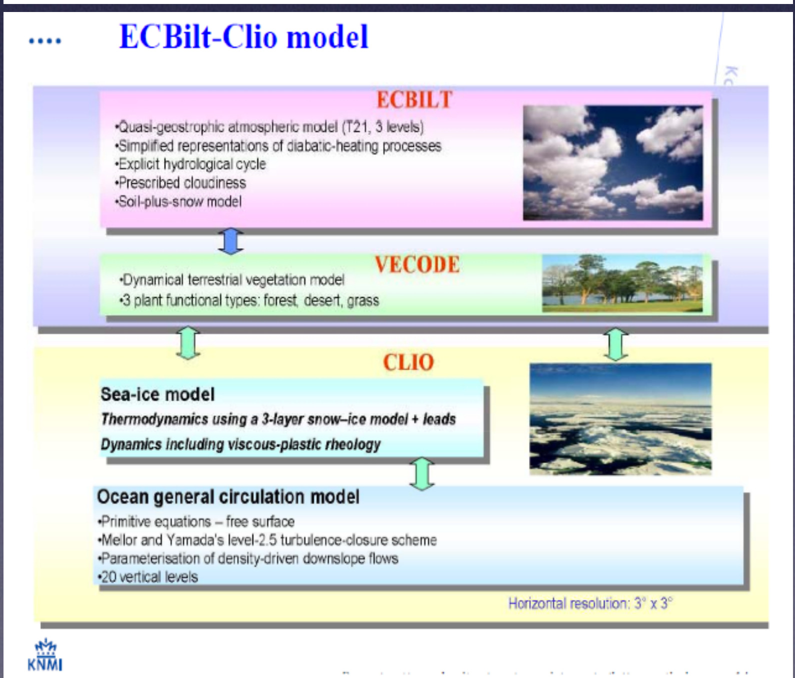
- Pros:
  - Simple enough to be analyzed by a person
    - Can explore all possibilities
  - Intuition
- Cons:
  - Too simple to prove scientific results definitively
  - Adding more processes could destroy phenomenon

Intermediate Complexity and Process Models

- Some spatial resolution
- More processes (but not too many)
- Simple enough for some interpretation
- Too complex to analyze "by hand"

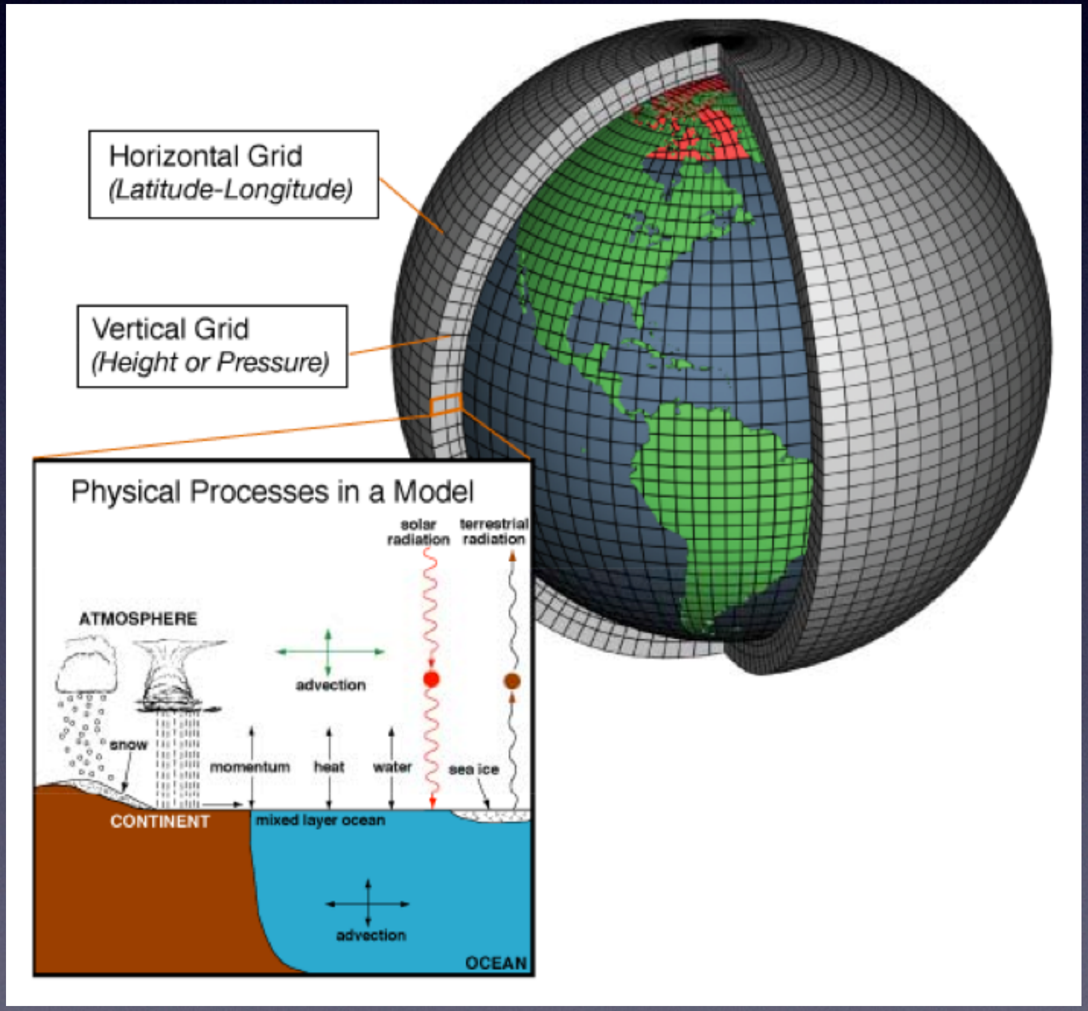


Climber-2 (PIK)



KNMI (The Netherlands)

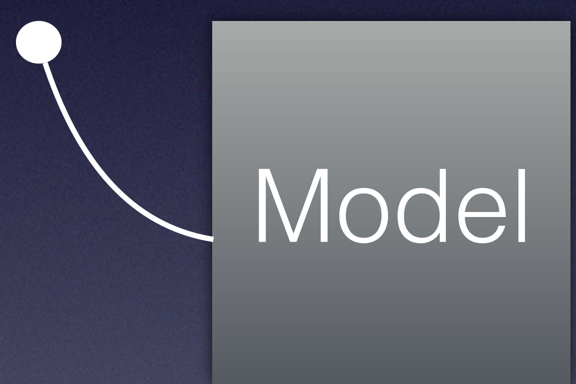
GCMs and ESMs



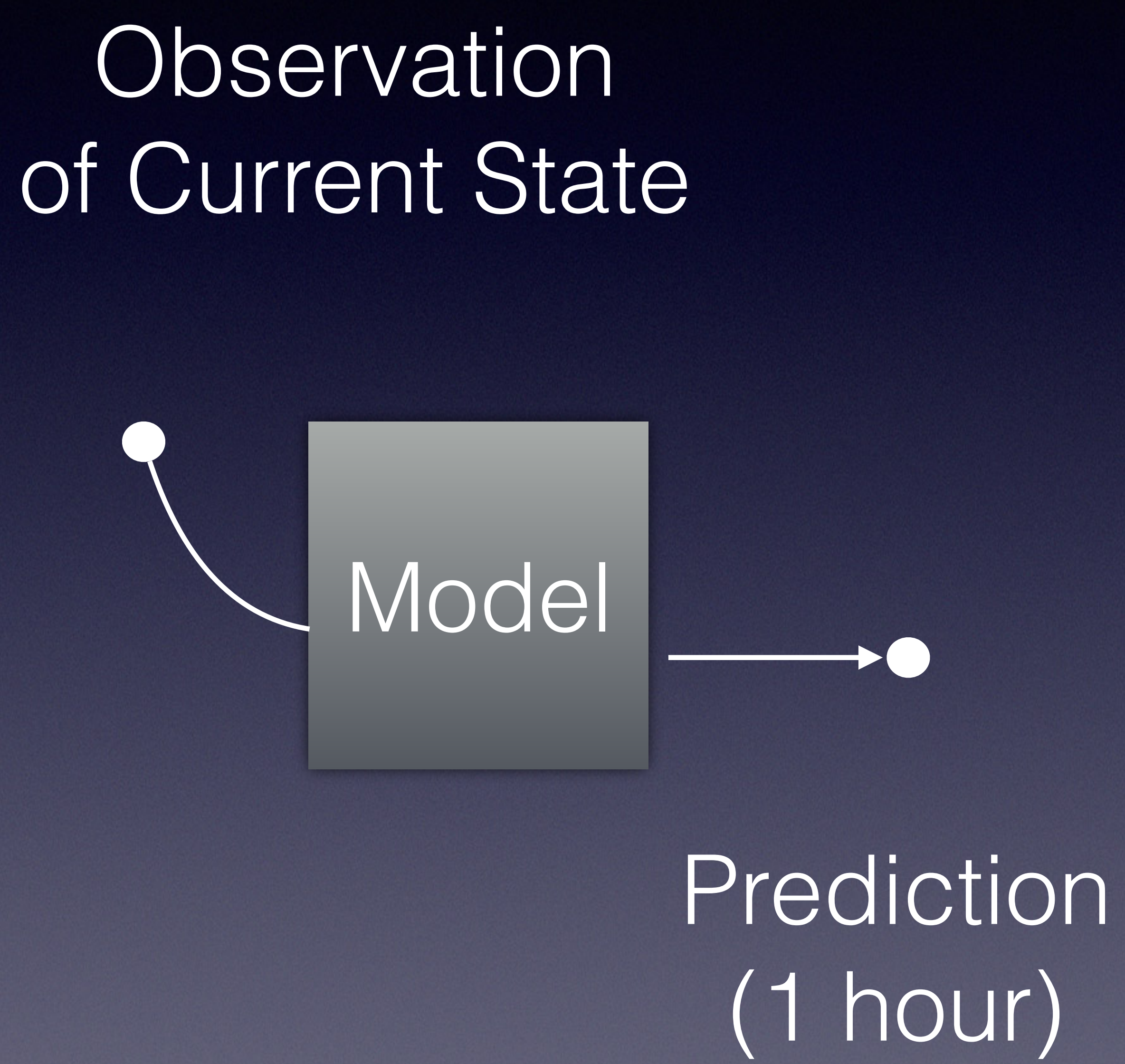
- Too complicated to interpret causality
- Too complicated to explore all possibilities (where do we look?)
- Millions of lines of code (bugs?)
- Expensive (financially and computationally)
- Treated as "experimental Earths"
- Useful for prediction*

Weather Prediction

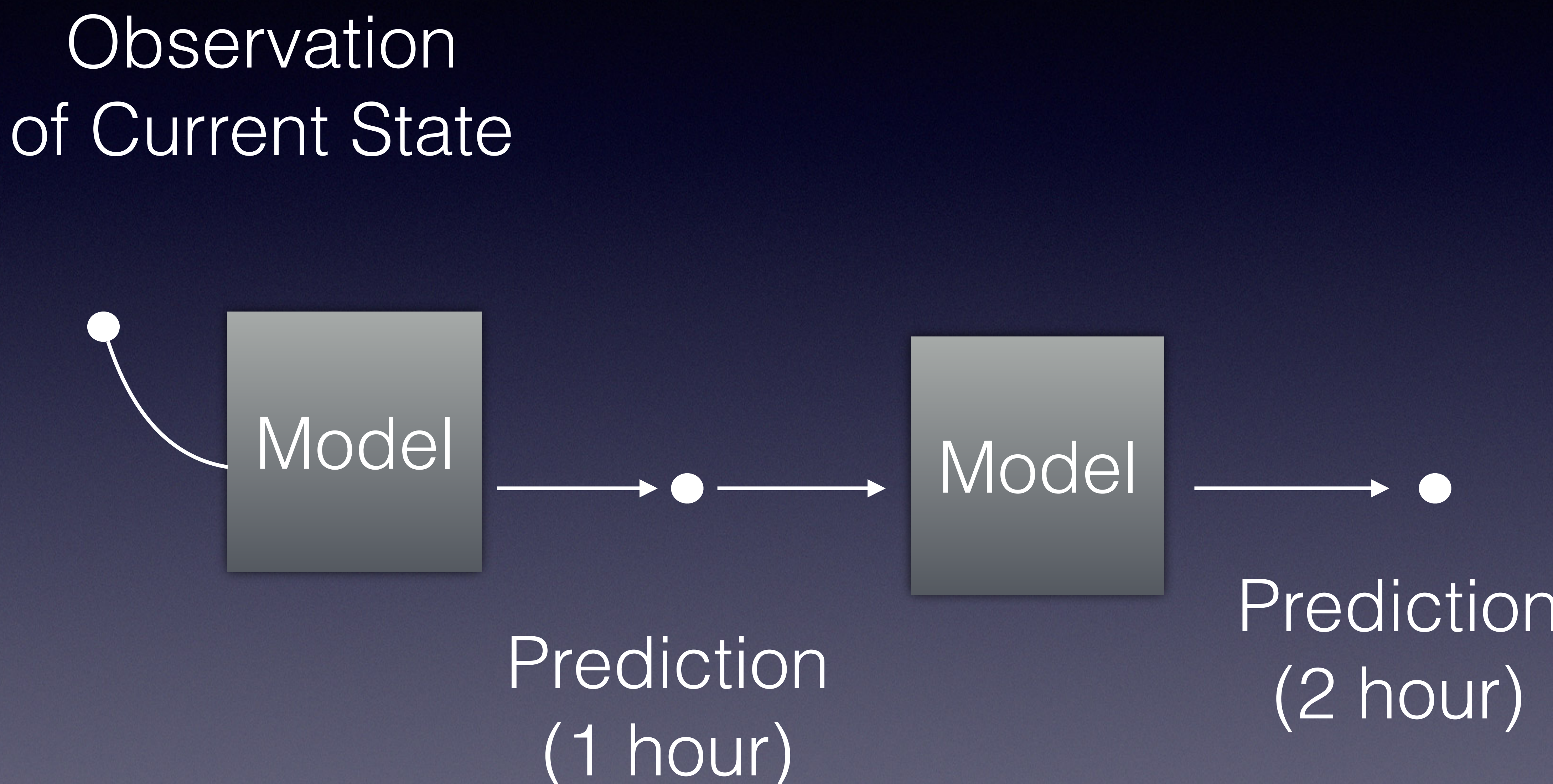
Observation
of Current State



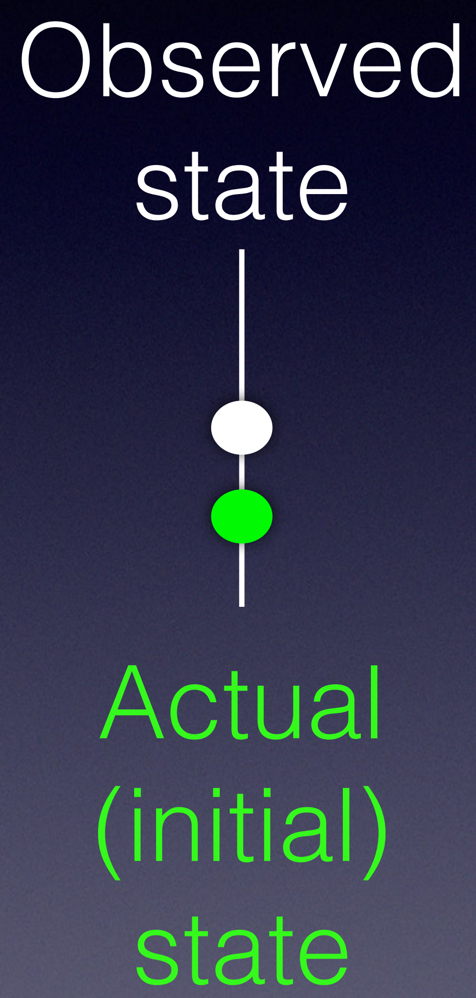
Weather Prediction



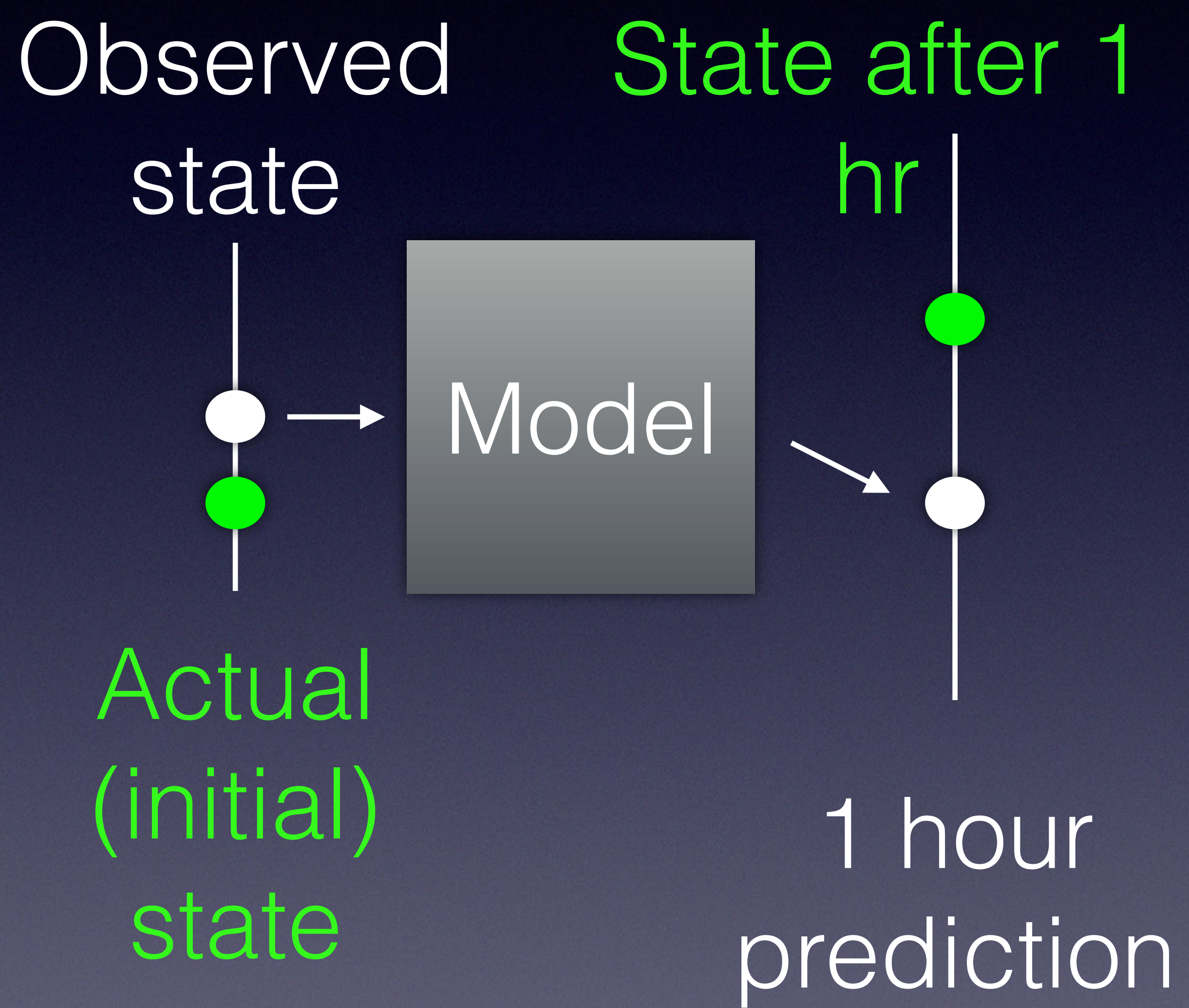
Weather prediction



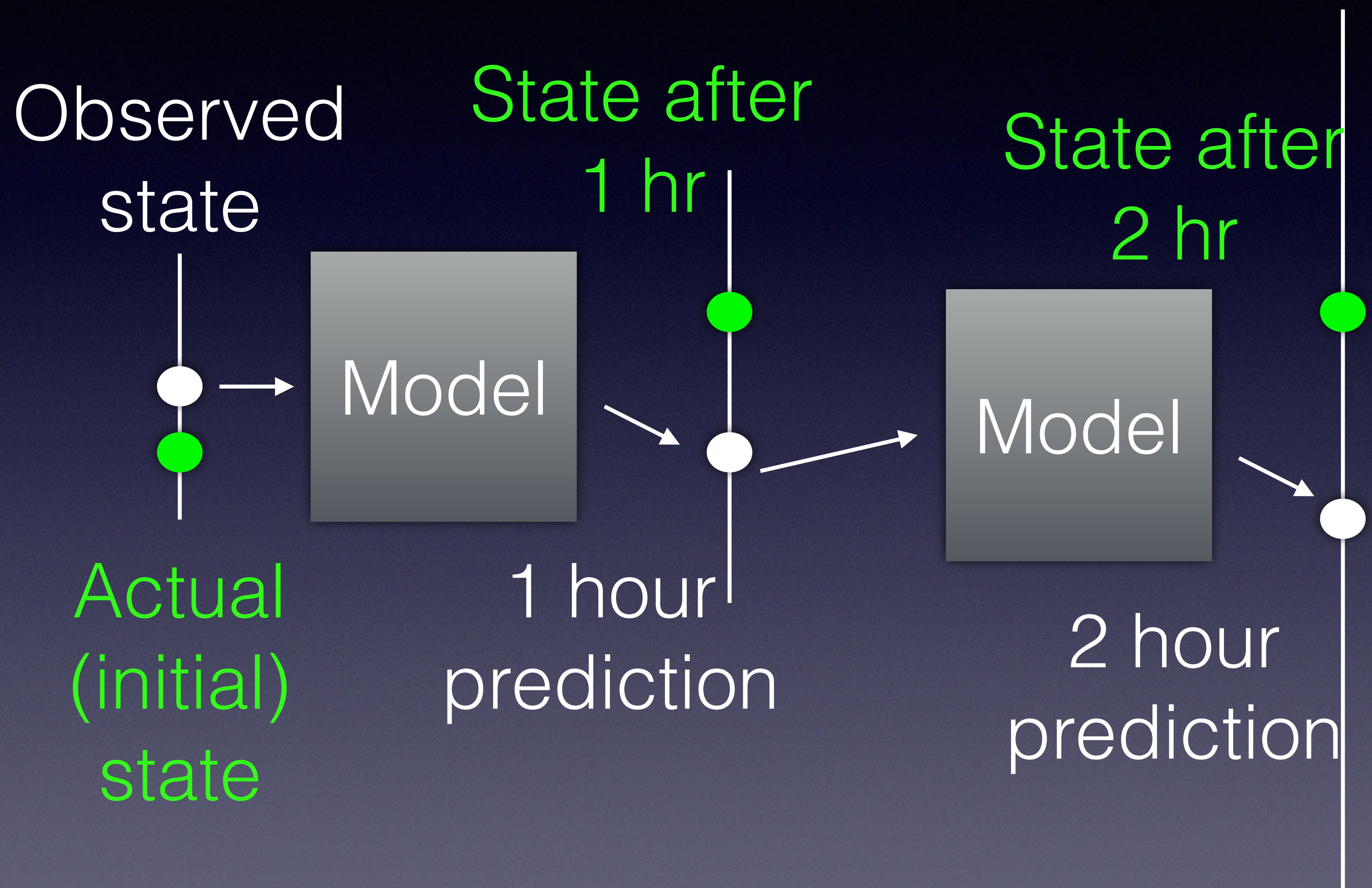
Observations have error



Error grows


Observed state
State after 1 hr
Model
Actual (initial) state
1 hour prediction

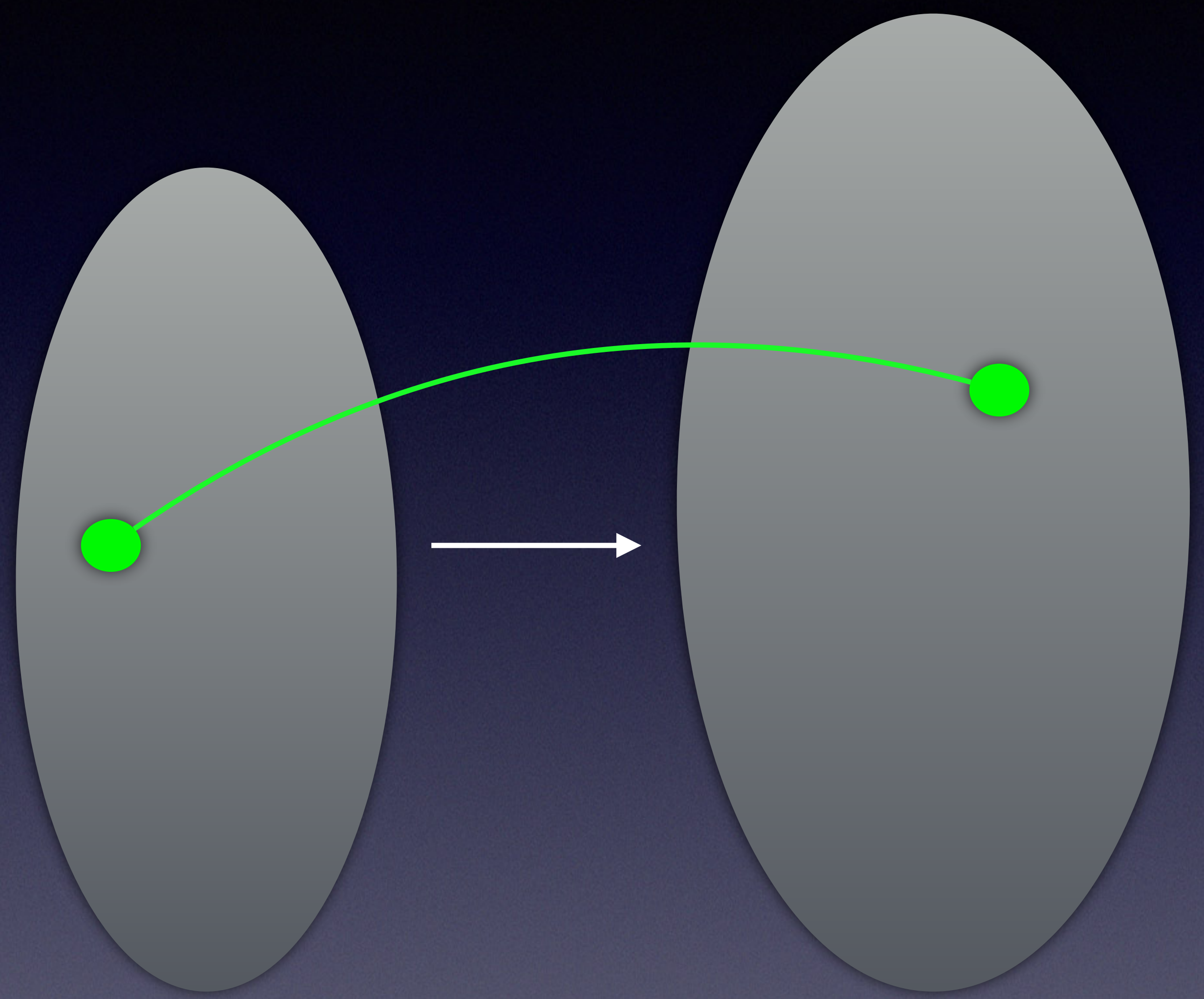
and grows…


Observed state
State after 1 hr
State after 2 hr
Model
Model
Actual (initial) state
1 hour prediction
2 hour prediction

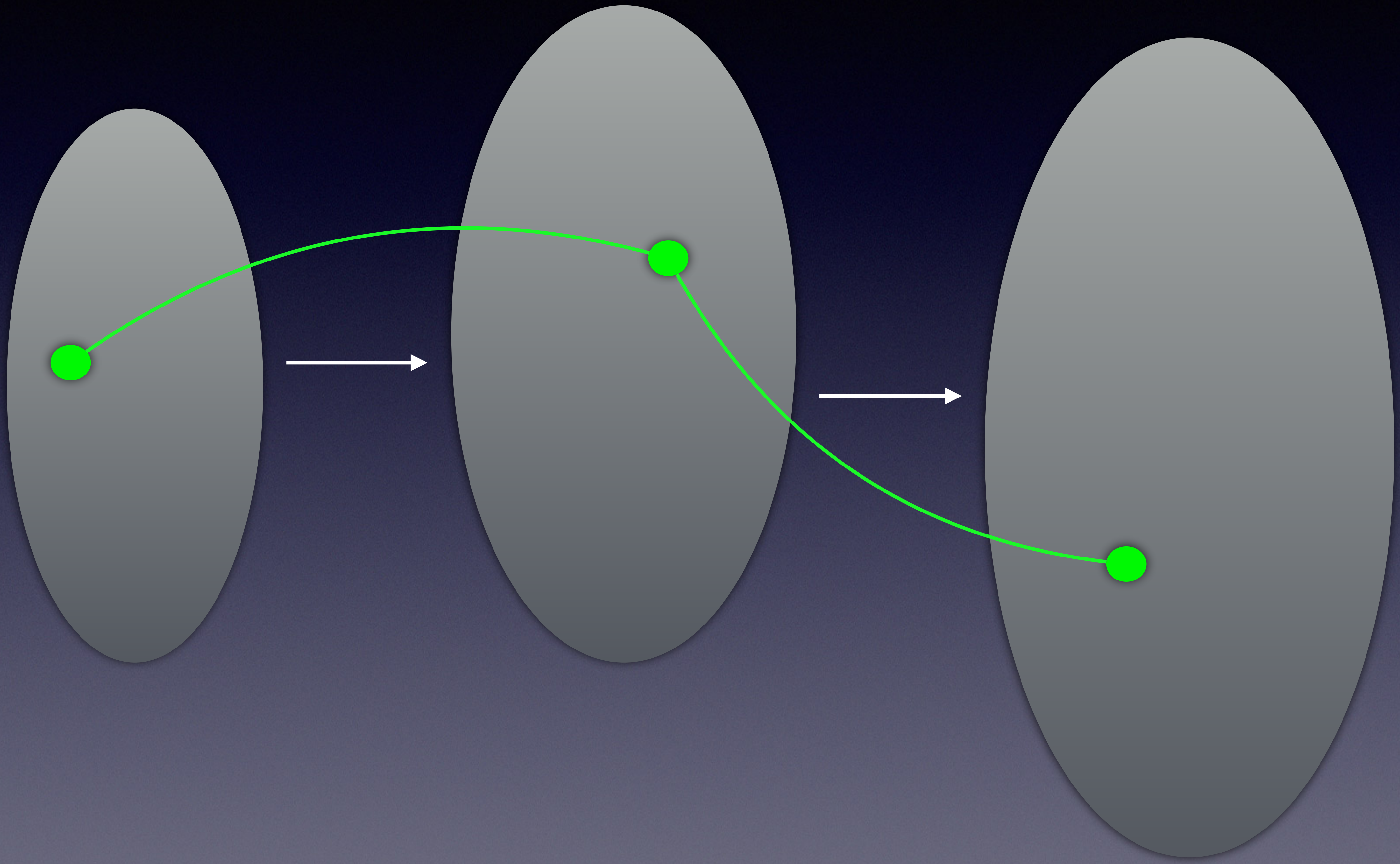
Lorenz Butterly

Climate Prediction

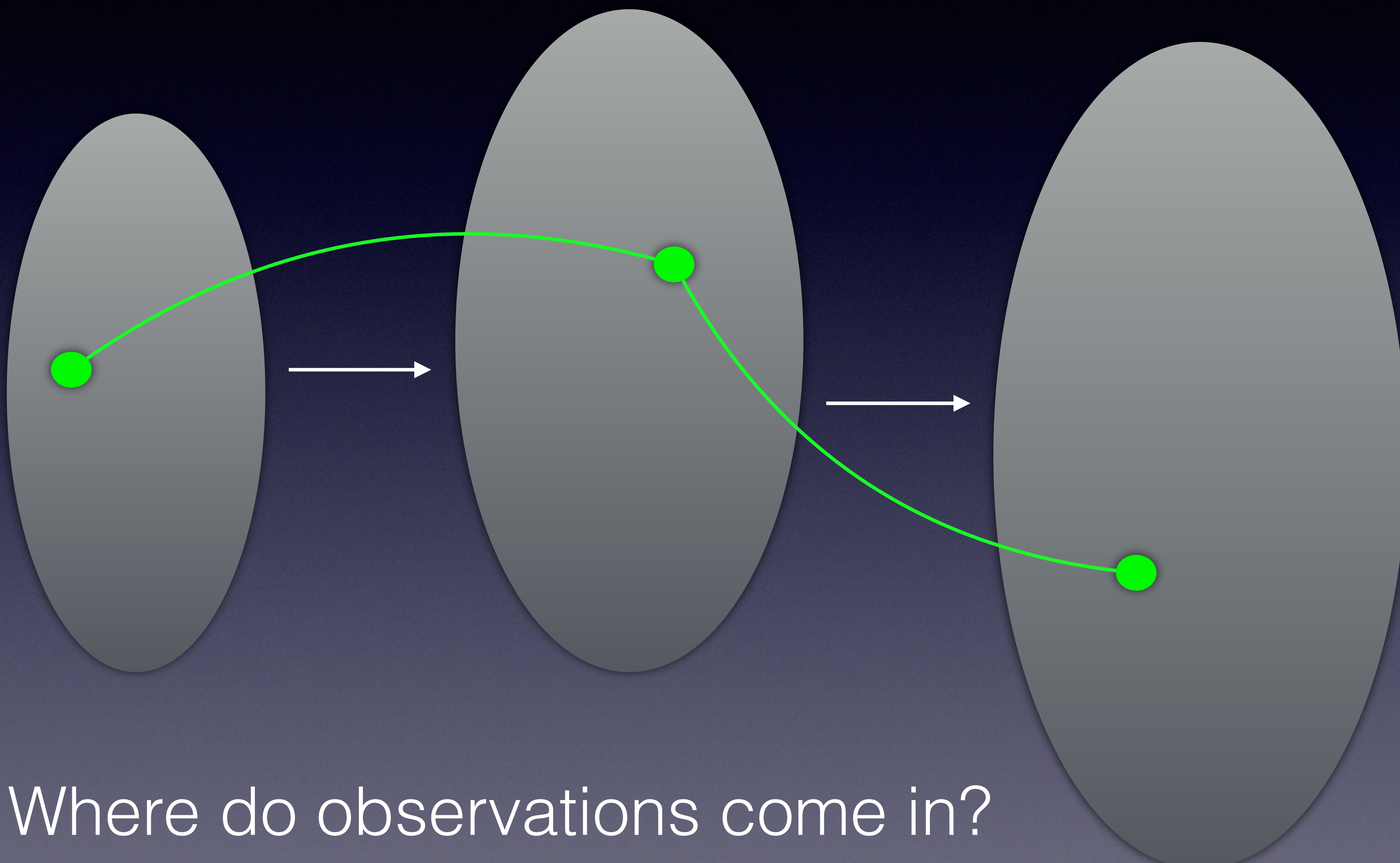
Climate Prediction



Climate Prediction



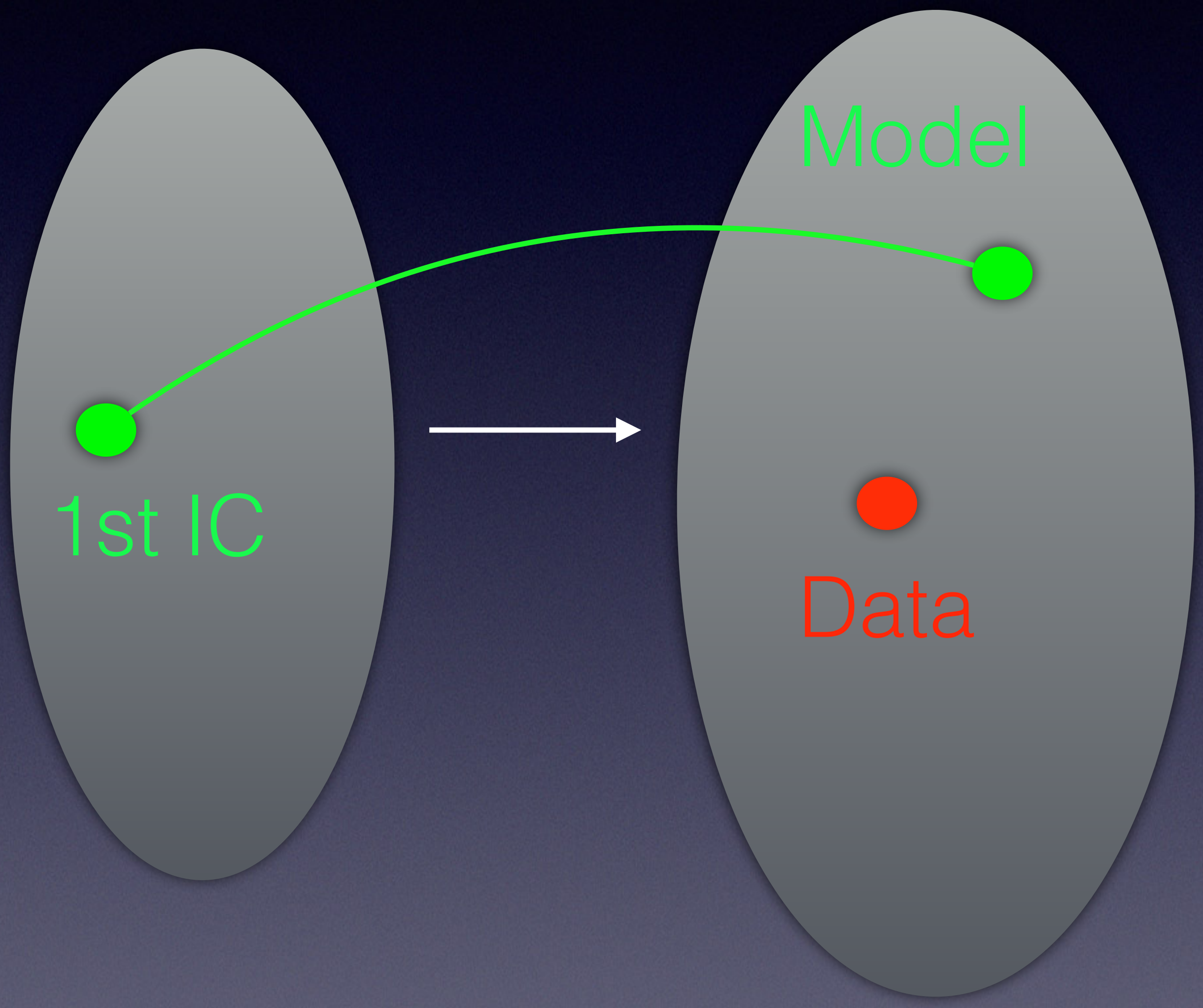
Climate Prediction



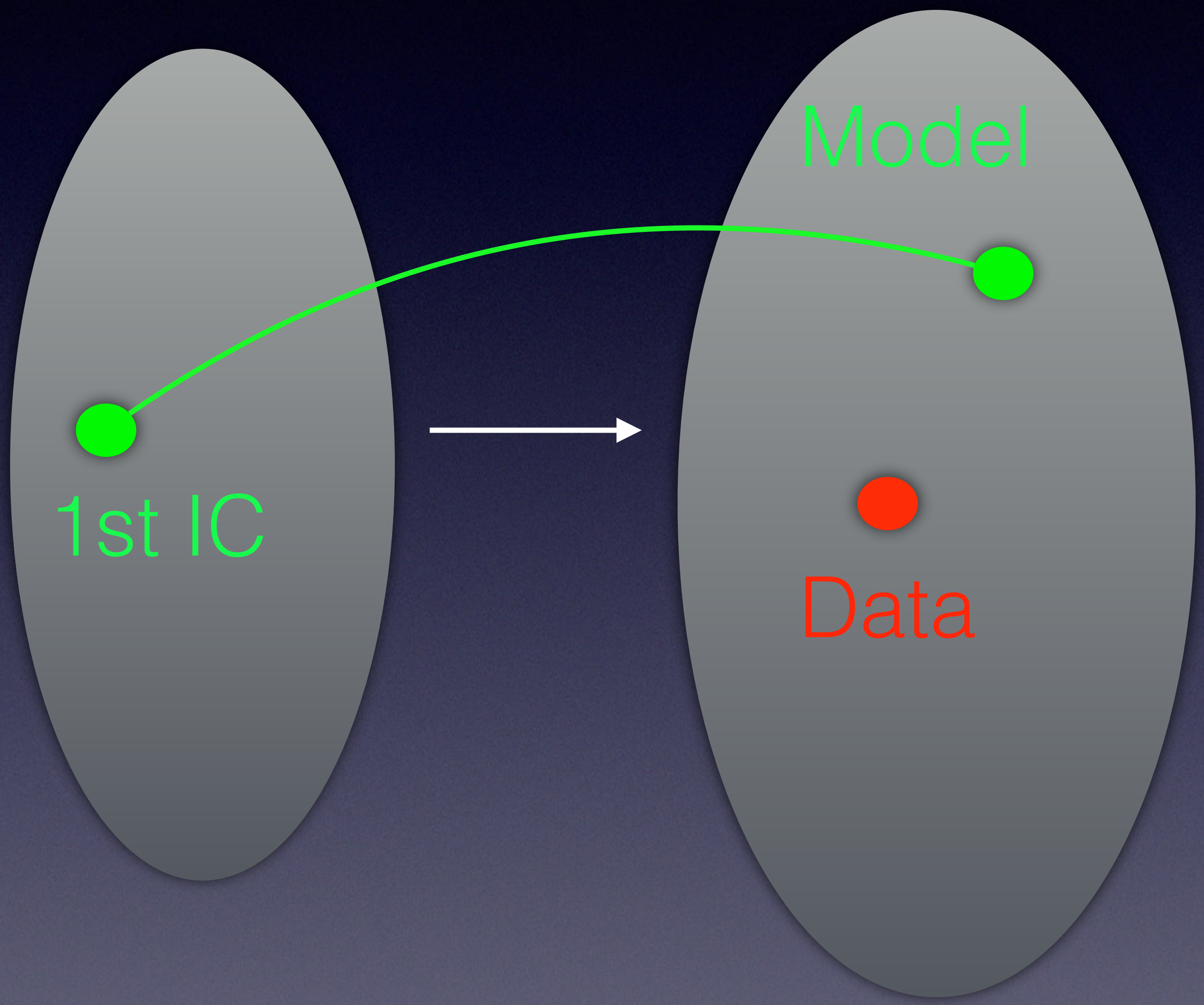
Where do observations come in?

Confronting Models with Data

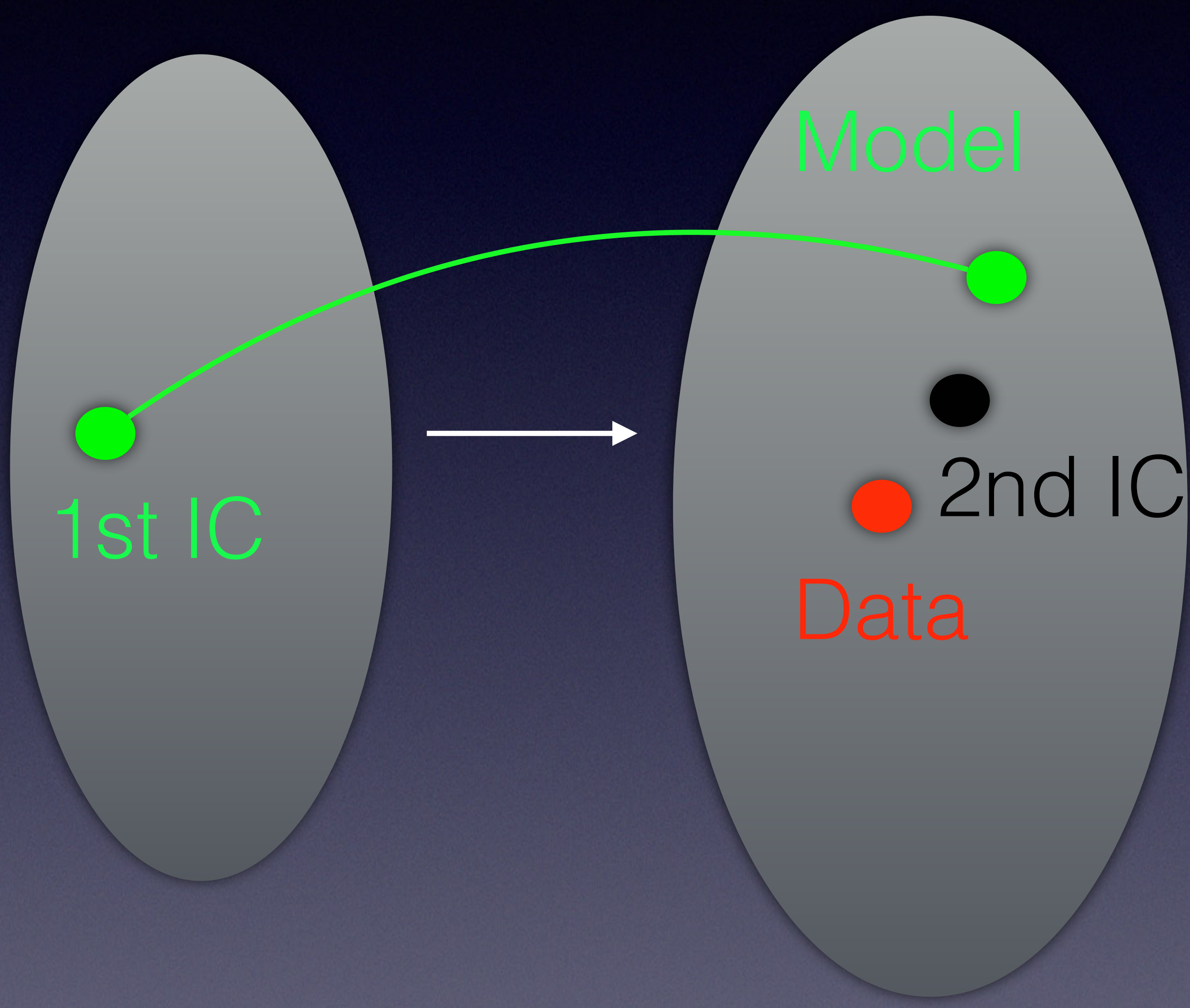
Confronting Models with Data



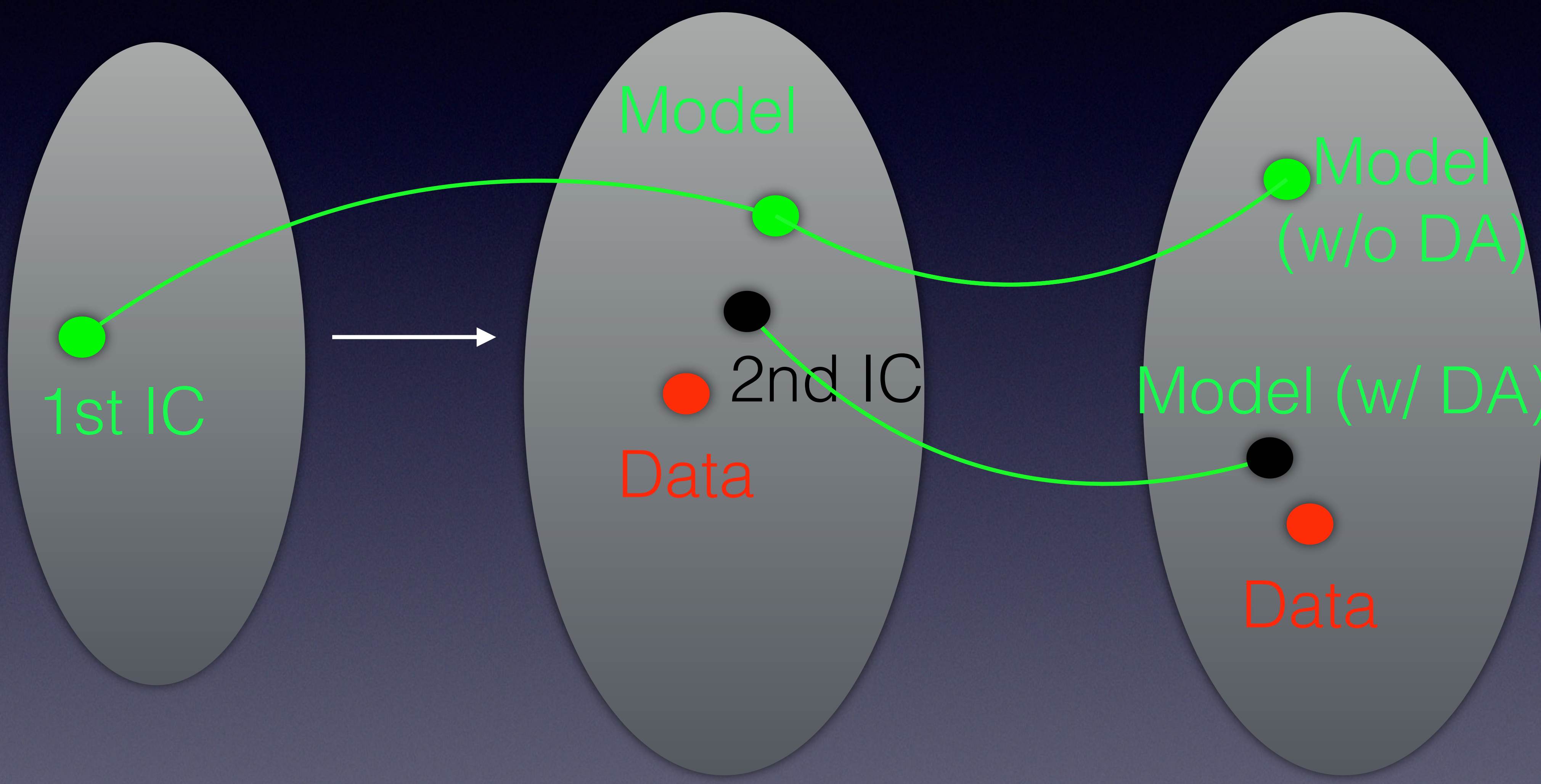
Confronting Models with Data



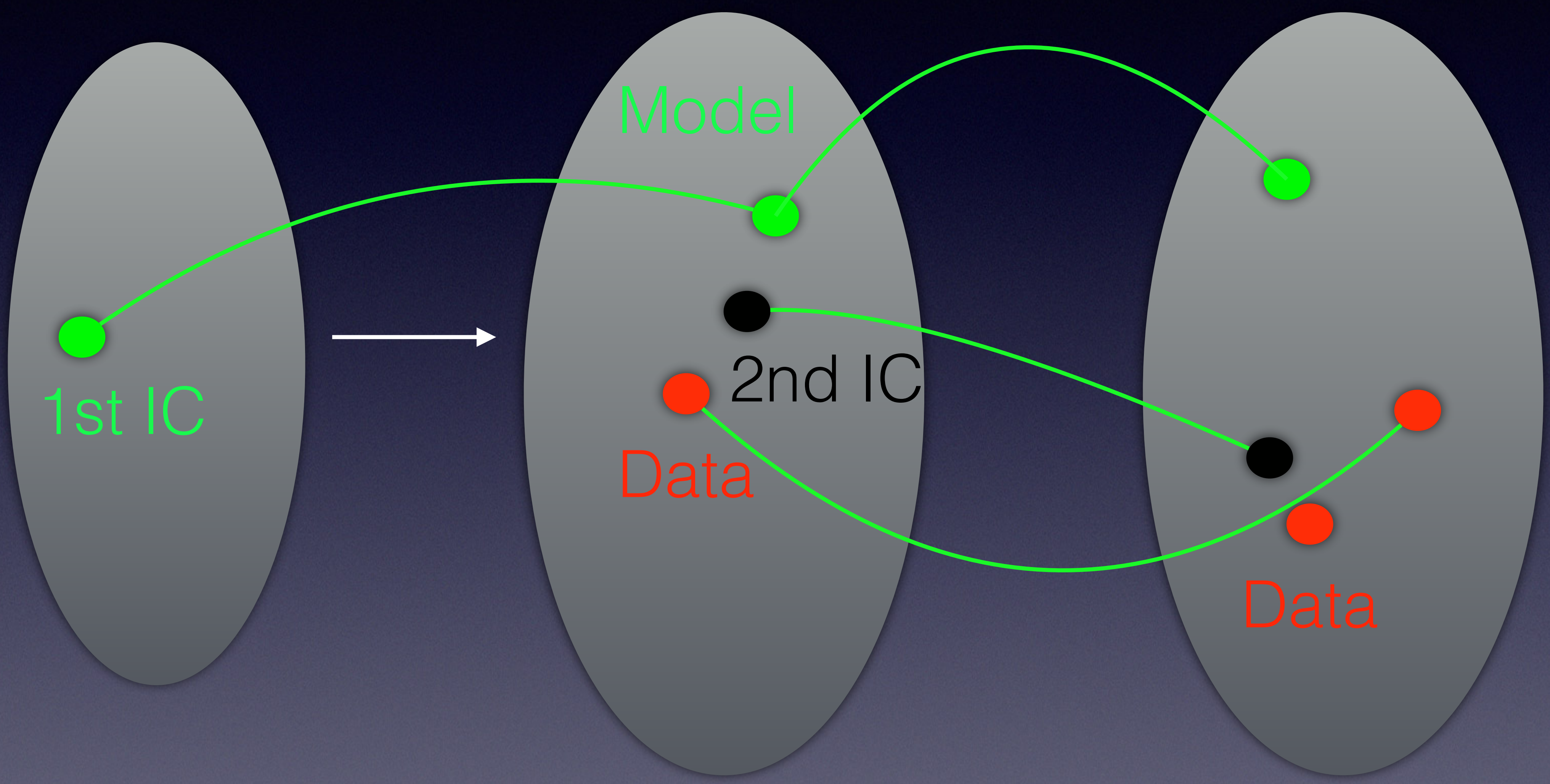
Data Assimilation



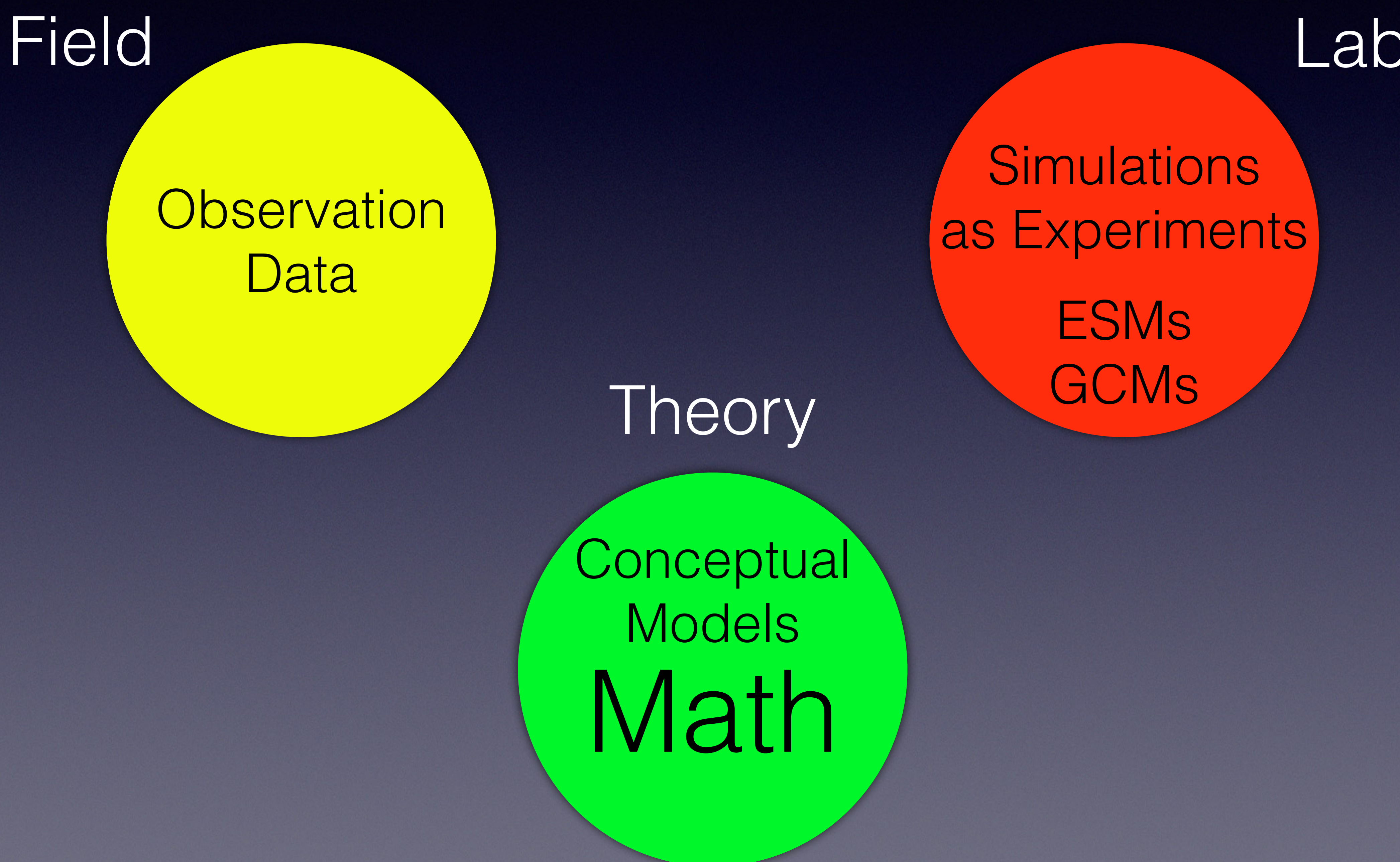
Data Assimilation



Data Assimilation



The Role of Mathematics in Climate Science



Dynamical Systems

Derivative
(from Calculus)

$$\frac{dx}{dt} = f(t)$$

Example

$$\frac{dx}{dt} = t^3 - t + k$$

$$x(t) = \frac{t^4}{4} - \frac{t^2}{2} + kt + C$$

Dynamical Systems

Derivative
(from Calculus)

$$\frac{dx}{dt} = f(t)$$

Example

$$\frac{dx}{dt} = t^3 - t + k$$

What if $\frac{dx}{dt} = f(x)$?

More than one variable?

System of
Differential Equations

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

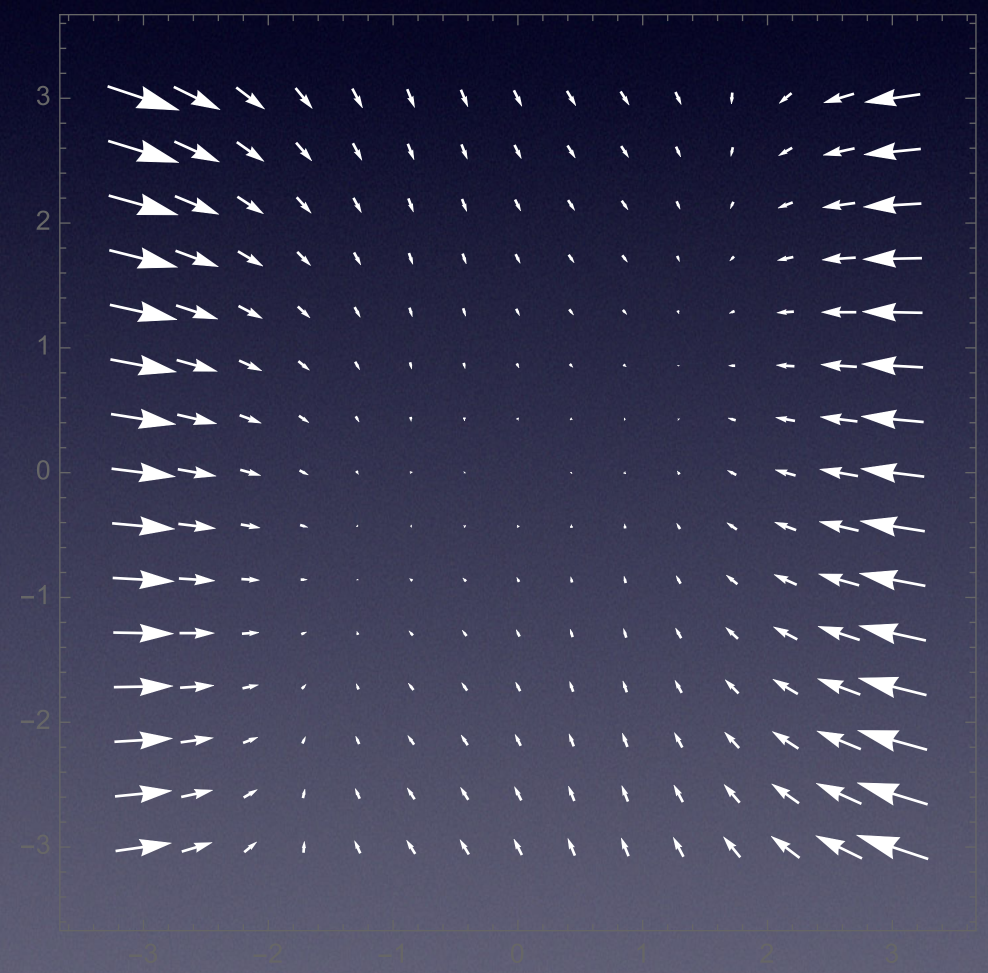
Defines a
Vector Field

Vector Fields

System of
Differential Equations

$$\dot{x} = y - x^3 + x$$
$$\dot{y} = x - 2y + k$$

Defines a
Vector Field

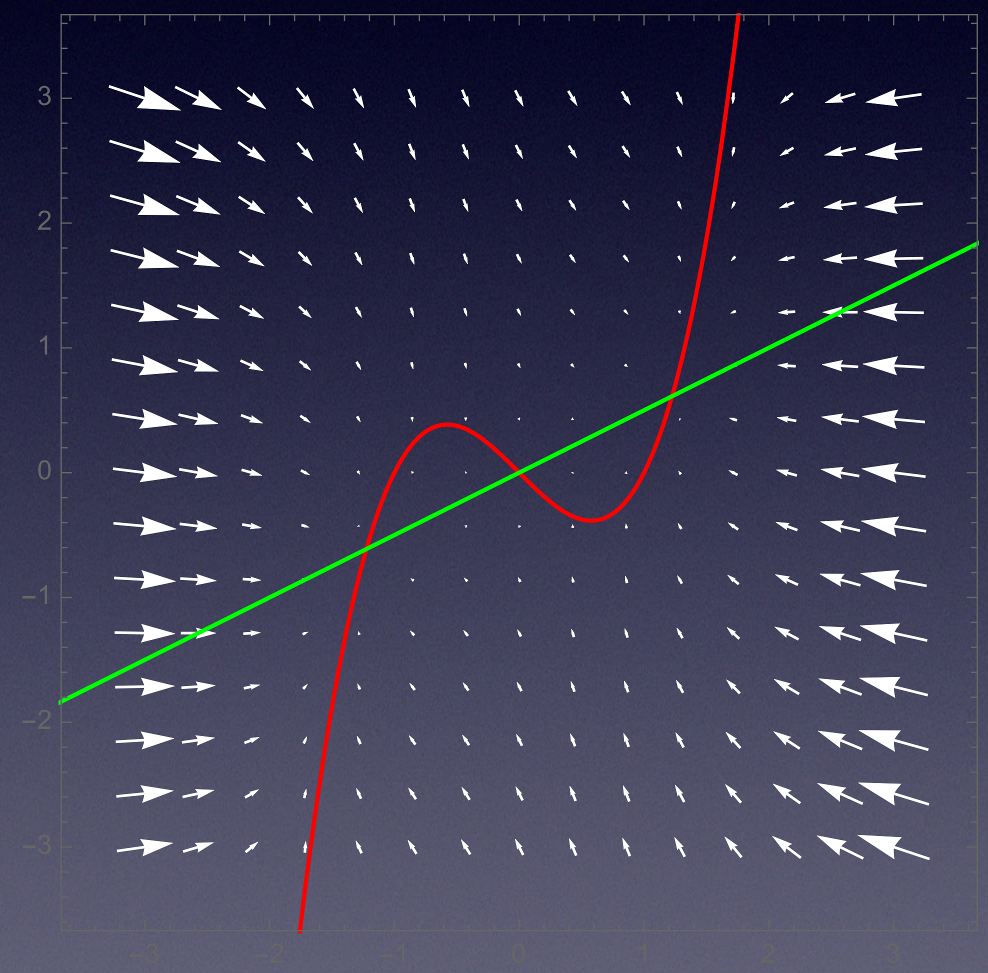


Equilibrium Points

$$\dot{x} = y - x^3 + x$$
$$\dot{y} = x - 2y + k$$

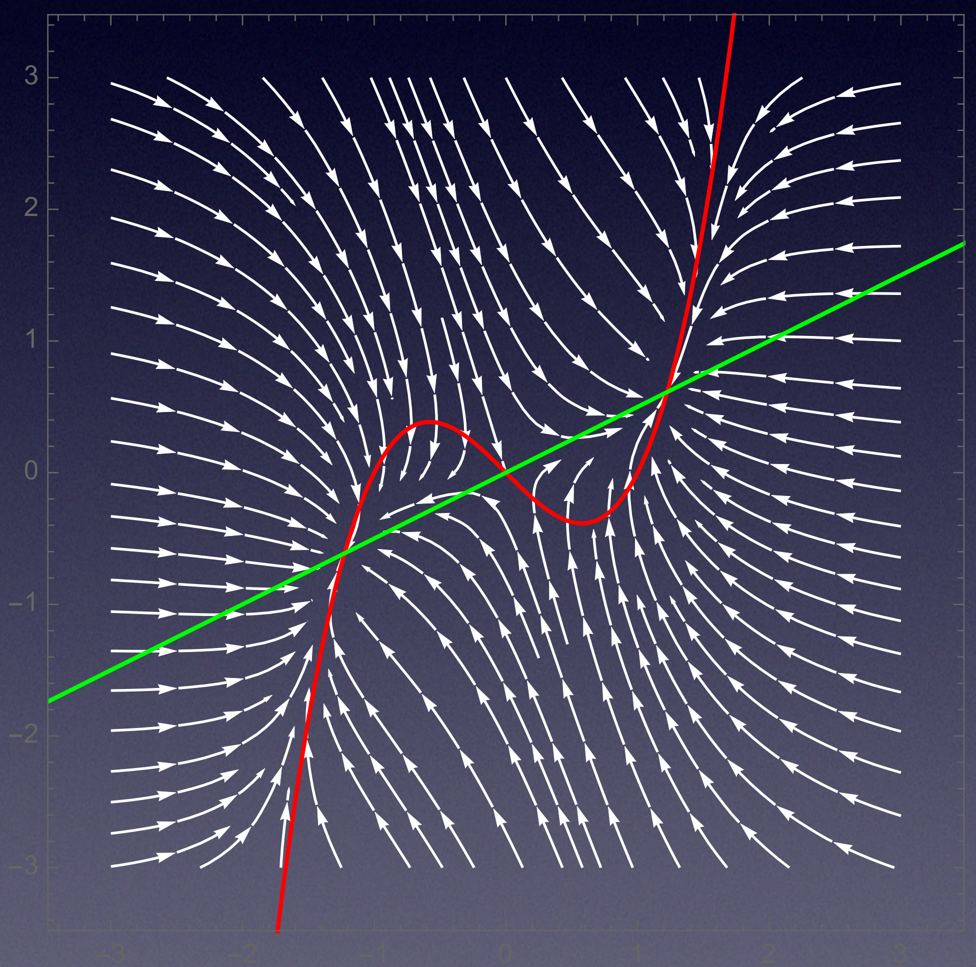
Equilibrium points occur when

$$\dot{x} = 0$$
$$\dot{y} = 0$$



Solutions

$$\dot{x} = y - x^3 + x$$
$$\dot{y} = x - 2y + k$$



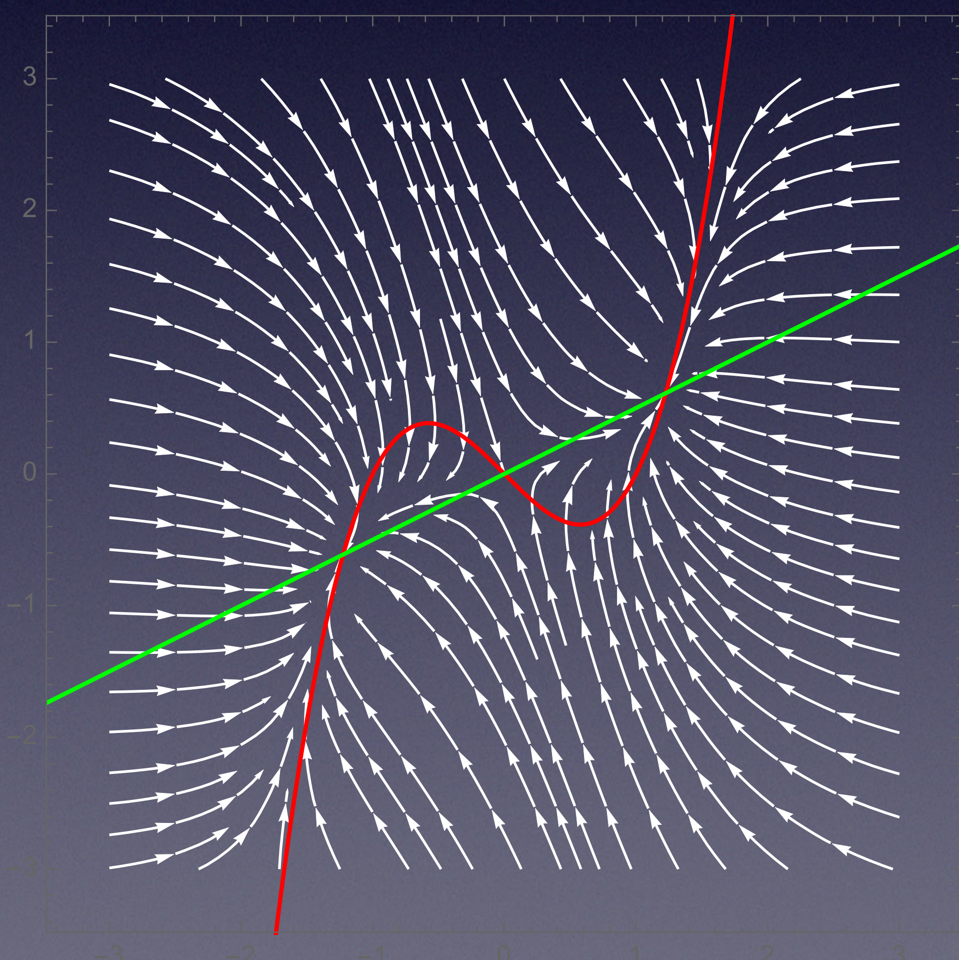
Even if equations
can't be solved,
we can understand

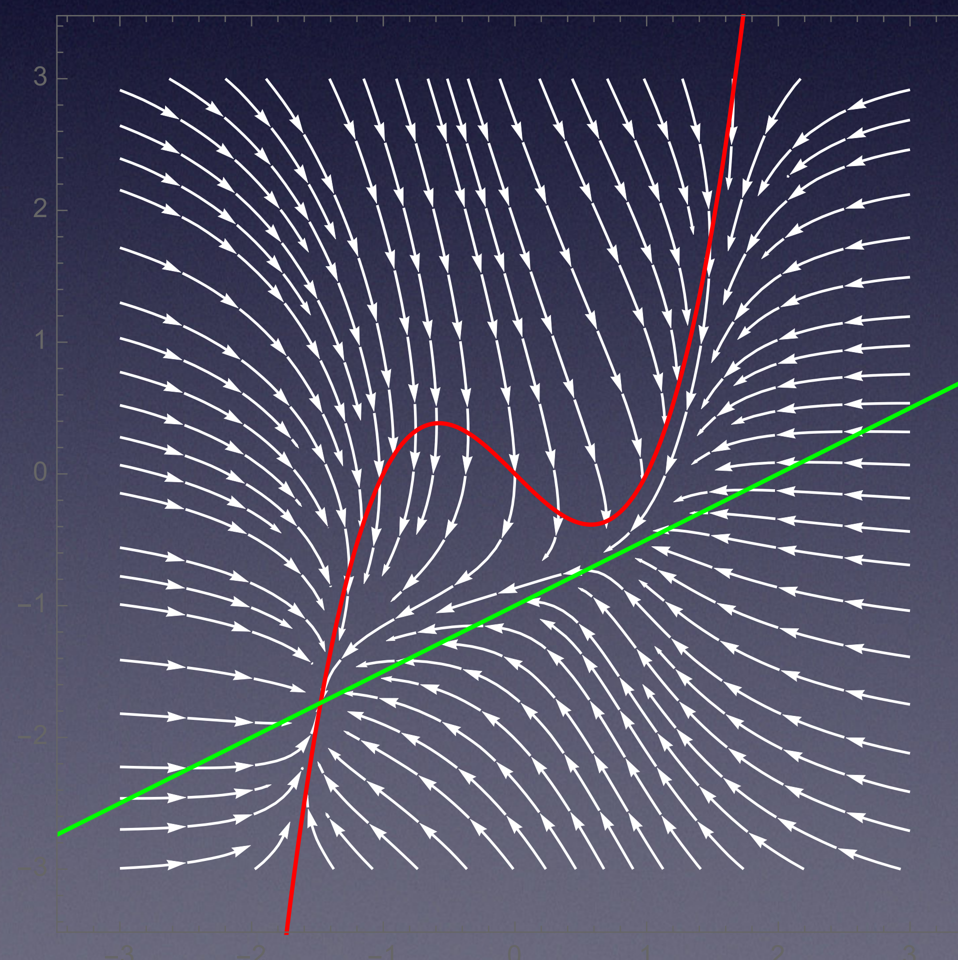
Qualitative Behavior

Varying k

$$\dot{x} = y - x^3 + x$$
$$\dot{y} = x - 2y + 0$$

$$\dot{x} = y - x^3 + x$$
$$\dot{y} = x - 2y - 2$$





Bifurcation in Algebra I

Quadratic equation

$$ax^2 + bx + c = 0$$

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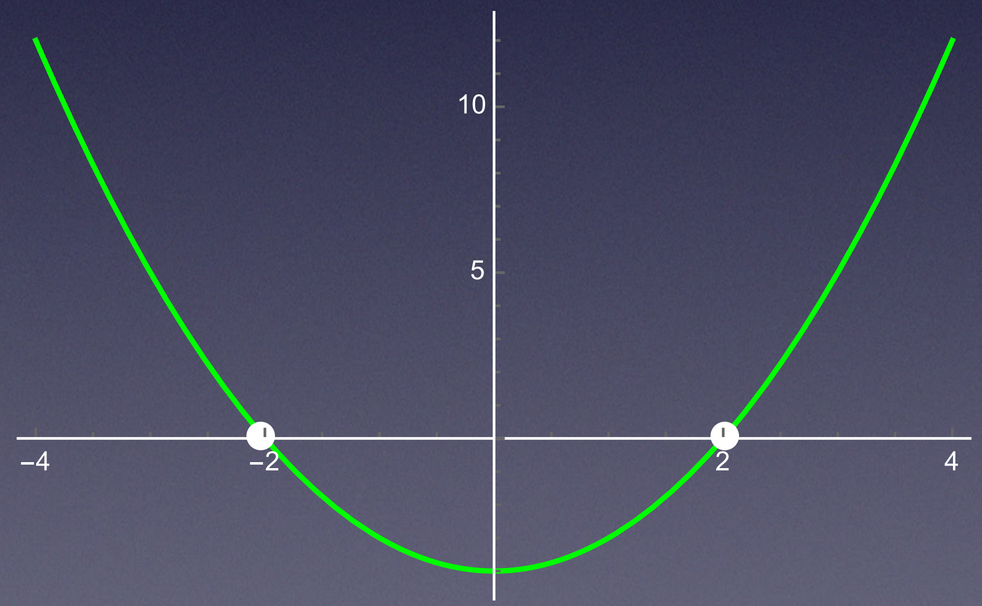
Quadratic equation

$$ax^2 + bx + c = 0$$

Bifurcation parameter:
Discriminant

$$b^2 - 4ac > 0$$

2 Real Roots



Bifurcation in Algebra I

Quadratic equation

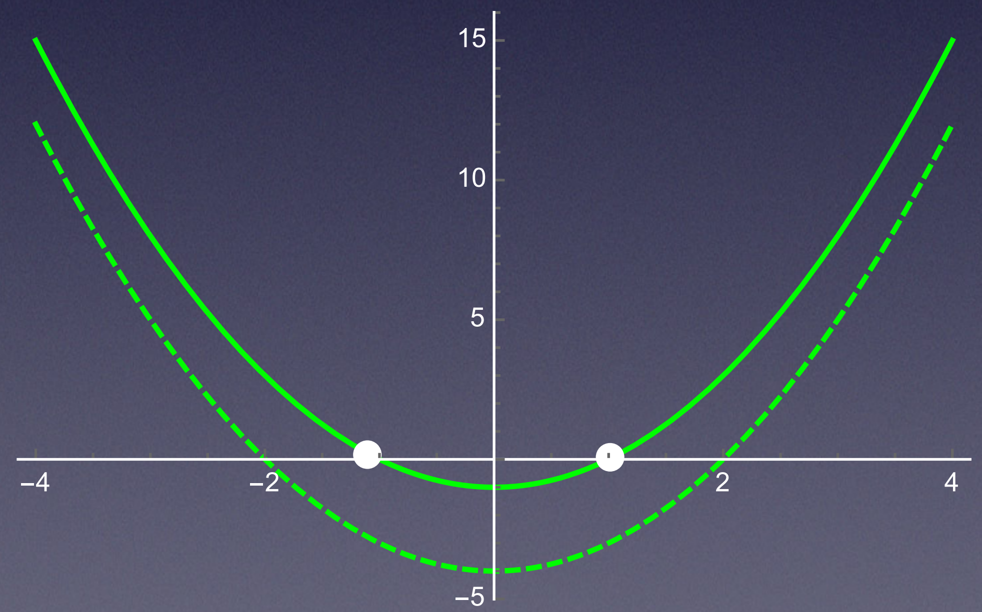
$$ax^2 + bx + c = 0$$

Bifurcation parameter:
Discriminant

$$b^2 - 4ac > 0$$

2 Real Roots

No qualitative change for small change in equation



Bifurcation in Algebra I

Quadratic equation

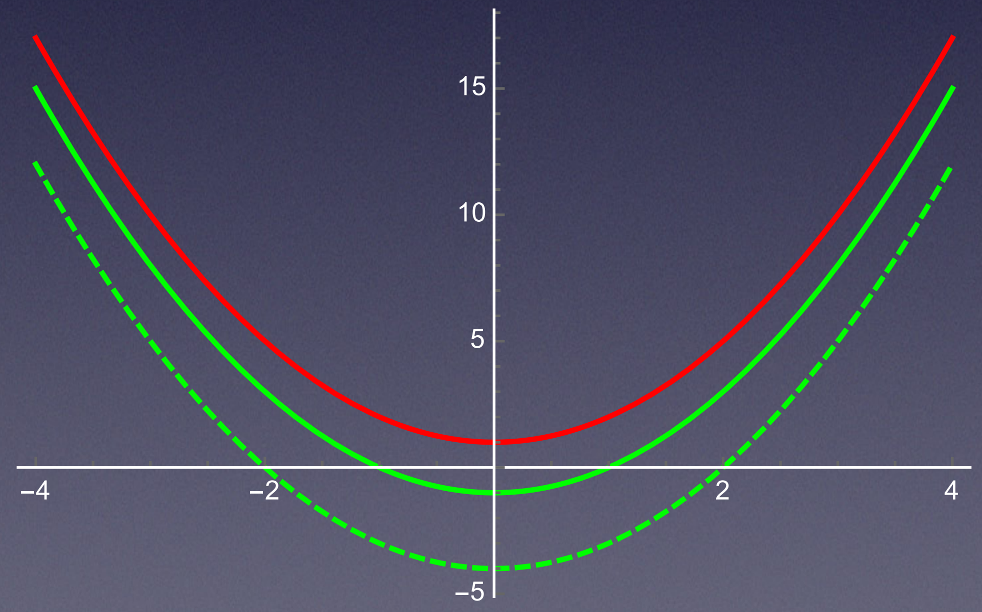
$$ax^2 + bx + c = 0$$

Bifurcation parameter: Discriminant

$$b^2 - 4ac < 0$$

0 Real Roots

Big enough change in system leads to qualitatively different solutions



Bifurcation in Algebra I

Quadratic equation

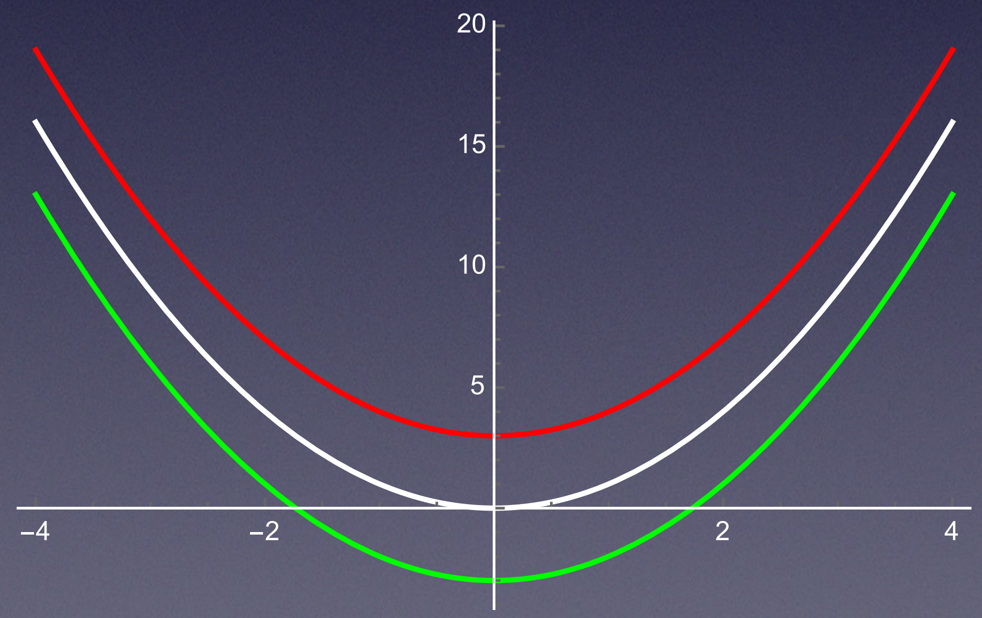
$$ax^2 + bx + c = 0$$

Bifurcation parameter: Discriminant

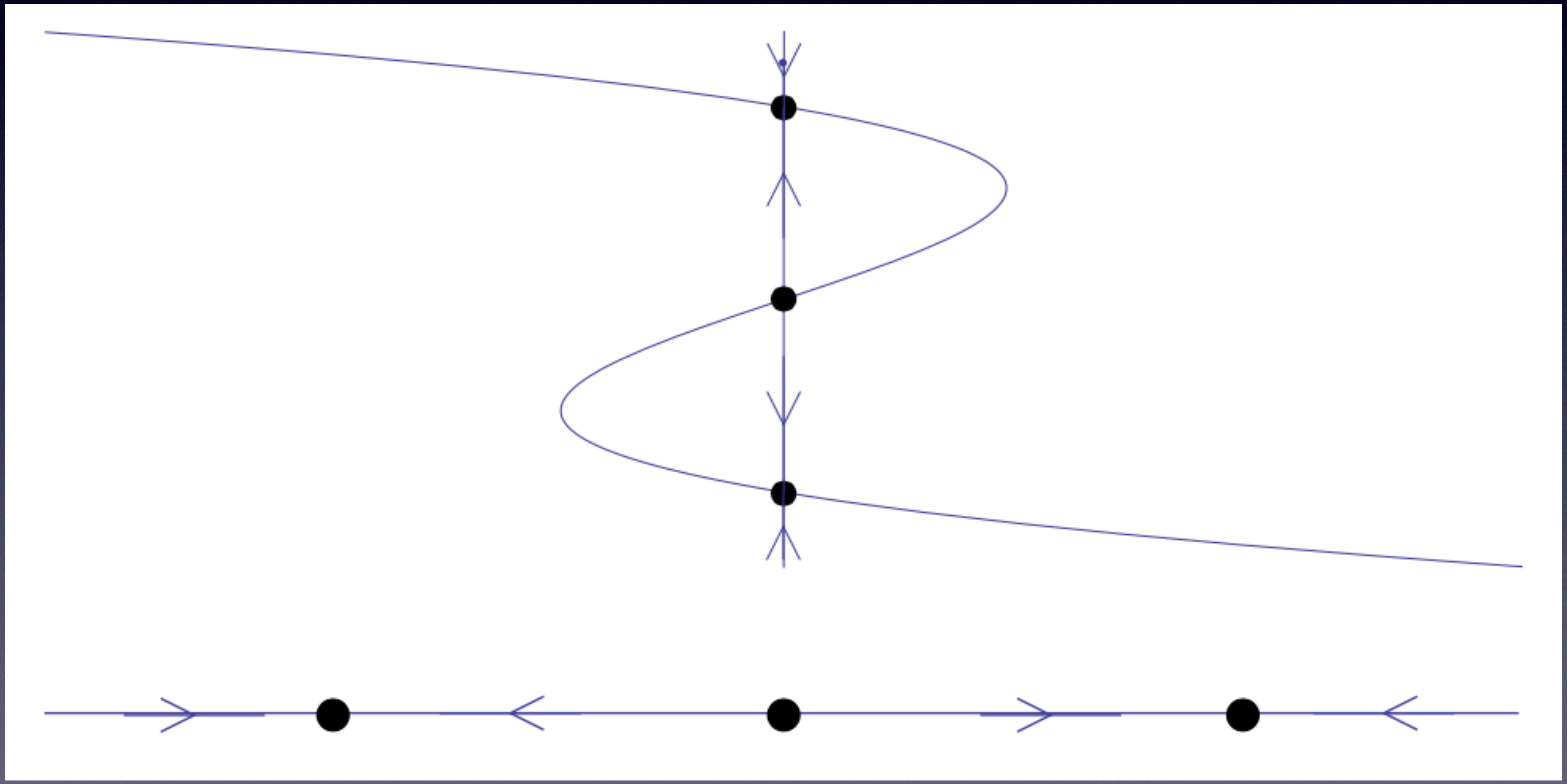
$$b^2 - 4ac = 0$$

1 Real Root

Bifurcation occurs when solutions collide



Bifurcations as Tipping Points



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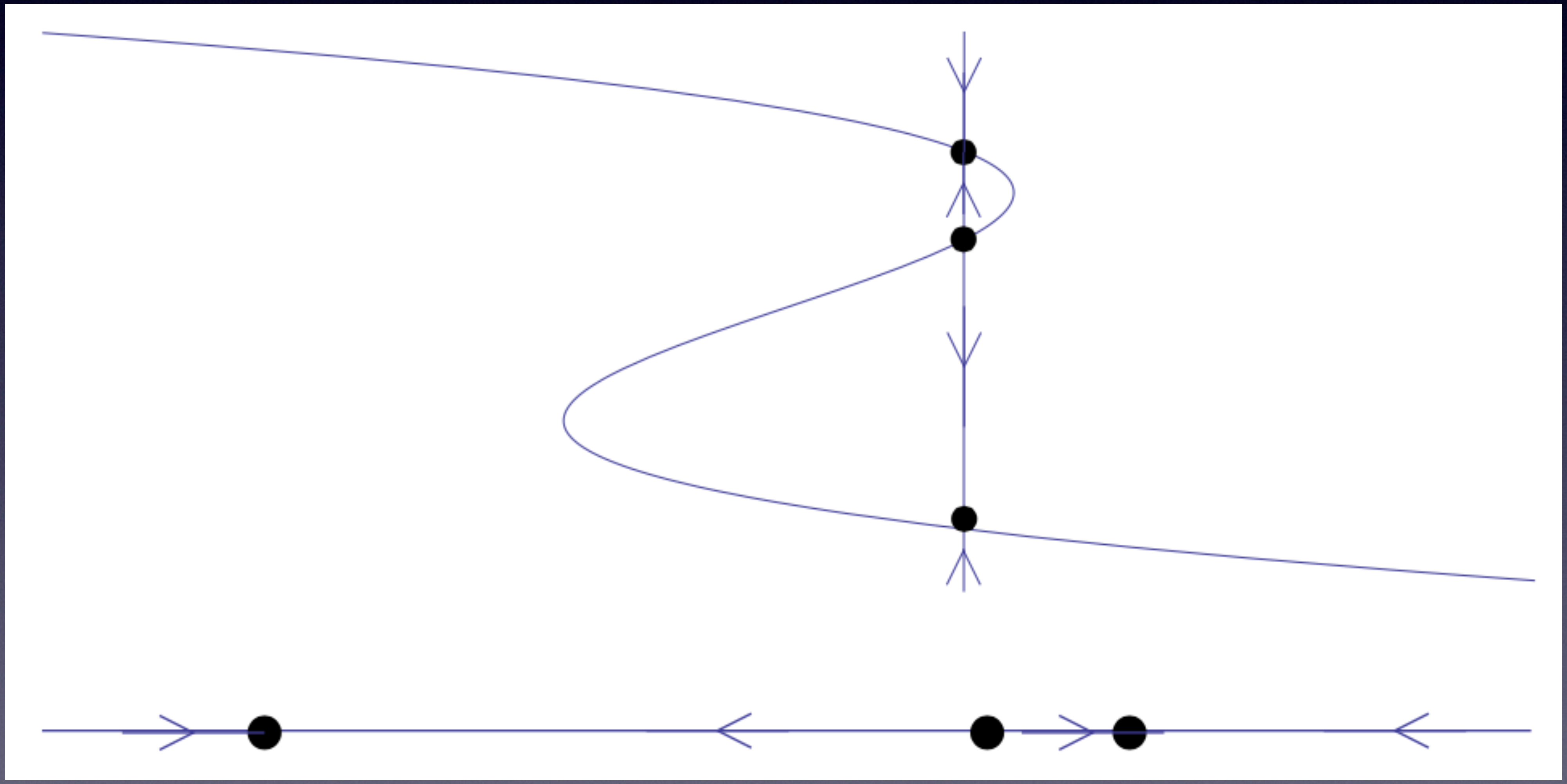
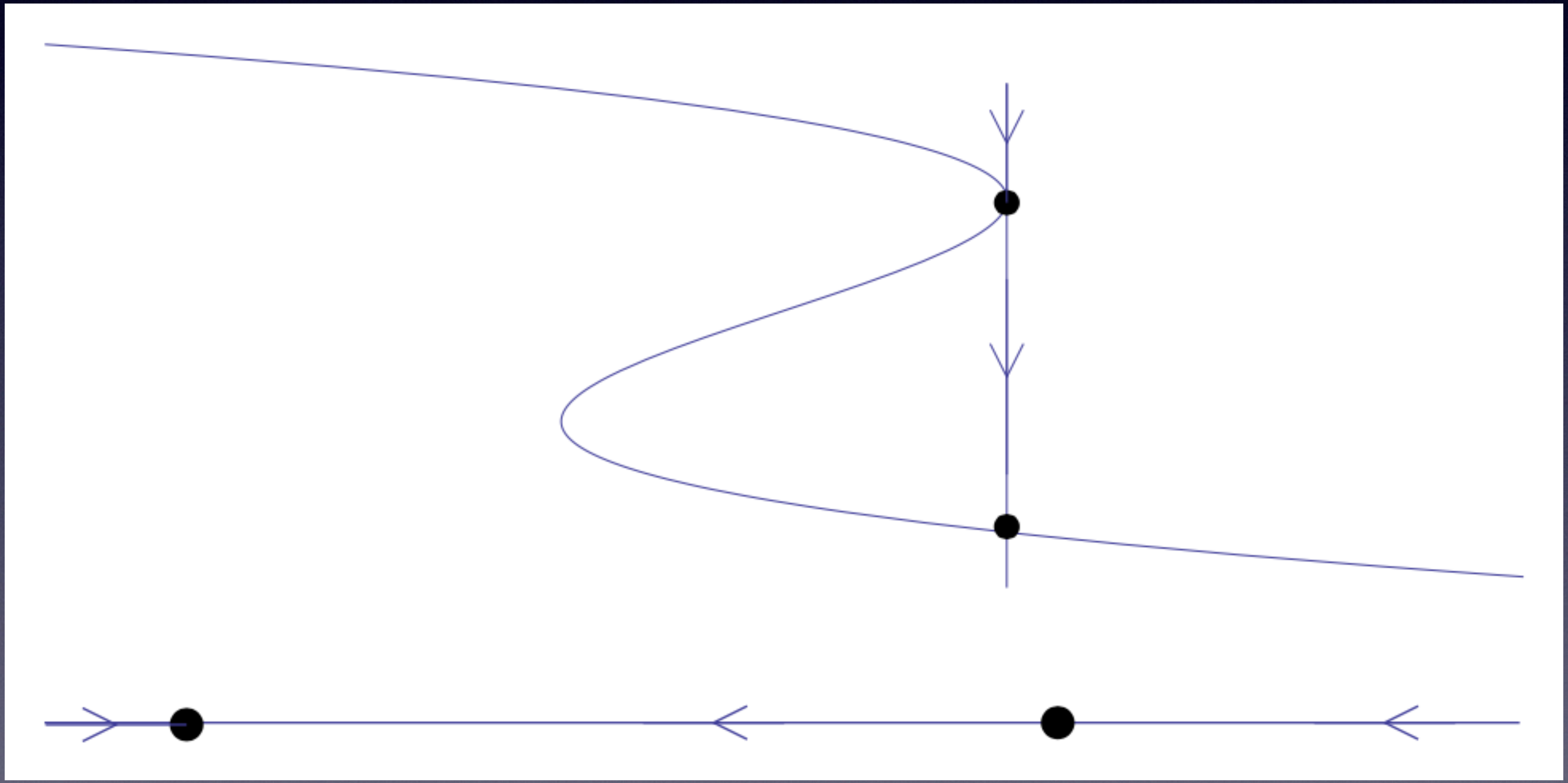
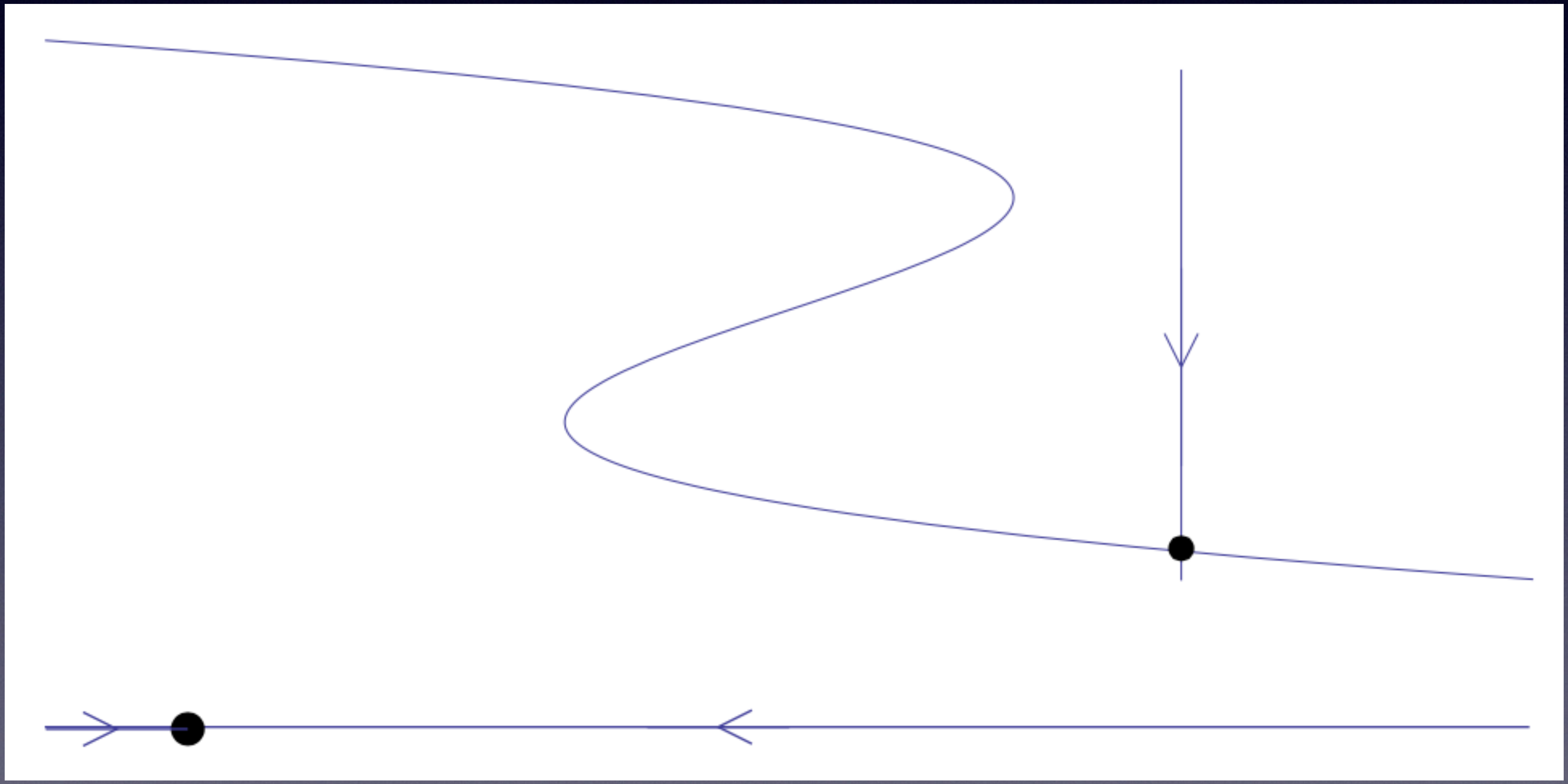


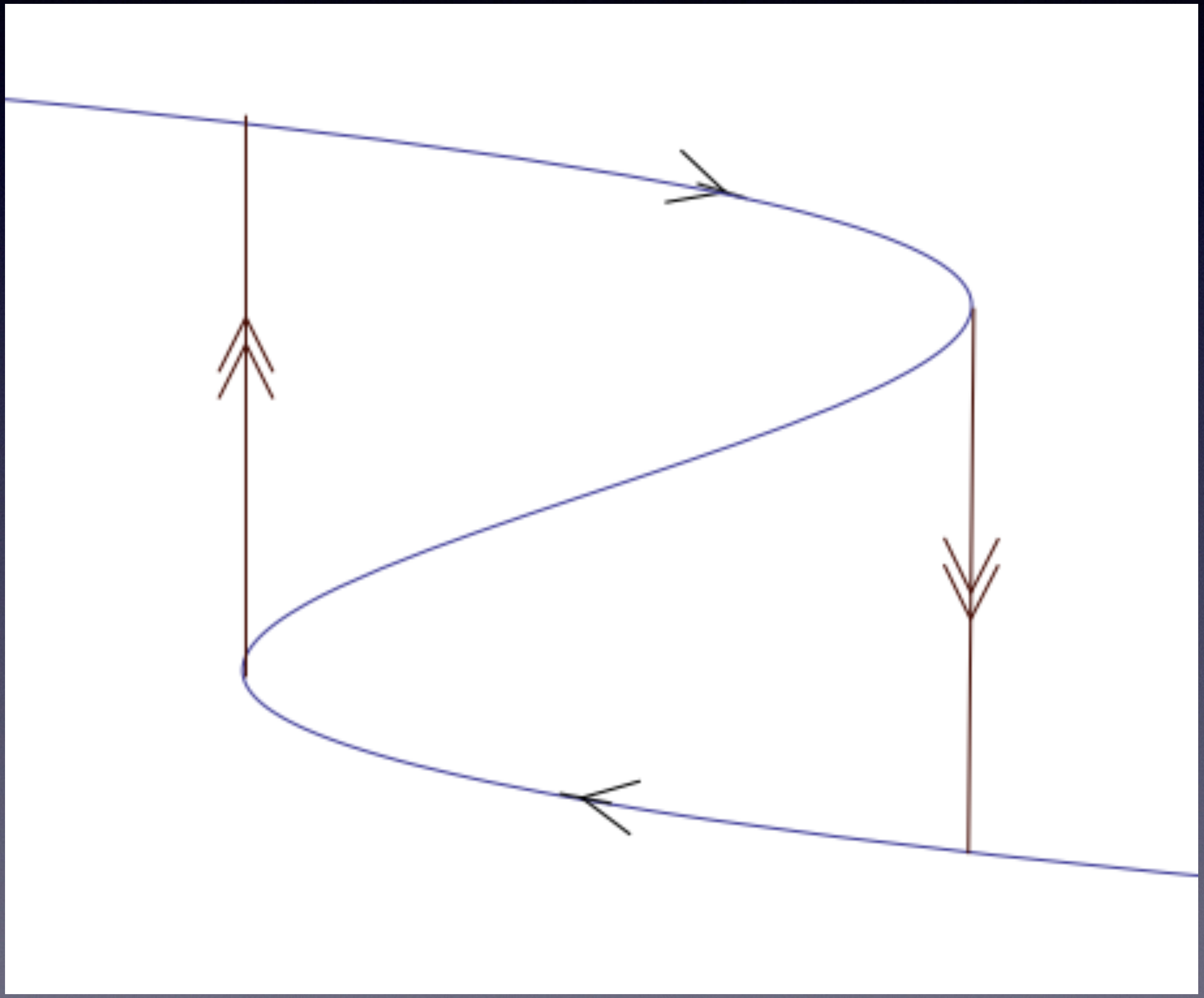
Bifurcations as Tipping Points



Bifurcations as Tipping Points



Hysteresis

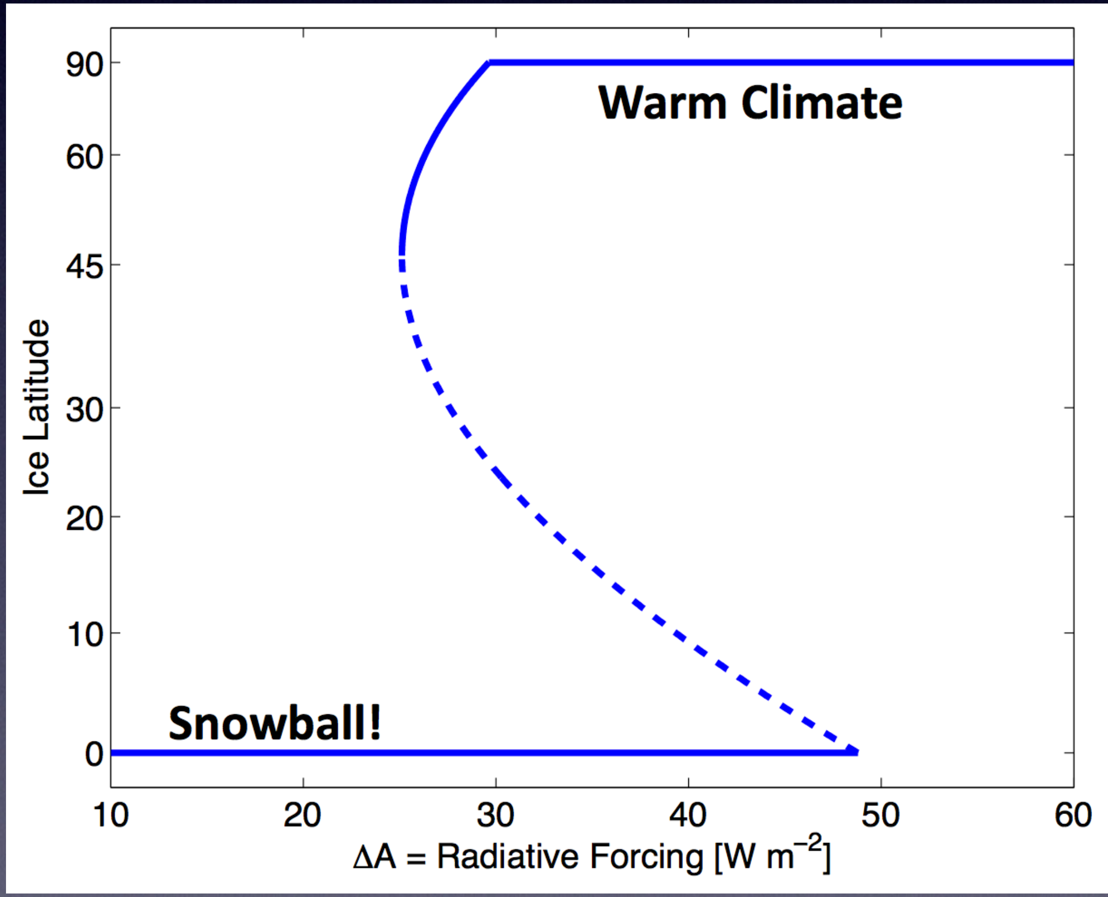


Bifurcation vs. Intrinsic Dynamics

- Idea of bifurcations assumes modeler has control over how parameters change — i.e., do NOT depend on state of system

- Snowball Earth: bifurcation “parameter” depends on GHGs (which in turn depend on temperature and ice)

- How does behavior change?



Fast/Slow Dynamics

- Fast variable is like state of system as before
- Slow variable is acts partly like parameter, partly like state variable
- Example of parameter: Milankovic cycles depend only on time (influence climate, but not influenced by climate)
- Examples of slow variable: GHGs, Ice coverage

$$\dot{x} = f(x; \lambda)$$

$$\lambda(t) = \tilde{g}(t)$$

$$\downarrow$$

$$\dot{x} = f(x, y)$$

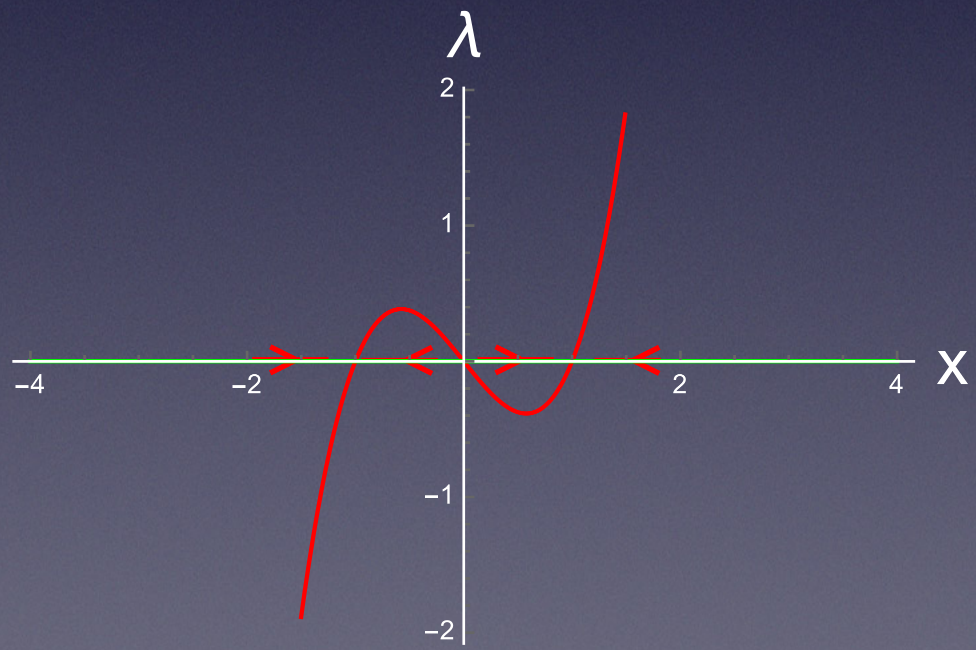
$$\dot{y} = \varepsilon g(x, y)$$

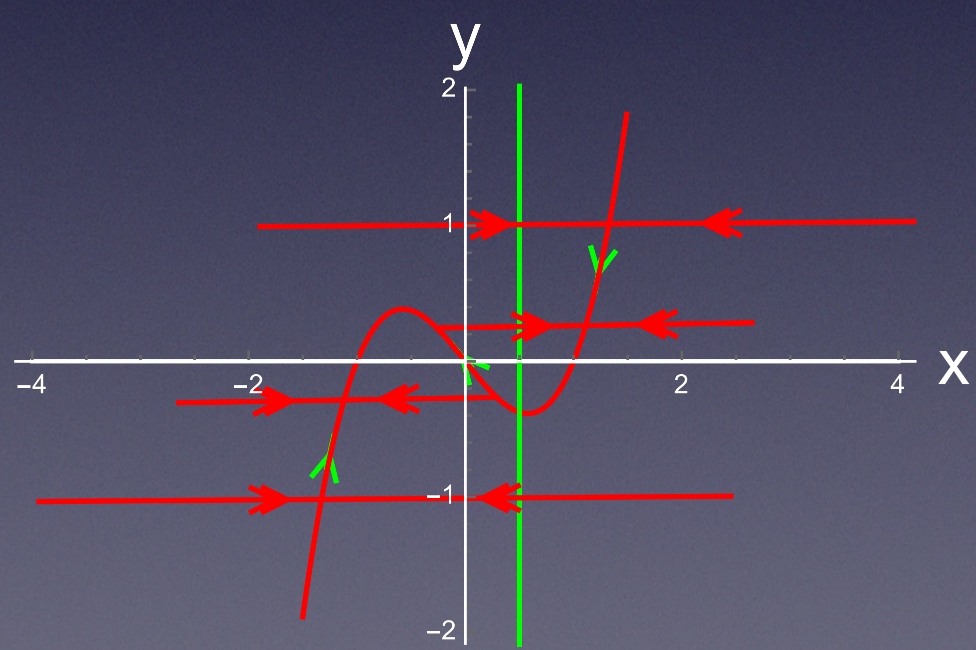
$$\varepsilon \ll 1$$

Picturing the difference

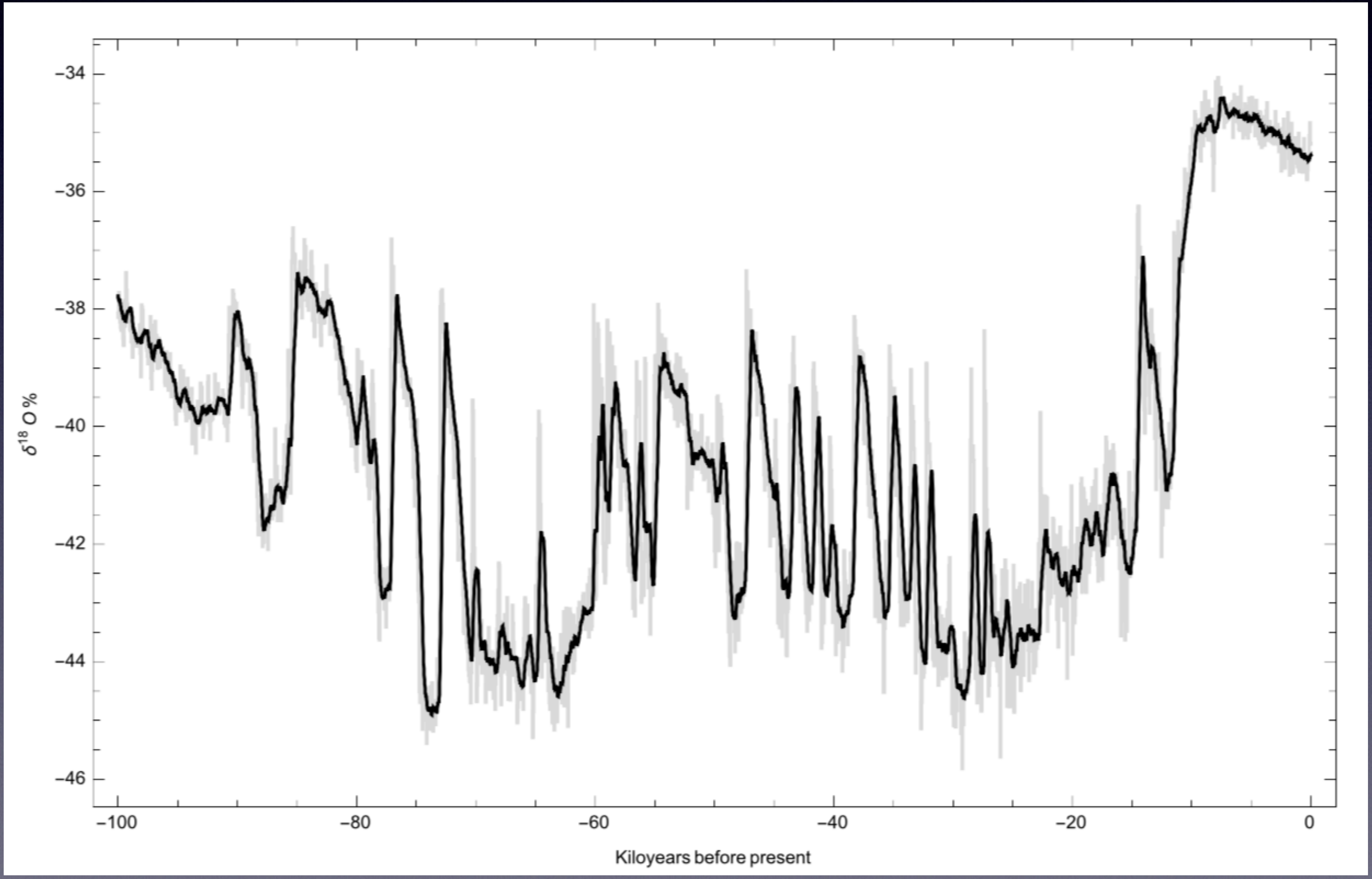
$$\dot{x} = f(x;\lambda)$$
$$\lambda(t) = \tilde{g}(t)$$

$$\dot{x} = f(x,y)$$
$$\dot{y} = \varepsilon g(x,y)$$
$$\varepsilon \ll 1$$





Example in Ocean Circulation



Stommel's Circulation Model

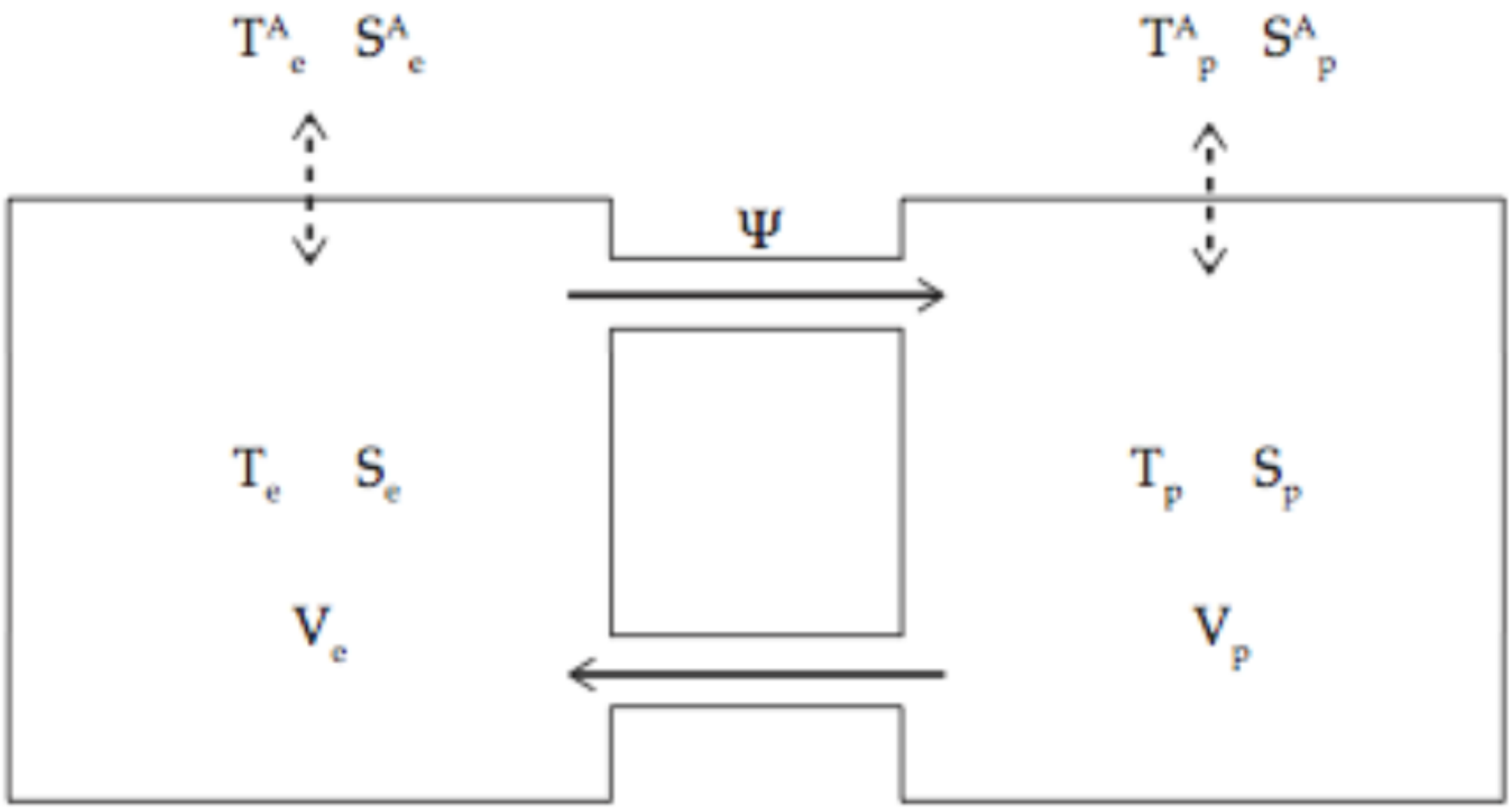


Figure: Schematic of Stommel's model (1961)—from Saha (2011).

Circulation variable: ψ

Stommel's Circulation Model

Model Reduces:

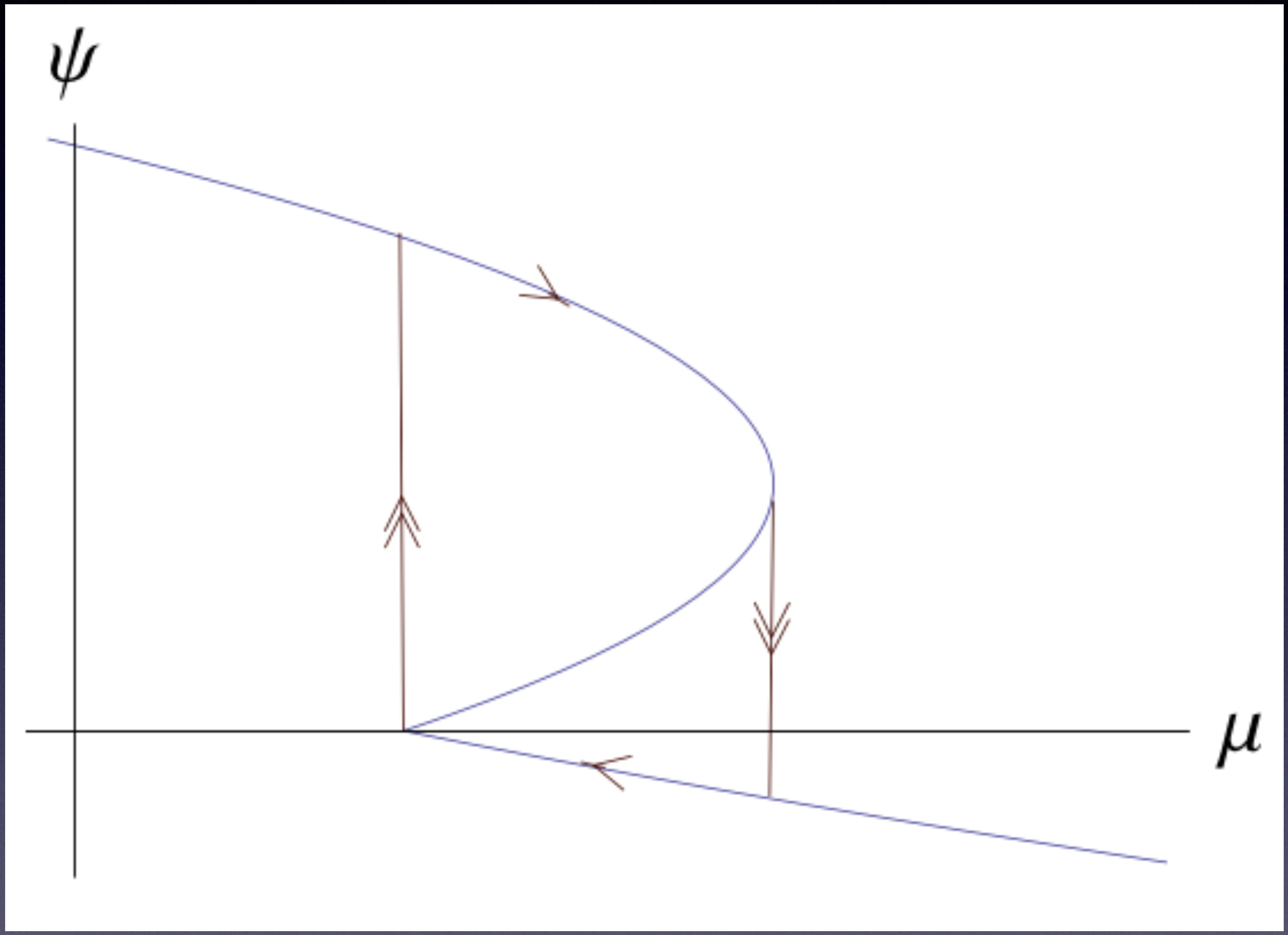
$x \sim T_e - Tp \to 1$

Get one state variable:

$y \sim S_e - S_p$

Bifurcation parameter:

$$\mu \sim \frac{\Delta S^A}{\Delta T^A}$$

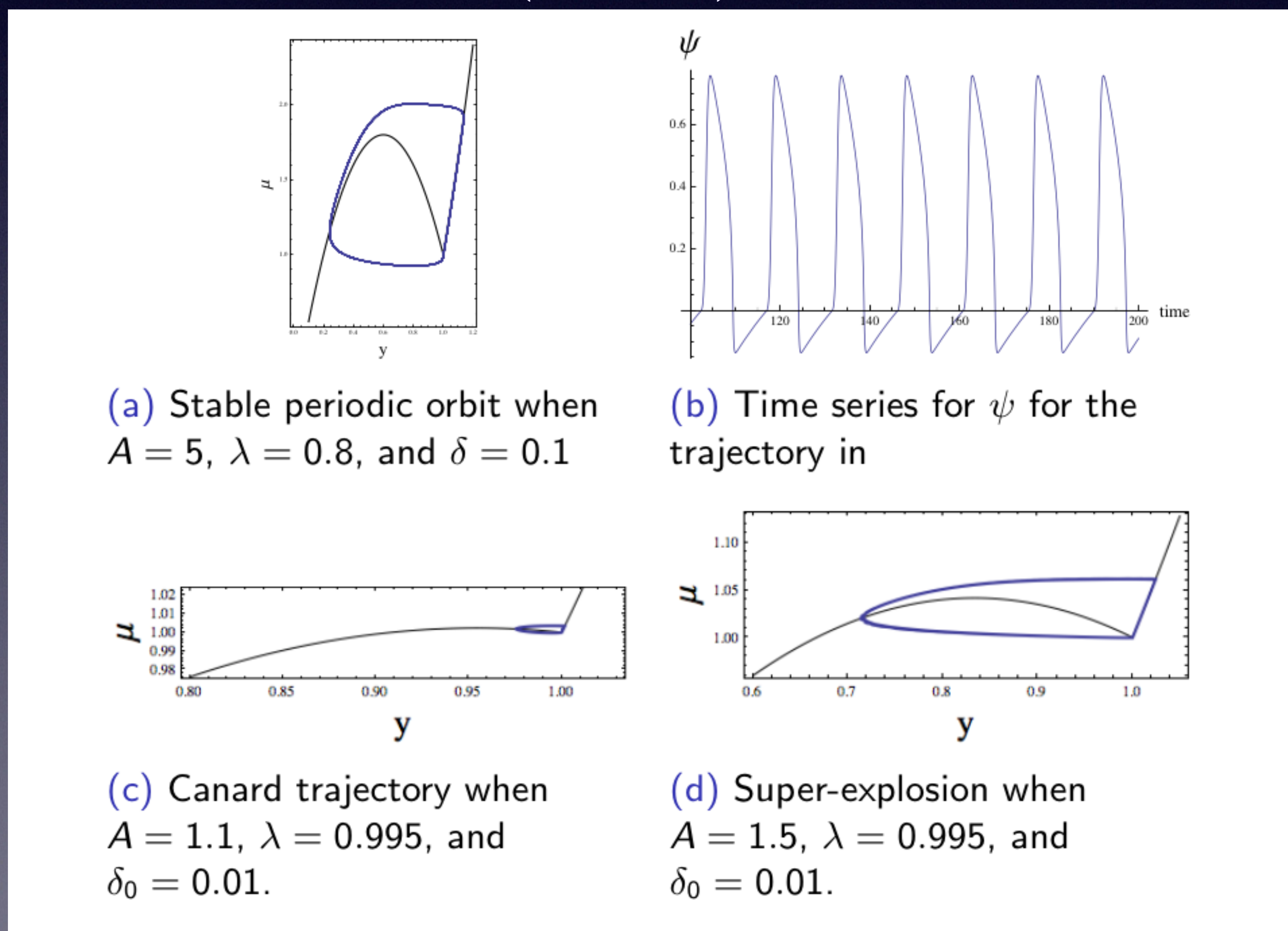


$$\dot{y} = \mu - y - A|1 - y|y$$

$\mu \rightarrow$ slow variable

$$\dot{y} = \mu - y - A|1-y|y$$

$$\dot{\mu} = \delta_0(\lambda - y)$$



(a) Stable periodic orbit when $A = 5$, $\lambda = 0.8$, and $\delta = 0.1$

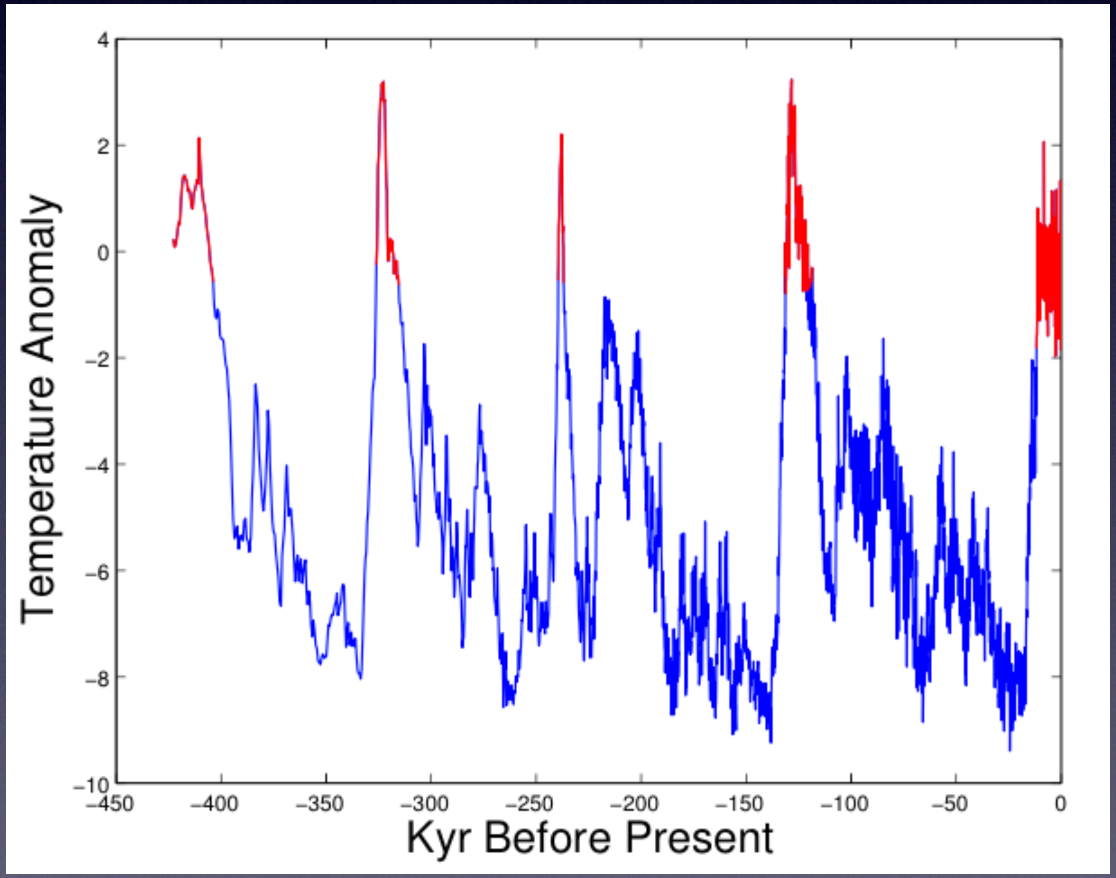
(b) Time series for ψ for the trajectory in

(c) Canard trajectory when $A = 1.1$, $\lambda = 0.995$, and $\delta_0 = 0.01$.

(d) Super-explosion when $A = 1.5$, $\lambda = 0.995$, and $\delta_0 = 0.01$.

Mixed-mode Oscillations

- 2D dynamical systems can have up to 3 end states:
  - Fixed equilibrium
  - Periodic equilibrium (with fixed amplitude and period)
  - Run-away behavior
- MMOs have big and small oscillations—need 3D system!
  - 3D dynamics much more complicated (chaos)



Ice Ages over the last 400 kyr

$$\dot{x} = y - x^3 + 3x - k$$

$$\dot{y} = \varepsilon[p(x-a)^2 - b - my - (\lambda + y - z)]$$

$$\dot{z} = \varepsilon r(\lambda + y - z)$$

Ice Ages over the last 400 kyr

$$\dot{x} = y - x^3 + 3x - k$$
$$\dot{y} = \varepsilon[p(x-a)^2 - b - my - (\lambda + y - z)]$$
$$\dot{z} = \varepsilon r(\lambda + y - z)$$

x ~ice volume

z ~ oceanic carbon

y ~ atmospheric carbon

Ice Ages over the last 400 kyr

$$\dot{x} = y - x^3 + 3x - k$$

$$\dot{y} = \varepsilon[p(x-a)^2 - b - my - (\lambda + y - z)]$$

$$\dot{z} = \varepsilon r(\lambda + y - z)$$

Fast

Slow

Ice Ages over the last 400 kyr

$$\dot{x} = y - x^3 + 3x - k$$
$$\dot{y} = \varepsilon[p(x-a)^2 - b - my - (\lambda + y - z)]$$
$$\dot{z} = \varepsilon r(\lambda + y - z)$$

Change in ice volume depends on temperature, but temperature depends on the amount of ice and how much GHGs are in the atmosphere

Ice Ages over the last 400 kyr

$$\dot{x} = y - x^3 + 3x - k$$
$$\dot{y} = \varepsilon[p(x-a)^2 - b - my - (\lambda + y - z)]$$
$$\dot{z} = \varepsilon r(\lambda + y - z)$$

Land-atmosphere carbon flux

Ice Ages over the last 400 kyr

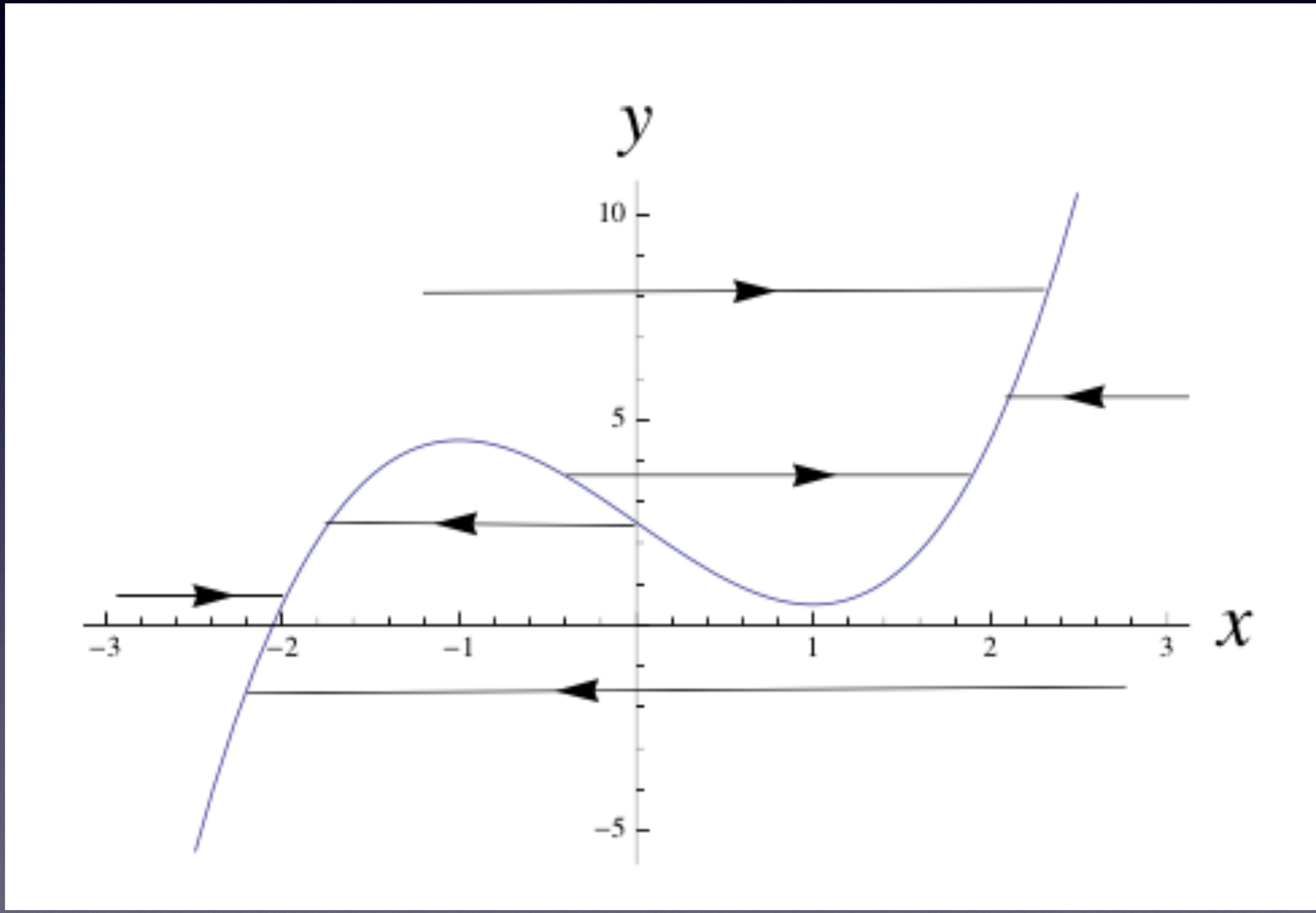
$$\dot{x} = y - x^3 + 3x - k$$

$$\dot{y} = \varepsilon[p(x-a)^2 - b - my - (\lambda + y - z)]$$

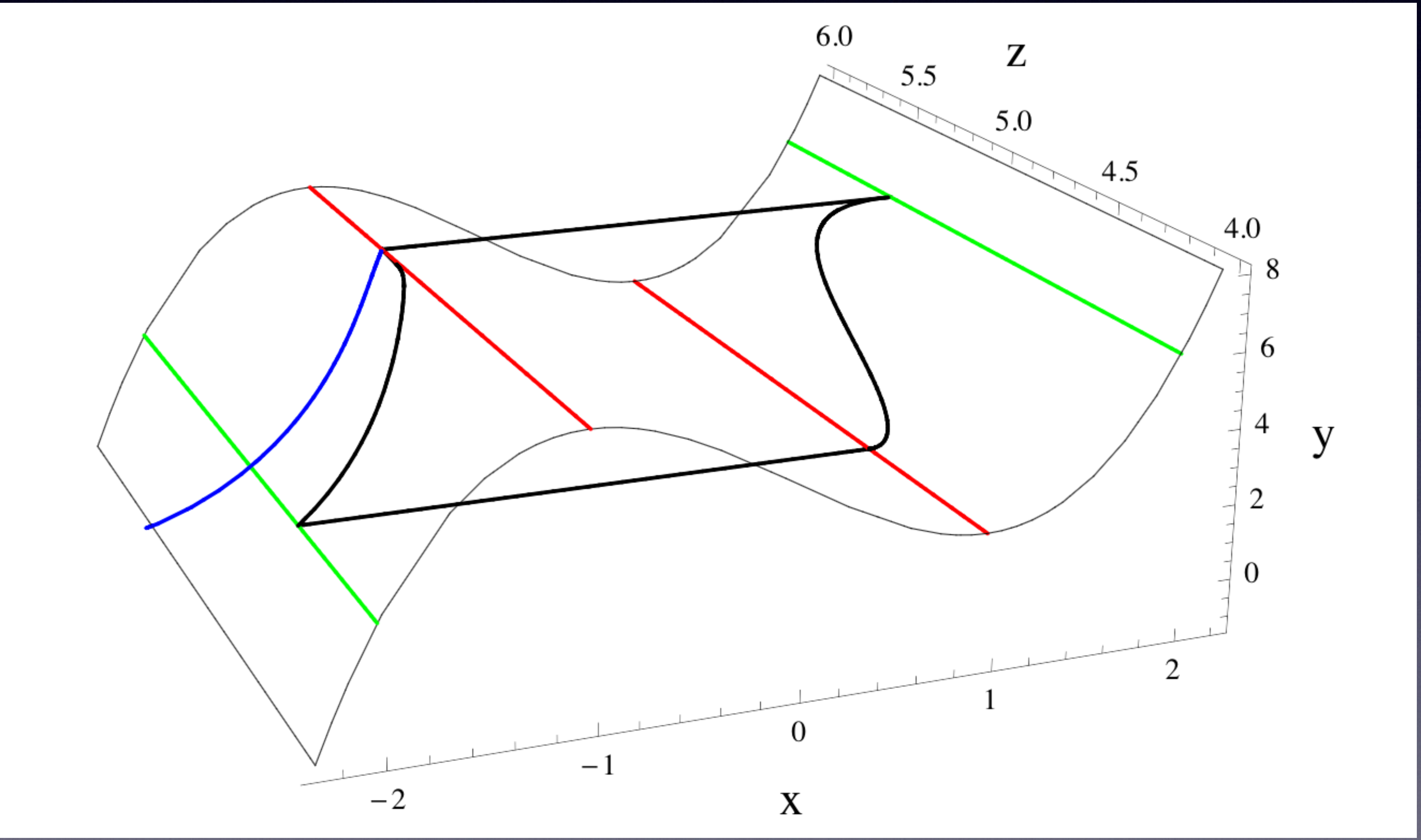
$$\dot{z} = \varepsilon r(\lambda + y - z)$$

Ocean-atmosphere carbon flux

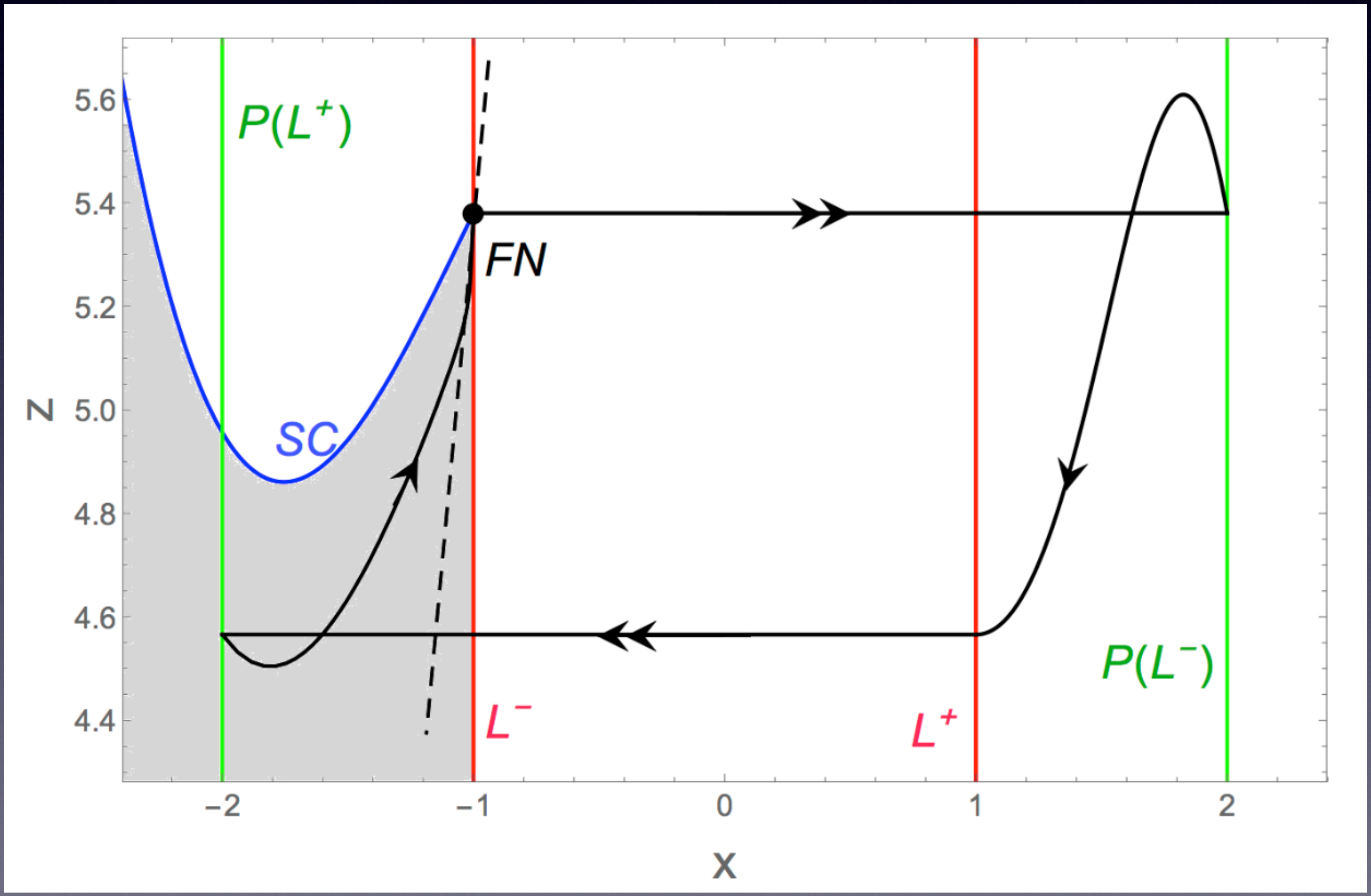
Ice Ages over the last 400 kyr



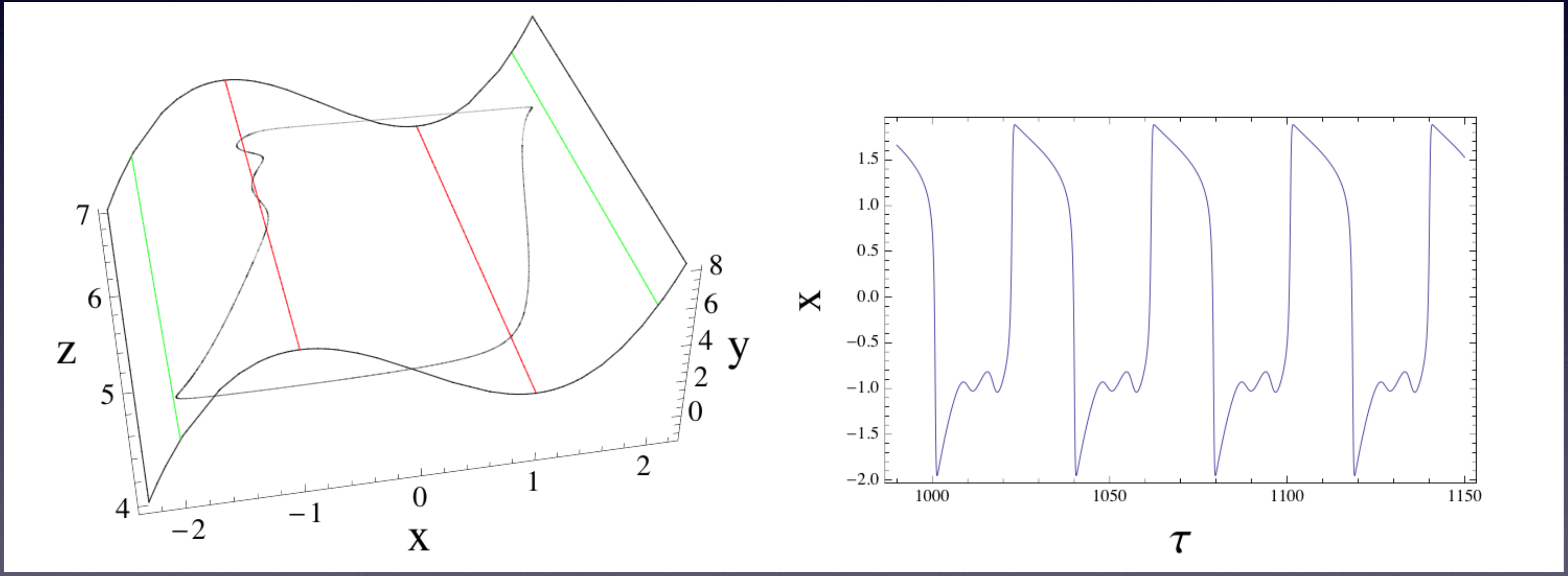
Ice Ages over the last 400 kyr



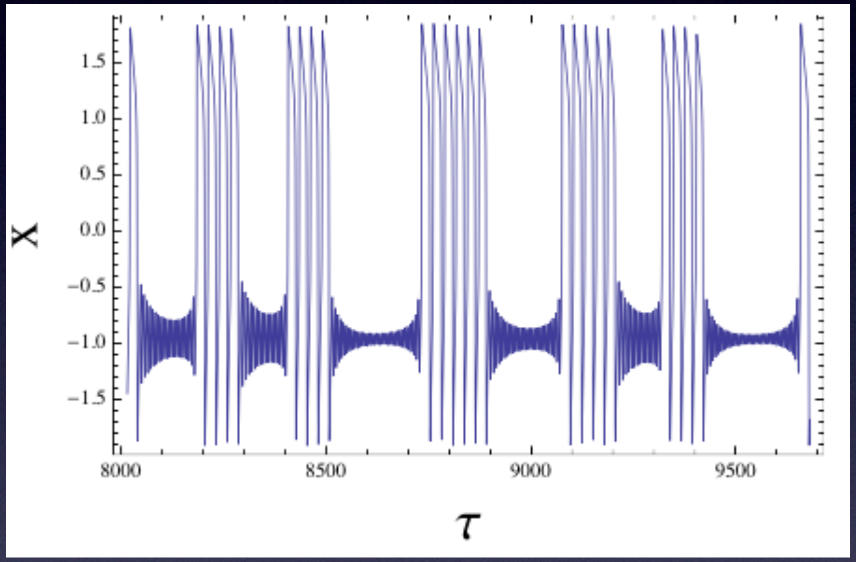
Ice Ages over the last 400 kyr

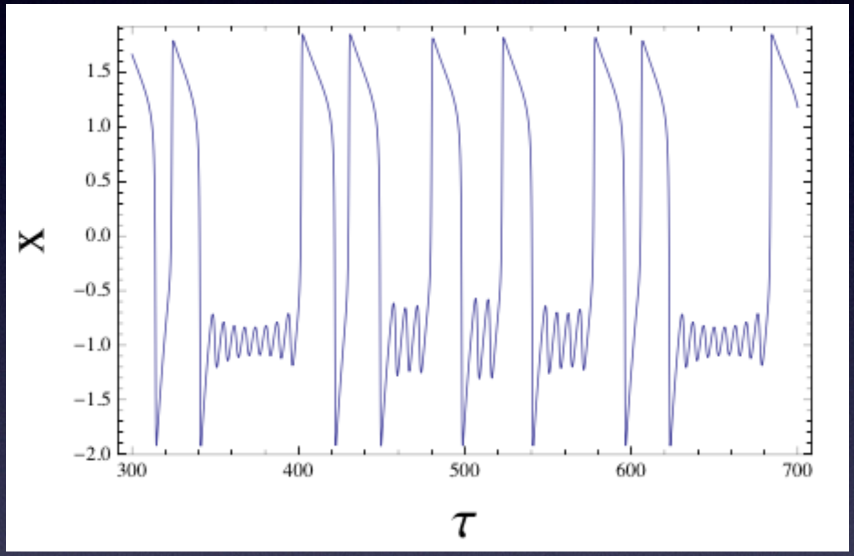


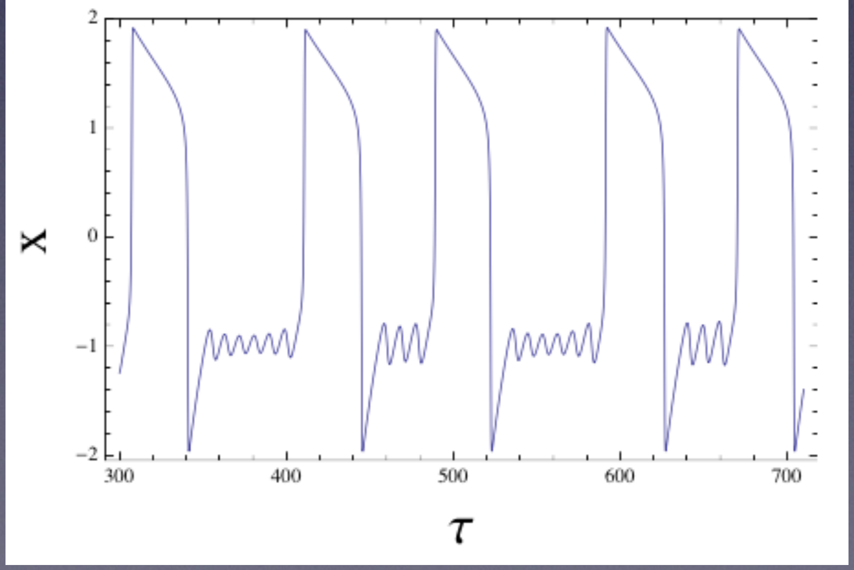
Ice Ages over the last 400 kyr



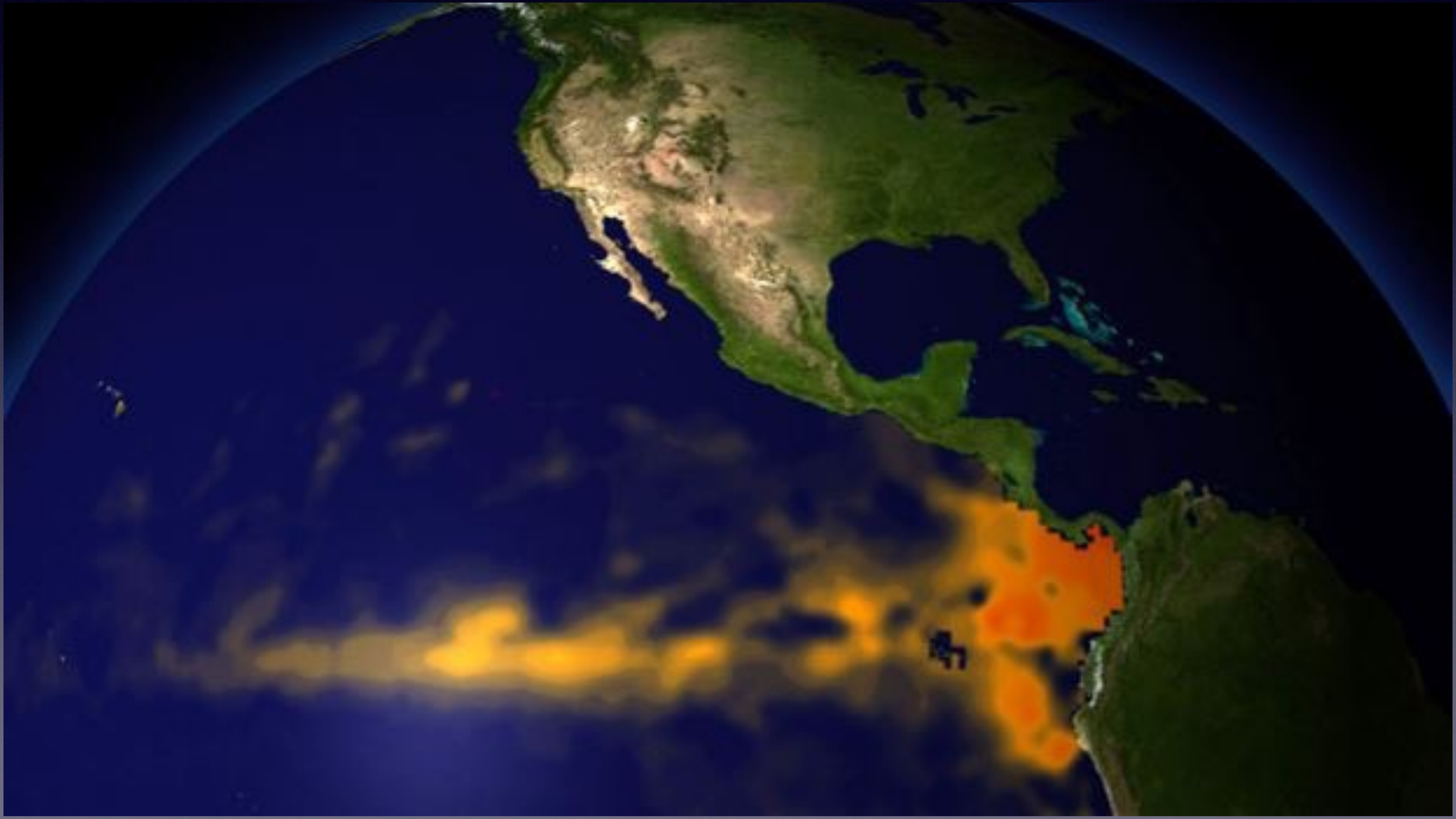
Ice Ages over the last 400 kyr



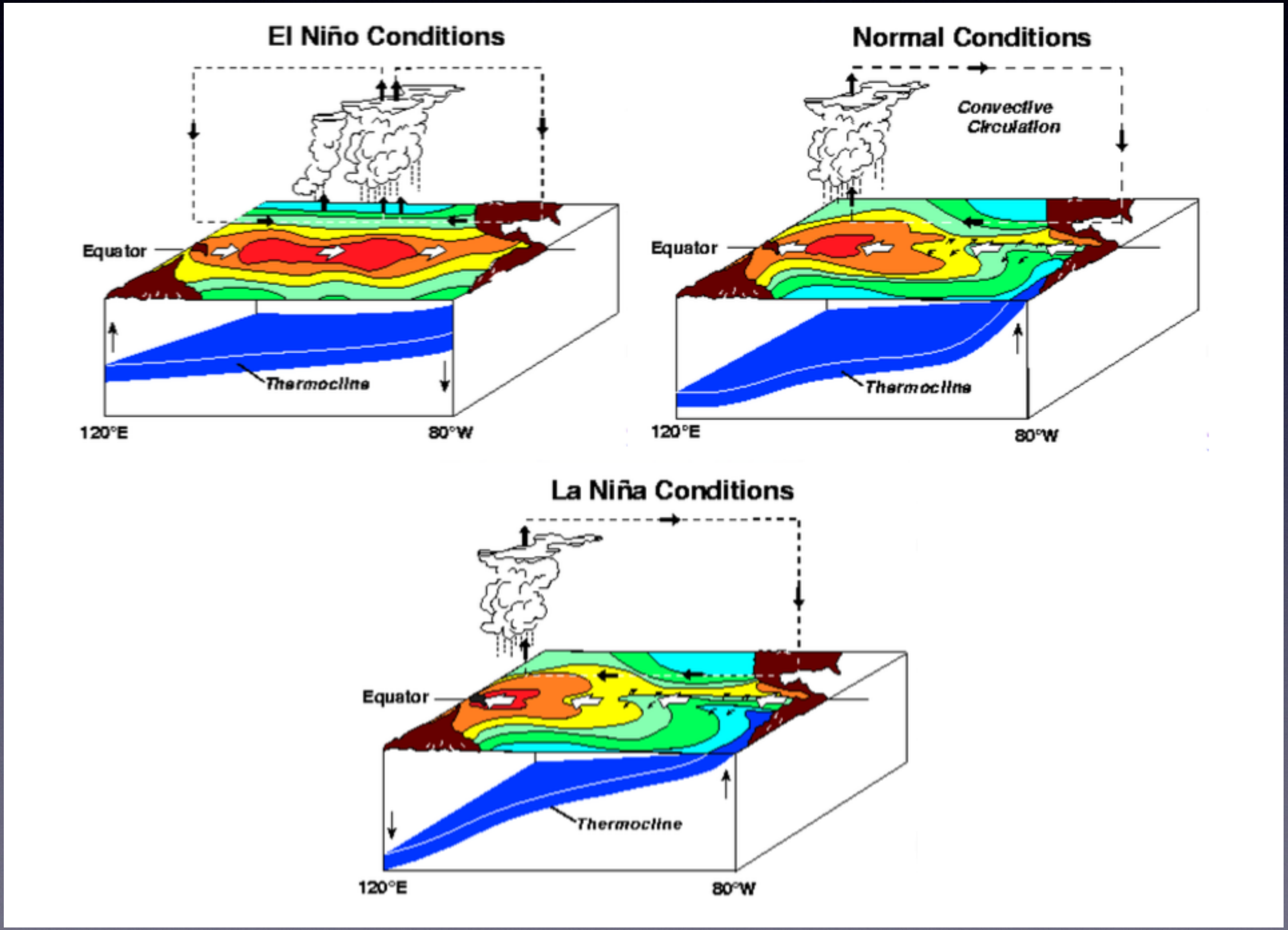




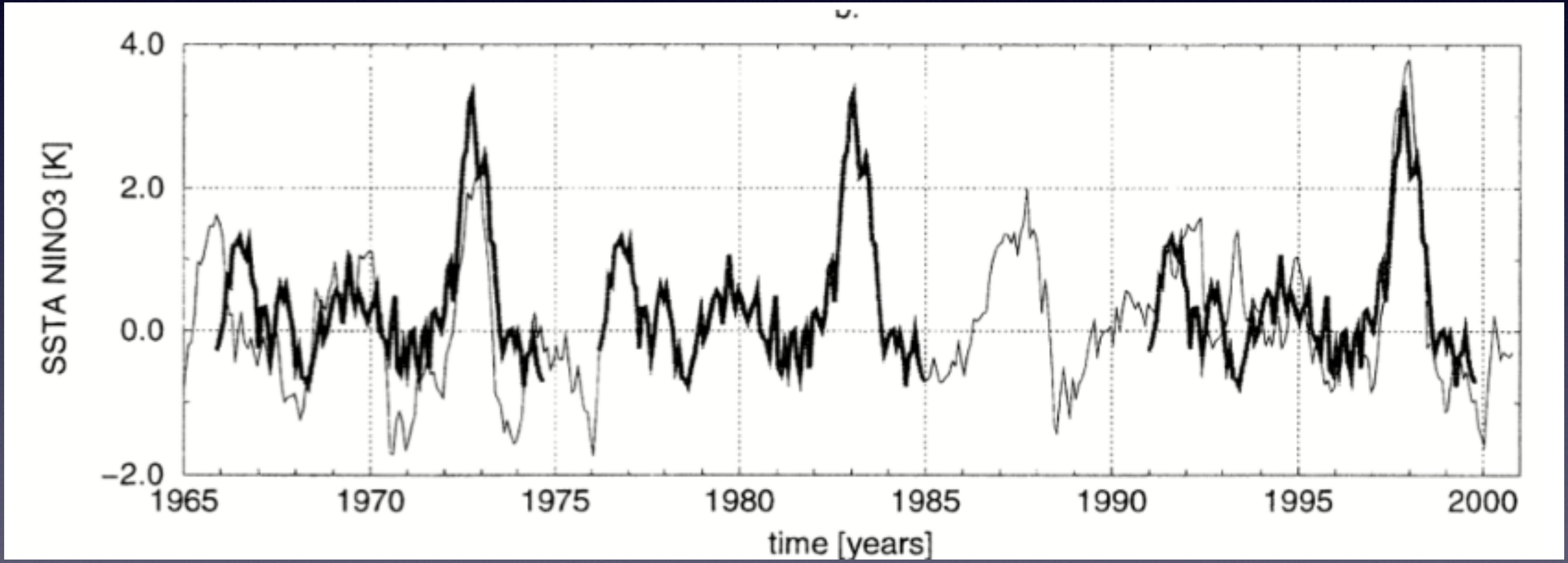
El Niño-Southern Oscillation



How does ENSO work?



The Data


4.0
2.0
0.0
−2.0
SSTA NINO3 [K]
1965
1970
1975
1980
1985
1990
1995
2000
time [years]

Predicting ENSO

August predictions

Talk of an El Niño year cools, but don’t despair yet about winter

While a 'super' El Niño looks to be off the table, what does develop this year might not deliver what many Canadians are hoping for

Don’t dismiss a 2014 ‘super’ El Niño just yet

Predicting ENSO

- August Prediction
  - ?
- September prediction
  - Probability of ENSO: low
- October Prediction
  - Probability of ENSO: 0.68
- November Prediction
  - 58% chance of ENSO
  - Normal to weak ENSO

ENSO

$$\dot{x} = \varepsilon(x^2 - ax) + x\left[x + y - nz + d - c\left(x - \frac{x^3}{3}\right)\right]$$

$$\dot{y} = -\varepsilon(ay + x^2)$$

$$\dot{z} = m\left(k - z - \frac{x}{2}\right)$$

ENSO

$$\dot{x} = \varepsilon(x^2 - ax) + x\left[x + y - nz + d - c\left(x - \frac{x^3}{3}\right)\right]$$

$$\dot{y} = -\varepsilon(ay + x^2)$$

$$\dot{z} = m\left(k - z - \frac{x}{2}\right)$$

x temperature gradient

y temp of W Pacific

z thermocline dept in W Pacific

ENSO

$$\dot{x} = \varepsilon(x^2 - ax) + x\left[x + y - nz + d - c\left(x - \frac{x^3}{3}\right)\right]$$

$$\dot{y} = -\varepsilon(ay + x^2)$$

$$\dot{z} = m\left(k - z - \frac{x}{2}\right)$$

Upwelling feedback

ENSO

$$\dot{x} = \varepsilon(x^2 - ax) + x\left[x + y - nz + d - c\left(x - \frac{x^3}{3}\right)\right]$$

$$\dot{y} = -\varepsilon(ay + x^2)$$

$$\dot{z} = m\left(k - z - \frac{x}{2}\right)$$

Thermocline adjustment

ENSO

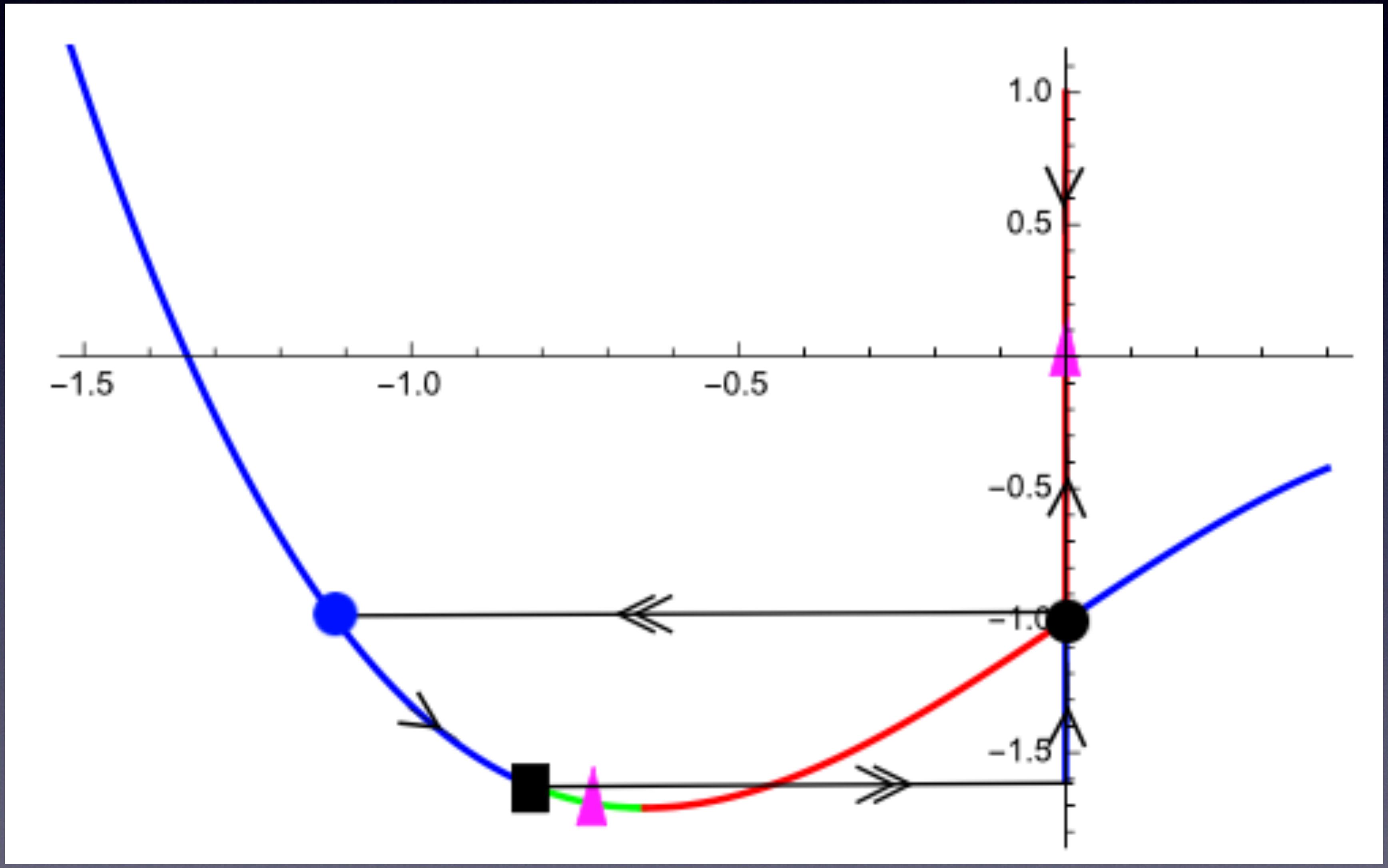
$$\dot{x} = \varepsilon(x^2 - ax) + x\left[x + y - nz + d - c\left(x - \frac{x^3}{3}\right)\right]$$

$$\dot{y} = -\varepsilon(ay + x^2)$$

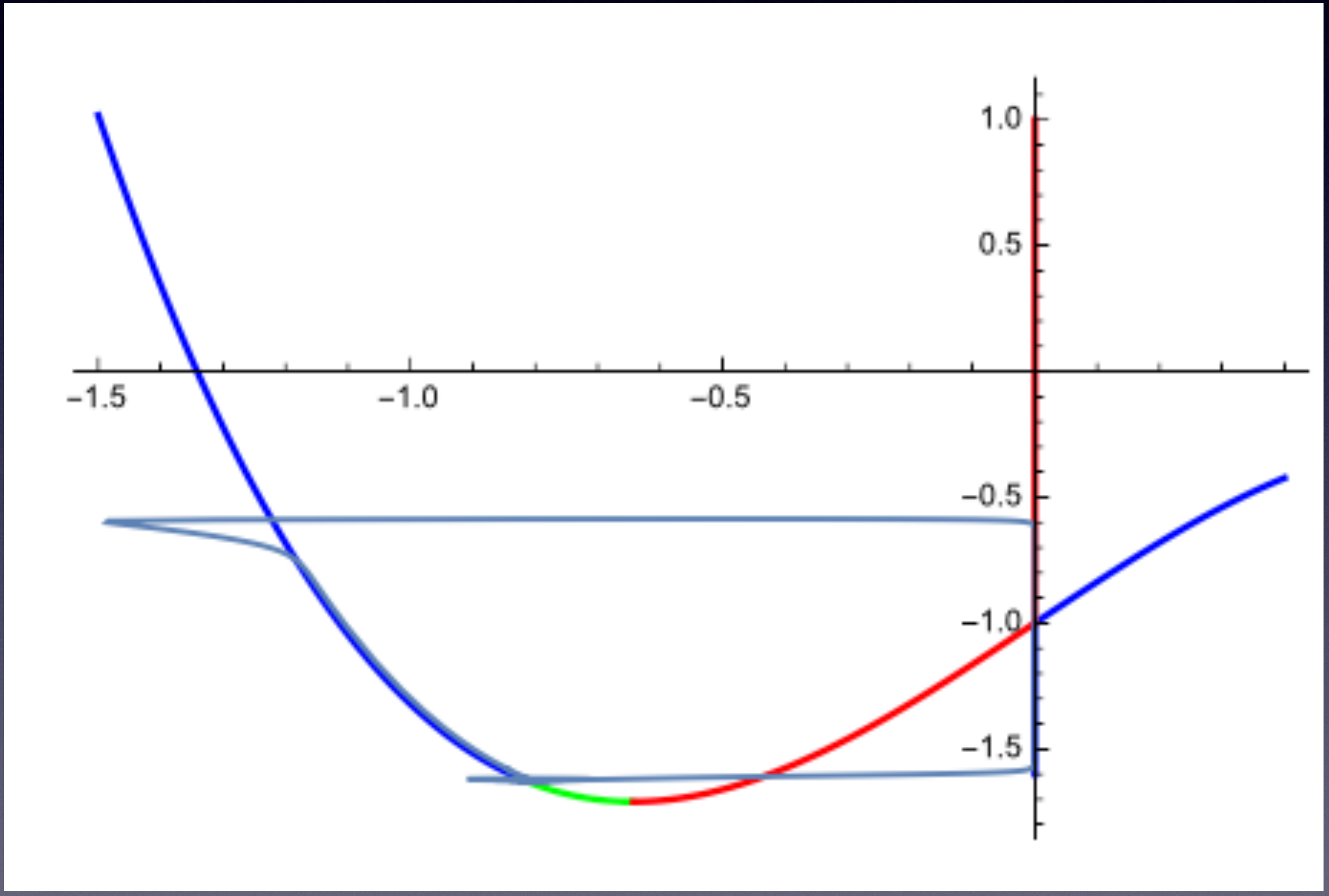
$$\dot{z} = m\left(k - z - \frac{x}{2}\right)$$

Advection

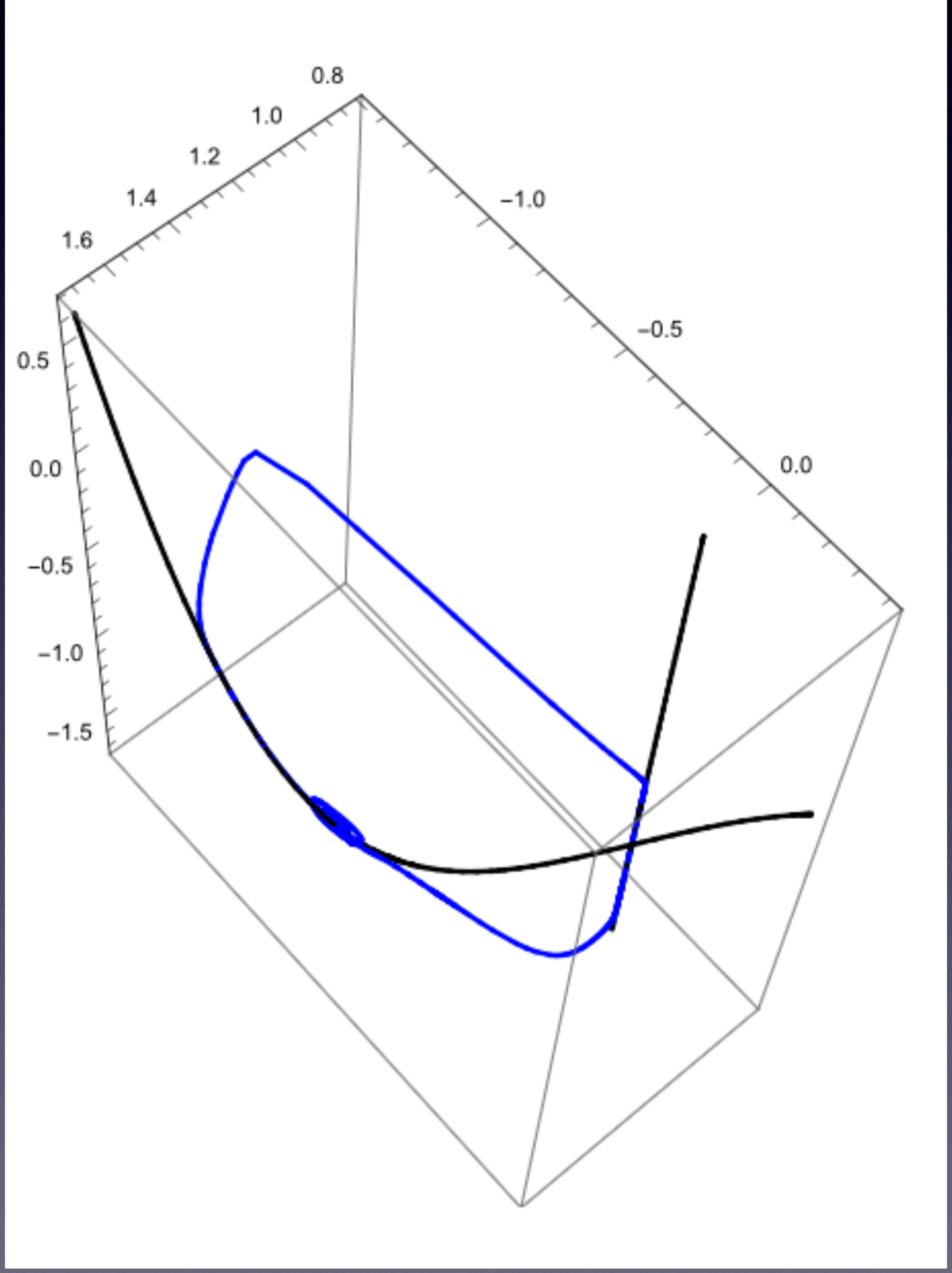
ENSO



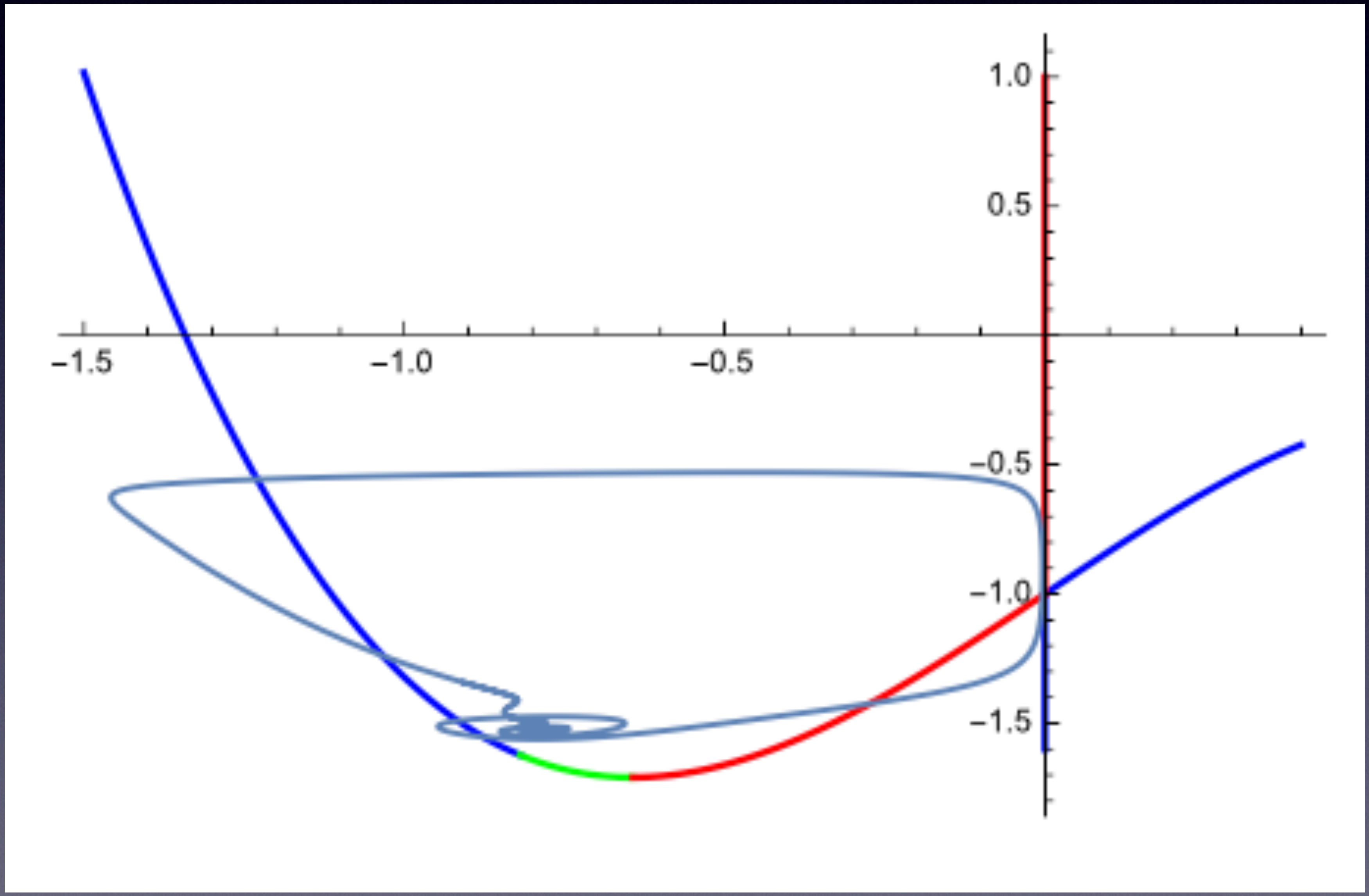
Simulation 1



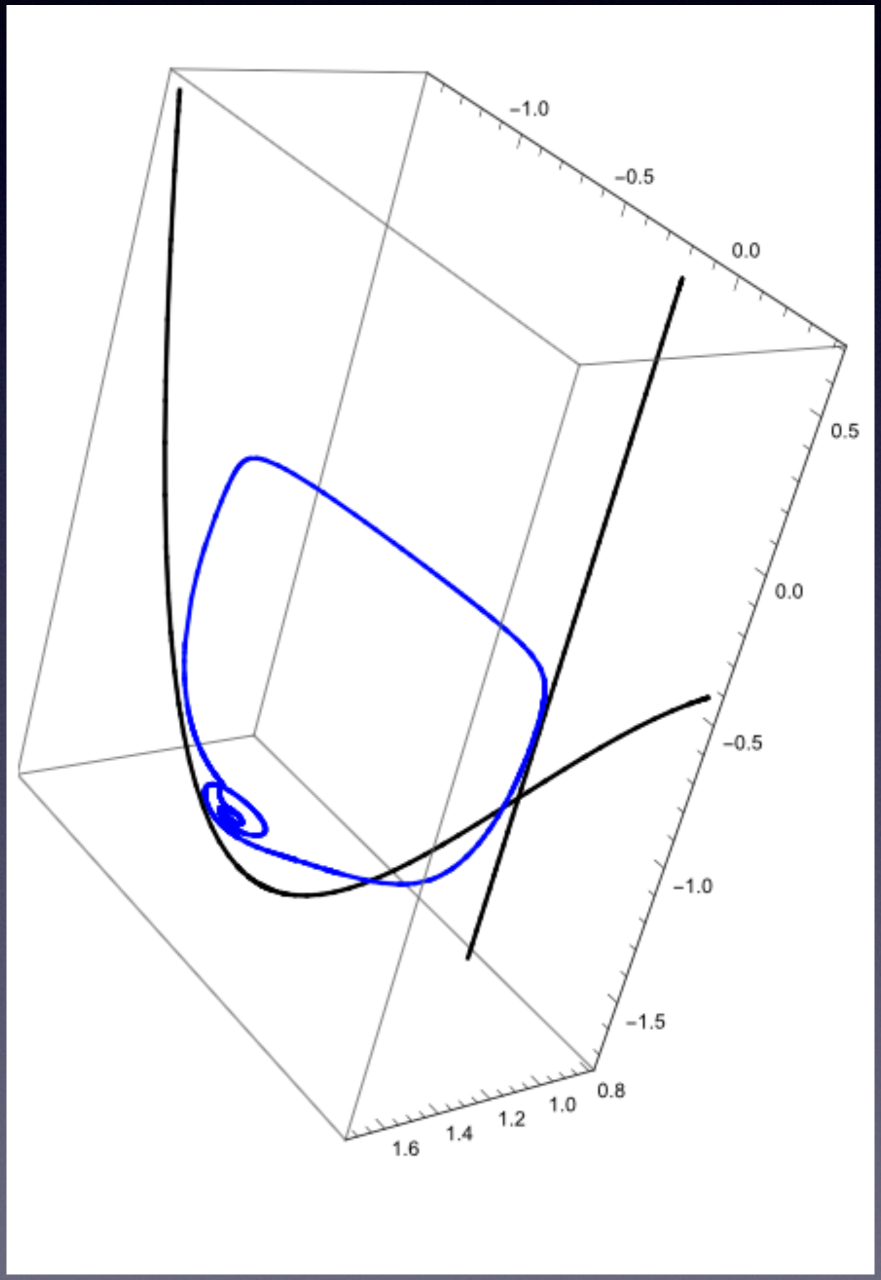
Simulation 1



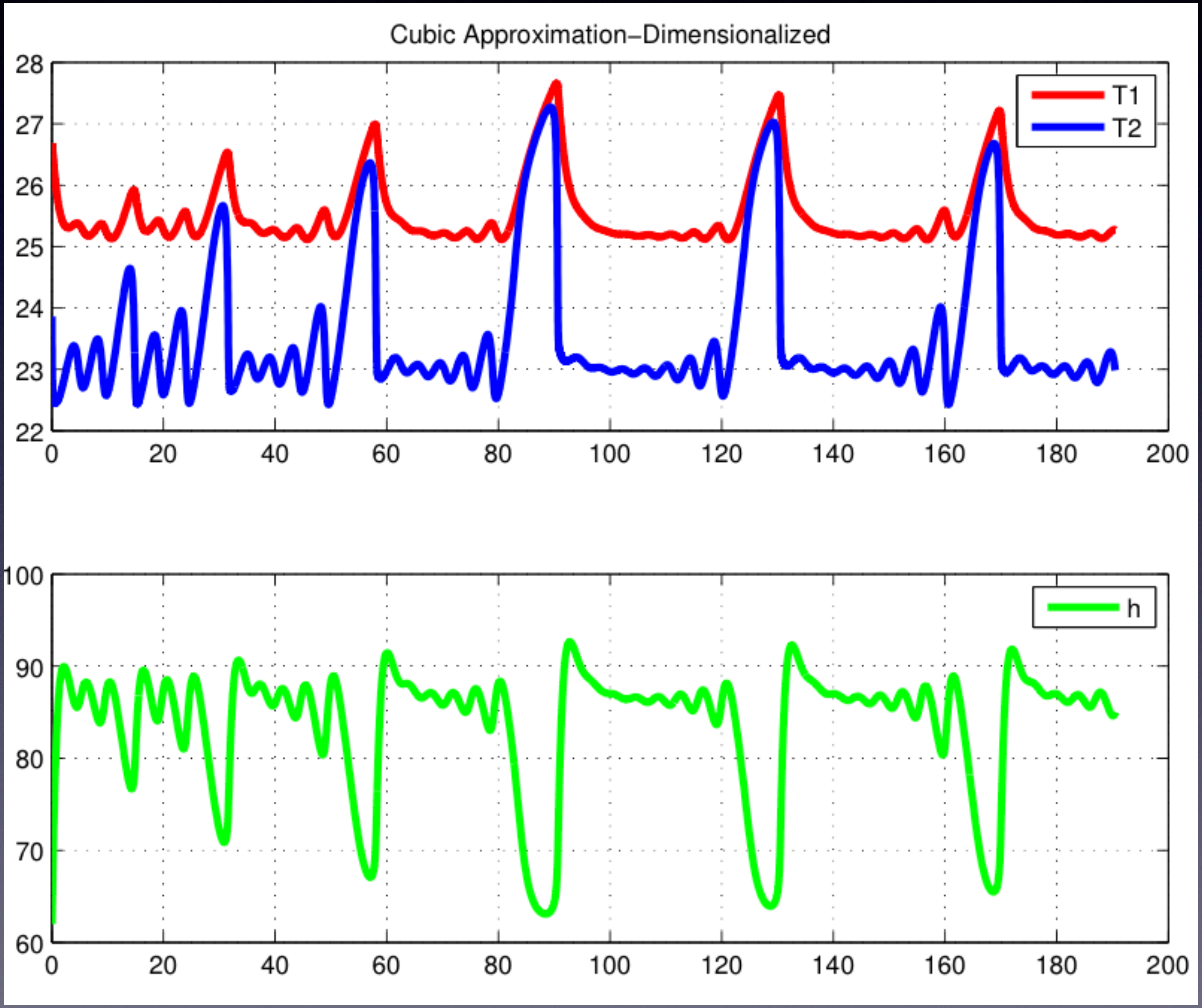
Simulation 2



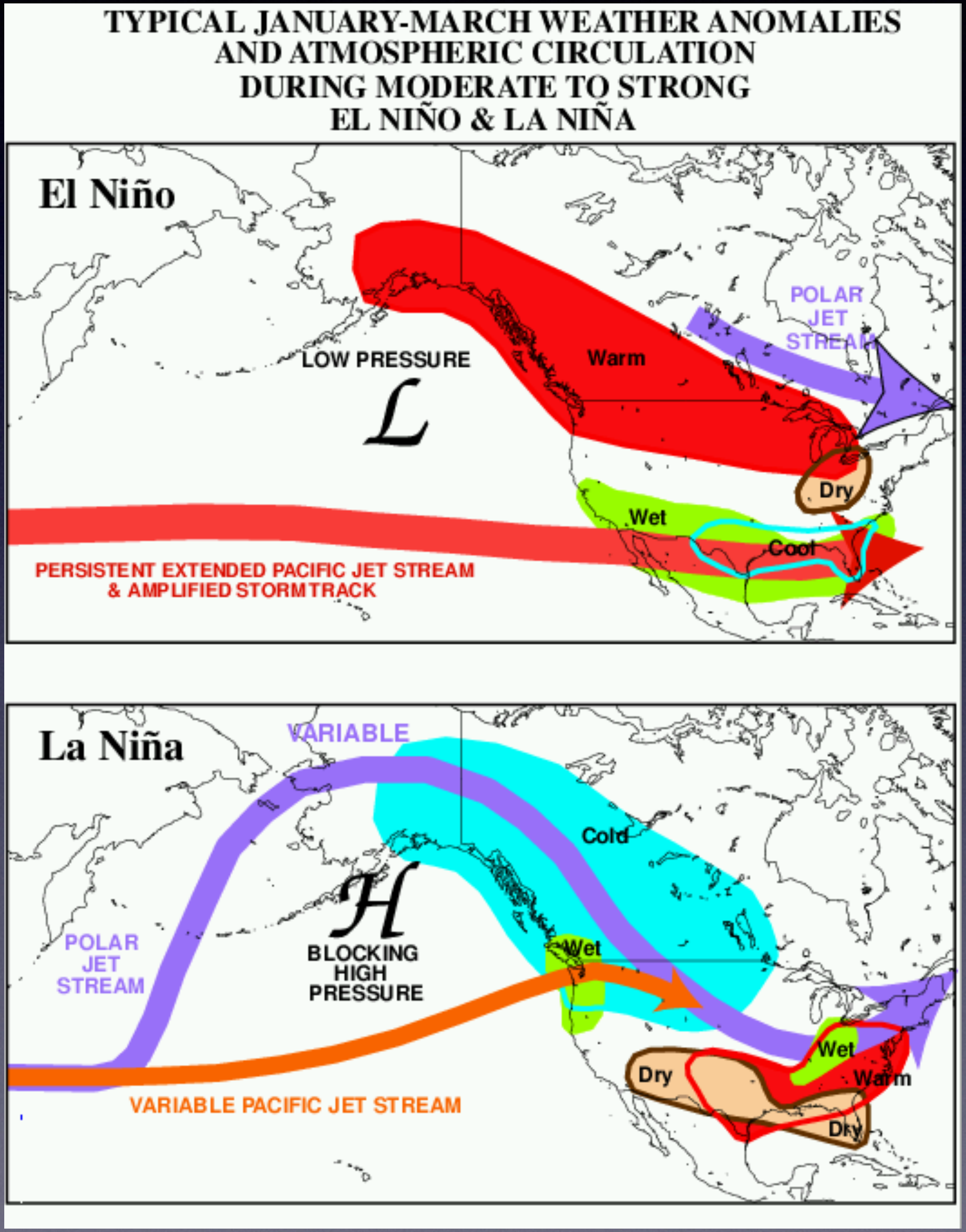
Simulation 2



Model Output



Winter?


TYPICAL JANUARY-MARCH WEATHER ANOMALIES
AND ATMOSPHERIC CIRCULATION
DURING MODERATE TO STRONG
EL NIÑO & LA NIÑA
El Niño
LOW PRESSURE
L
Warm
POLAR
JET
STREAM
Dry
Wet
Cool
PERSISTENT EXTENDED PACIFIC JET STREAM
& AMPLIFIED STORMTRACK
La Niña
VARIABLE
Cold
H
BLOCKING
HIGH
PRESSURE
POLAR
JET
STREAM
Wet
Wet
Warm
Dry
Dry
VARIABLE PACIFIC JET STREAM