Table Cloth Math,

from Balthazar van der Pol, early 20^{th} century

Beverly West
Cornell University
November 11, 2017

The Two Men Involved

John Bukowski, JMM 2015

Juniata College

Balthasar van der Pol

1889-1959

Balthasar van der Pol (1889-1959)

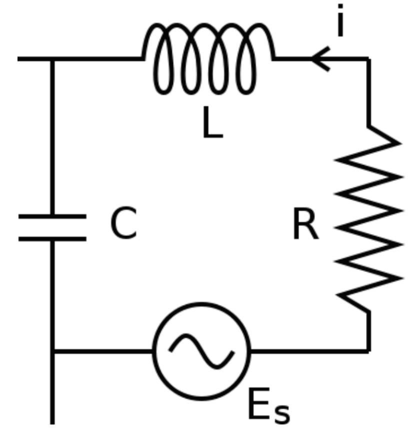
- Born in Utrecht 1889
- Universiteit Utrecht 1916, physics and math
- Doctoral degree Utrecht 1920 (electromagnetism and radio waves)
- Many years at Philips Lab in Eindhoven
- Taught at TU Delft
- Visitor at Berkeley 1957, Cornell 1958
- Died in Wassenaar 1959

Van der Pol was a pioneer in radio and telecommunications in the early 20th century, at the time when vacuum tubes were used to control flow of electricity in the circuitry of transmitters and receivers.

The basic differential equation for electrical current I is a *linear* one:

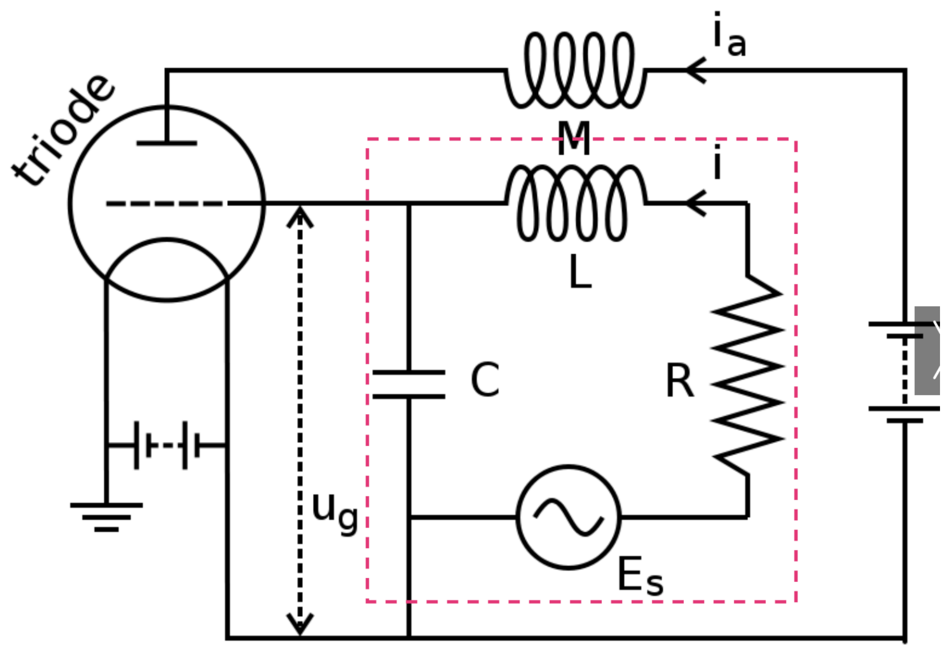
$LI'' - RI' + (1/C)I = 0$, with

Inductance L
Resistance R
Capacitance C
Forcing E_s
(= 0 for us)



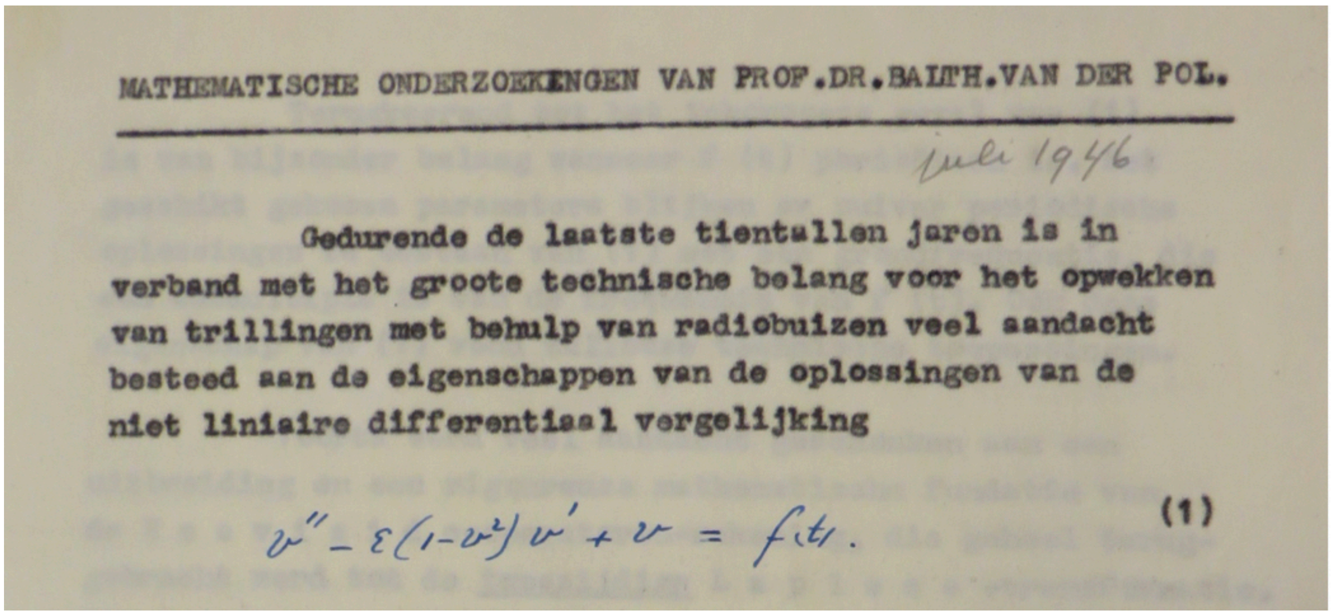
(Note that this has the same format as a mass-spring equation $mx'' - bx' + kx = 0$.)

Experimenting with oscillations in a vacuum tube triode circuit,



van der Pol observed that *all* initial conditions gave trajectories that converged to the *same* periodic orbit of finite amplitude, which cannot result from a *linear* differential equation ----

So, van der Pol then proposed as a model an equation with *nonlinear* resistance, which agreed with his observations.



MATHEMATISCHE ONDERZOEKINGEN VAN PROF.DR.BALTH.VAN DER POL.

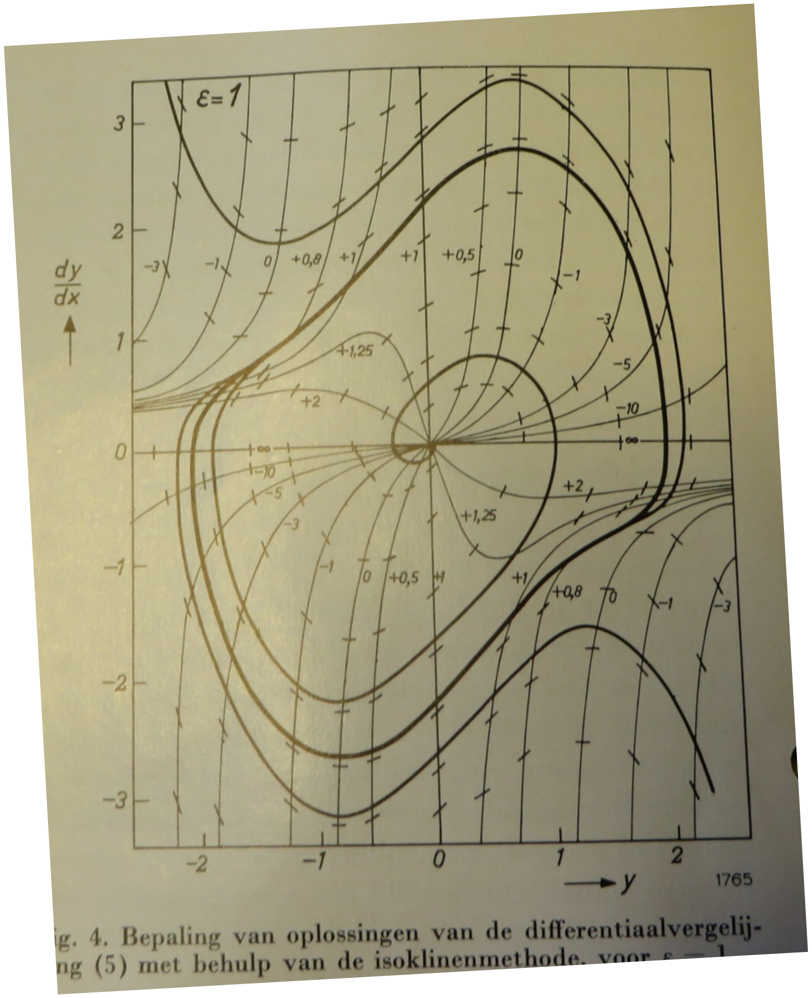
juli 1946

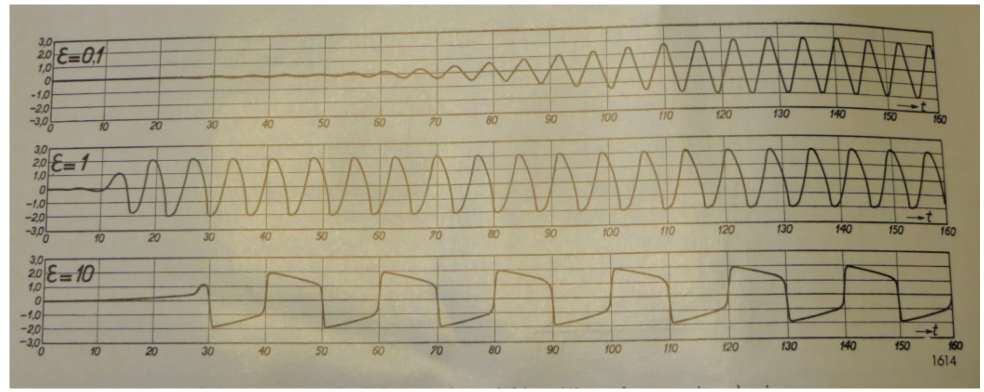
Gedurende de laatste tientallen jaren is in verband met het groote technische belang voor het opwekken van trillingen met behulp van radiobuizen veel aandacht besteed aan de eigenschappen van de oplossingen van de niet liniaire differentiaal vergelijking

$$v'' - \varepsilon(1-v^2)v' + v = f(t). \tag{1}$$

$$v'' - \boxed{\varepsilon(1-v^2)}\,v' + v = 0\,.$$

van der Pol equation results





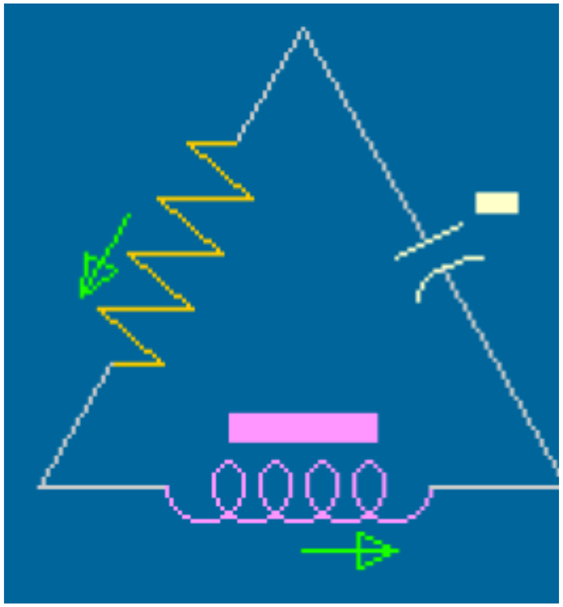
van der Pol originals as reproduced in

Philips Technical Mag.
Vol. 22, No. 2 (1960)

$$v''-\varepsilon(1-v^2)v'+v=0$$

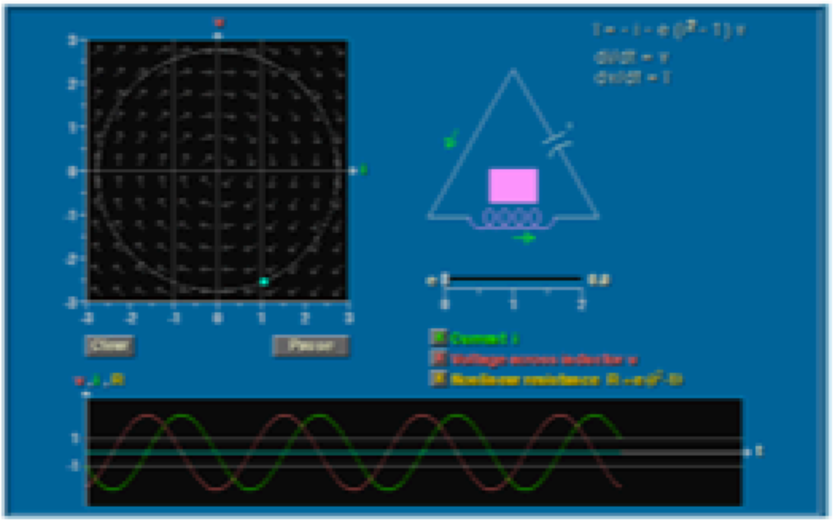
van der Pol in action

Using a simplified circuit, with no forcing, we can vary *e* (for epsilon) and *see* some of the results that van der Pol saw.

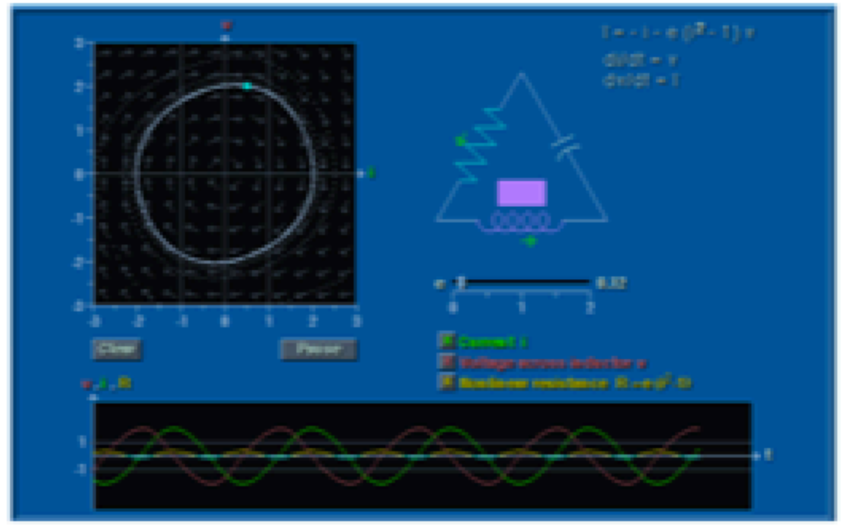


(Interactive Differential Equations, Pearson)

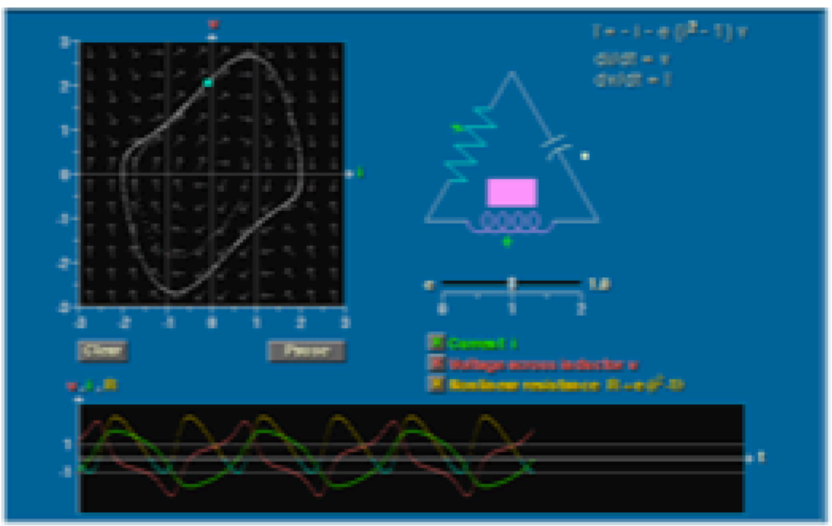
Simulations for different ε



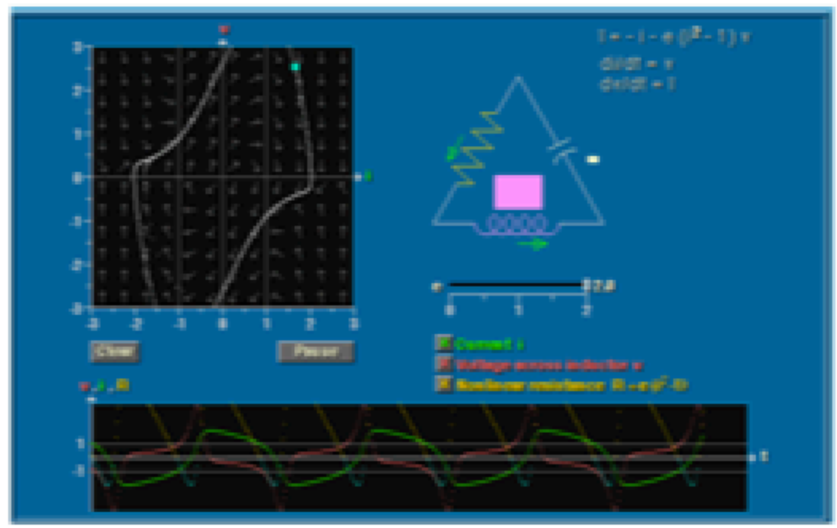
e = 0.0



e = 0.12



e = 1.0



e = 2.0

van der Pol equation

- Expanded to forced oscillations

$$v'' - \varepsilon(1 - v^2)v' + v = f(t)$$

- Used for modeling systems with self-excited limit cycle oscillations, e.g.
  - in biology, action potentials of neurons;
  - in seismology, interaction of tectonic plates in a geological fault.

Archive at Museum Boerhaave in Leiden:

Dutch National Museum for the
History of Science and Medicine

Leiden U. catalog

- Correspondence
- Biographies, in memoriams, etc.
- 4 tafelkleedjes (!)
- Musical compositions
- Mathematical works
- Chess problems with solutions

Leiden U. catalog

- Correspondence
- Biographies, in memoriams, etc.
- 4 tafelkleedjes (!) → tablecloths
- Musical compositions
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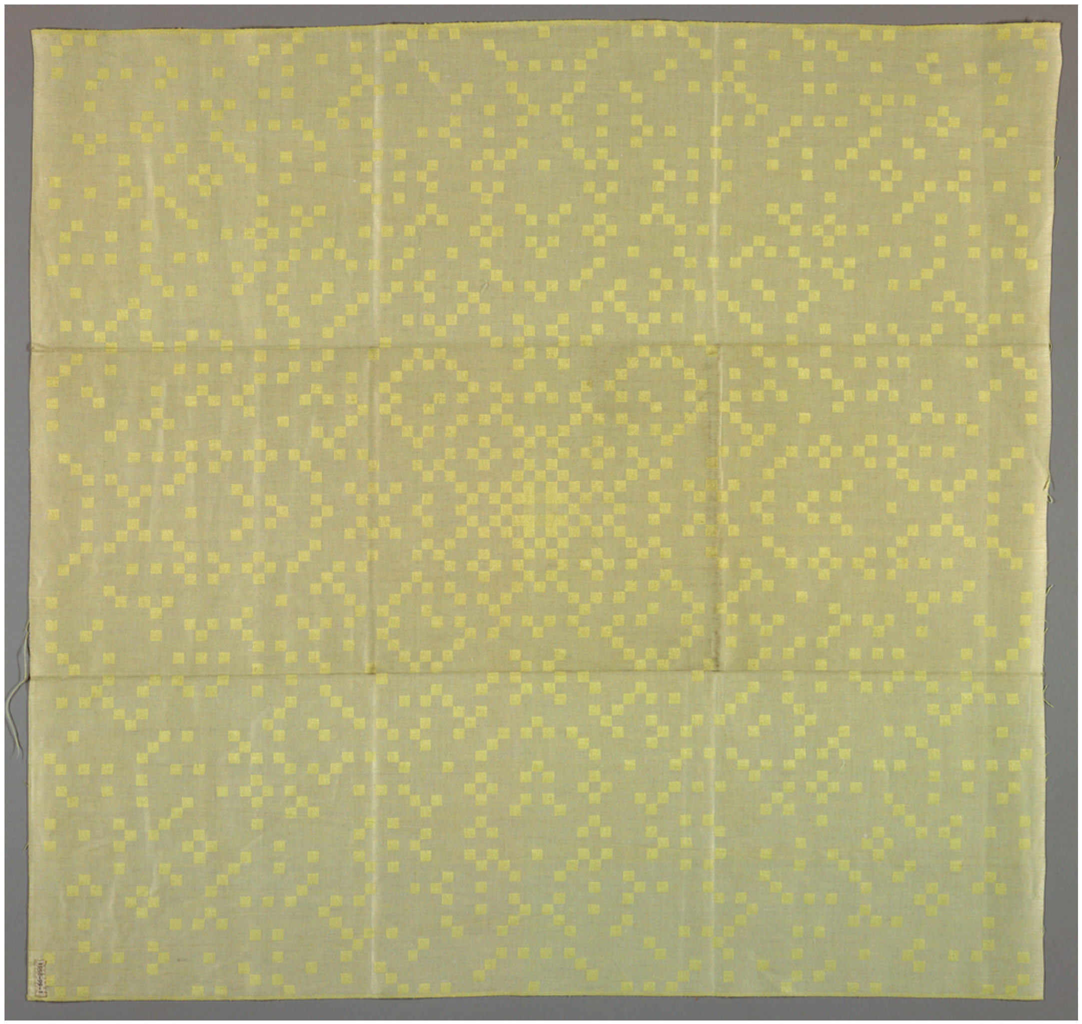
Original Damask Tablecloth

in Smithsonian Design Collection at Cooper Hewitt



Primes in the complex plane

- In the ordinary integers, it is well known that not every integer can be decomposed into smaller factors –
- The same is true in the complex numbers $a + bi$, where a and b *are* integers – these complex numbers are called Gaussian integers.
- Primes in the Gaussian integers are called Gaussian primes.

Gaussian primes in the complex plane

- *Finding* Gaussian primes is not so simple!
- Many ordinary (real) primes are *not* prime in the complex plane.

  E.g.

$$5 = (1 + 2i)\,(1 - 2i)$$

Gaussian primes in the complex plane

In fact, 5 has *more* combinations of complex prime factors:

$$5 = (1 + 2i)(1 - 2i), \text{ or } (2 + i)(2 - i), \text{ or } \ldots$$

Note: We see there is *no unique factorization* of Gaussian integers into prime factors, as there is in the *real* numbers!

Gaussian primes in the complex plane

$5 = (1 + 2i)\,(1 - 2i)$, or $(2 + i)\,(2 - i)$ or …

We find exactly *eight* Gaussian prime factors of 5 :

$(\pm 1 \pm 2i)$ and $(\pm 2 \pm i)$

Soon we will plot all eight on the tablecloth graph, and label each of those squares as prime factors of 5 .

Gaussian primes in the complex plane

- Back to the observation that many ordinary (real) primes are *not* prime in the Gaussian integers. A few others are

$$13 = (2 + 3i)\,(2 - 3i)$$

$$17 = (4 + i)\,(4 - i)$$

$$29 = (5 + 2i)\,(5 - 2i)$$

How do we find them?

Fortunately, there is a good (but strange) criterion:

Theorem

Part I:

- If $a = 0$ or $b = 0$, then the complex number $a + bi$ lies on an axis, and is a Gaussian prime if and only if the remaining term is a *prime*, $\pm p$, with $|p| \equiv 3 \pmod 4$.

Activity 1:

List the numbers $3 \pmod 4$ and cross out those that are not prime.

Theorem

Part II.

- If both $a, b \neq 0$, then $a + bi$ is a Gaussian prime if and only if

  $a^2 + b^2 = p$, where p is a prime.

Look again at our previous examples:

$13 = (2 + 3i)\,(2 - 3i)$

$17 = (4 + i)\,(4 - i)$

$29 = (5 + 2i)\,(5 - 2i)$

Theorem

Part II:

- If both $a, b \neq 0$, then $a + bi$ is a Gaussian prime if and only if

  $a^2 + b^2 = p$, where p is a prime.

  This is guaranteed to occur when

  $p = 1 (\text{mod } 4)$. It also occurs for $p = 2$.

Activity 2:

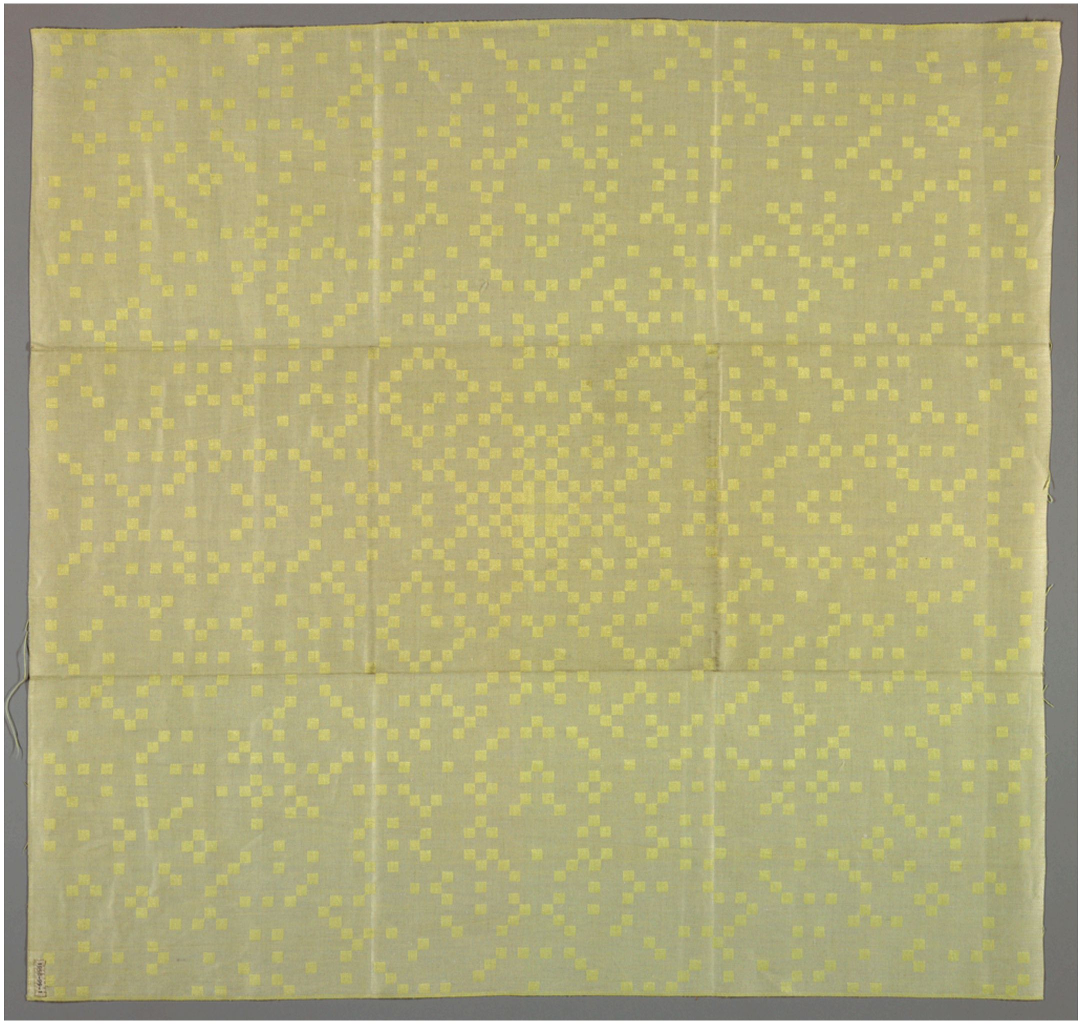
List the numbers 2, and those that are 1(mod 4), and cross out those that are not prime.

Theorem recap:

- If $a = 0$ or $b = 0$, then $a + bi$ is on an axis, and is a Gaussian prime if and only if the remaining term is a prime with $p = 3 \pmod 4$, i.e., $p = 4k - 1$.

- If $a, b \neq 0$, then $a + bi$ is a Gaussian prime if and only if $a^2 + b^2 = p$, where p is a prime. This occurs when $p = 2$, or $p = 1 \pmod 4$, i.e., $p = 4k + 1$.

van der Pol’s tablecloth!



van der Pol’s tablecloth, modern

Creating Tablecloth Graph

Activity 3:

For each prime p in your second list, figure out a and b to give $a^2 + b^2 = p$

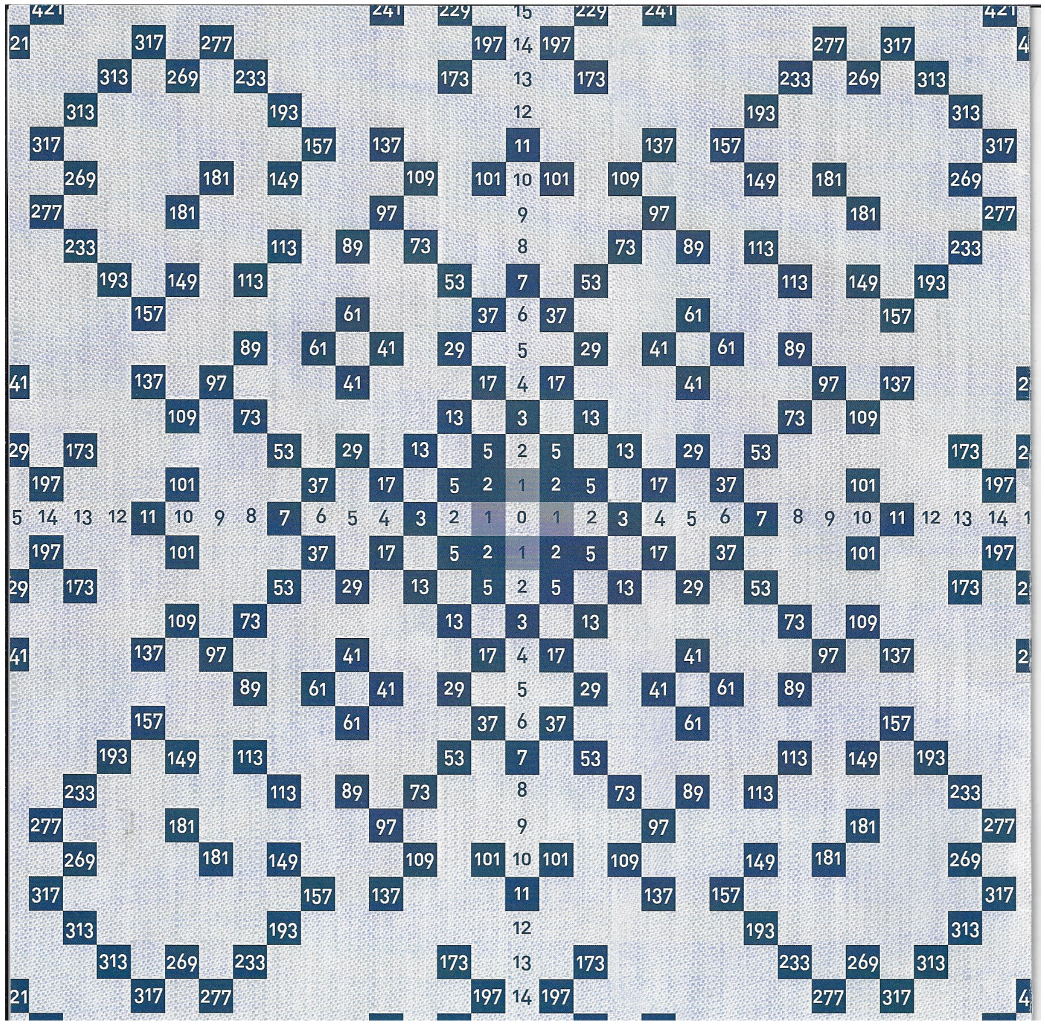
Now you will have *three* lists, which you will use to plot the Tablecloth Graph…

Creating Tablecloth Graph

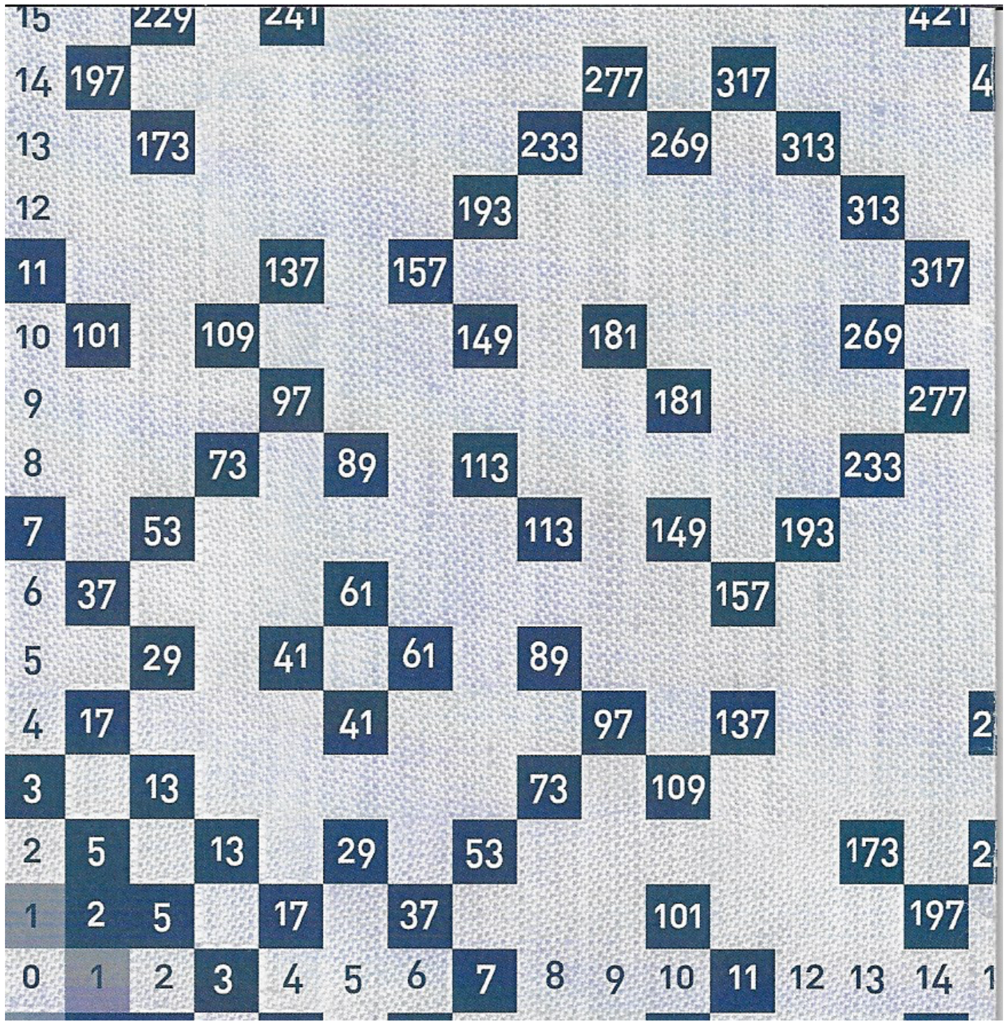
Activity 4:

- Turn to your graph paper and mark the center square for (0,0). Consider your axes as the strings of *boxes* to the right and left, above and below.
- For each prime p in your first list, put its number in the appropriate square on each axis.
- For each prime p in your second and third lists, locate each of the squares ($\pm a$, $\pm b$) for the prime factors, and label those squares with the appropriate p . *For each p there are 8 such squares, except for p = 2, where there are only 4.* This labelling is just a bookkeeping device, used in the graph provided with the tablecloth:

Tablecloth Graph

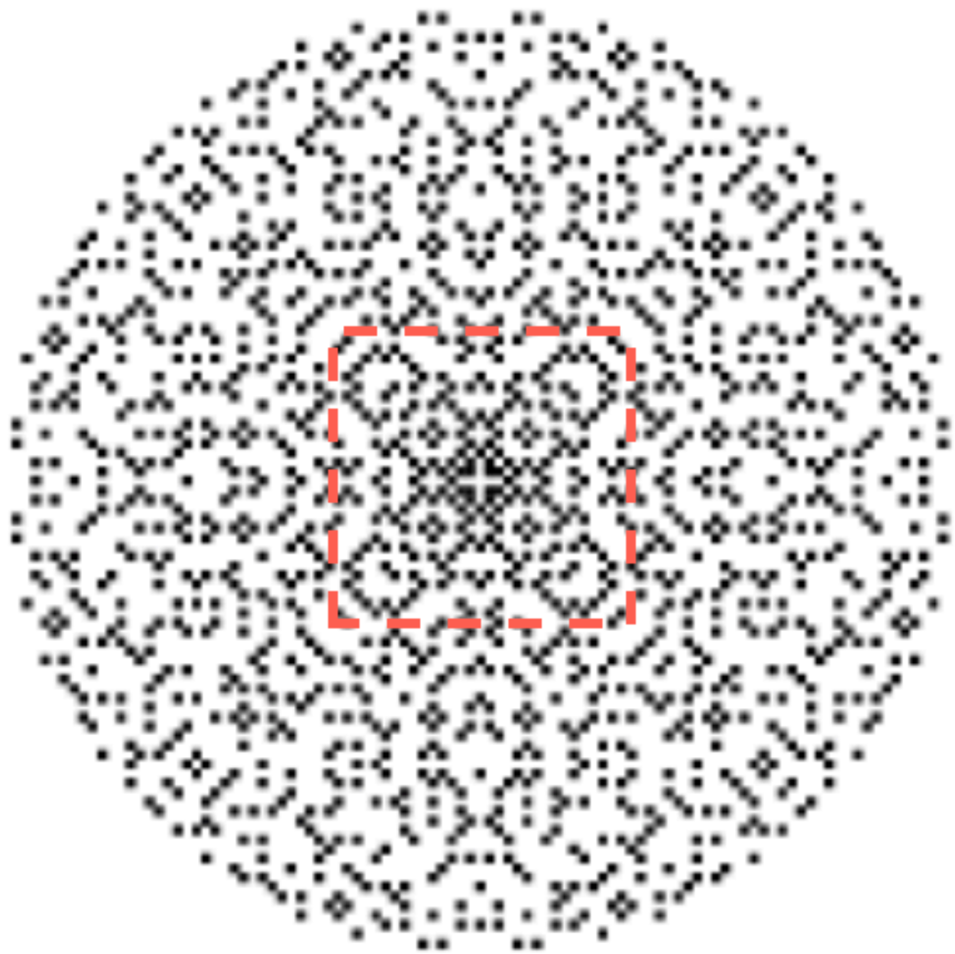


Tablecloth Graph
(first quadrant)



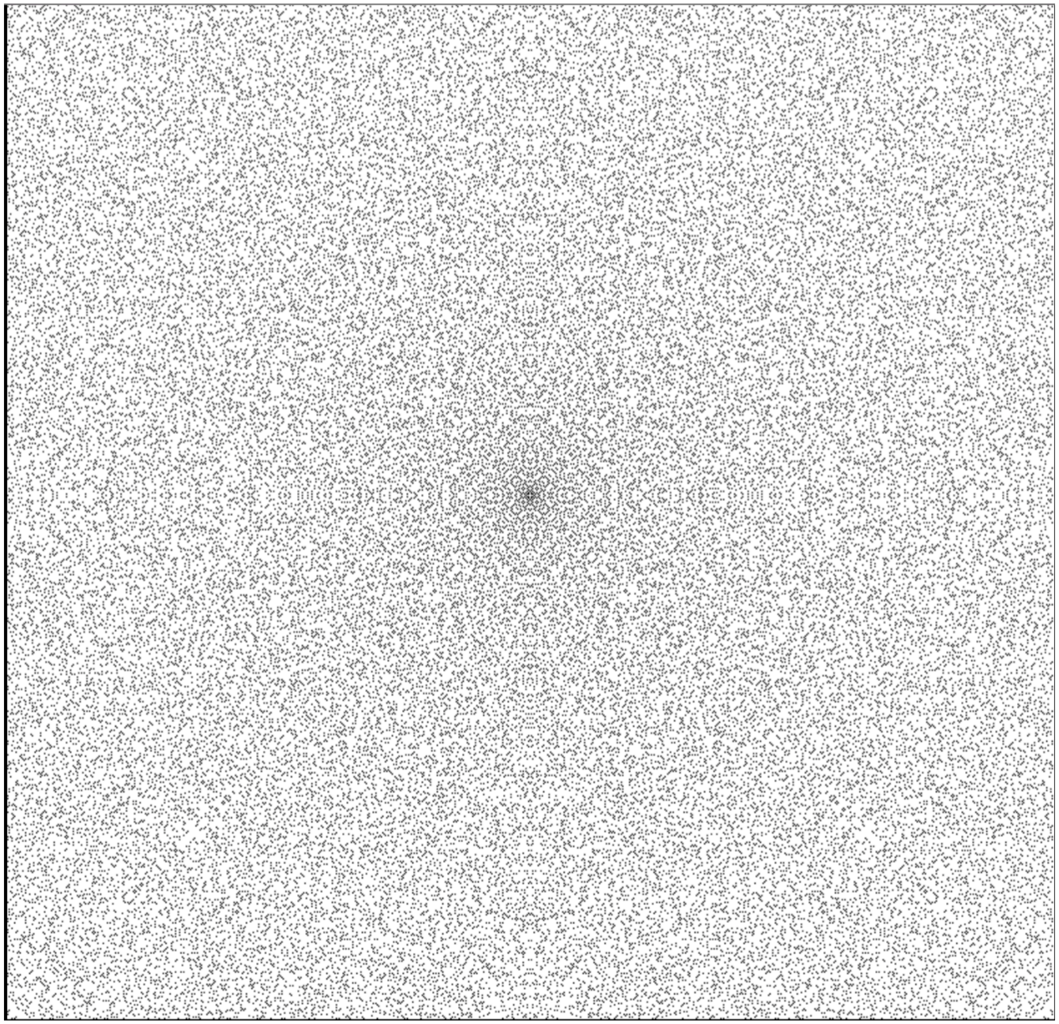
Gaussian primes of norm < 1000

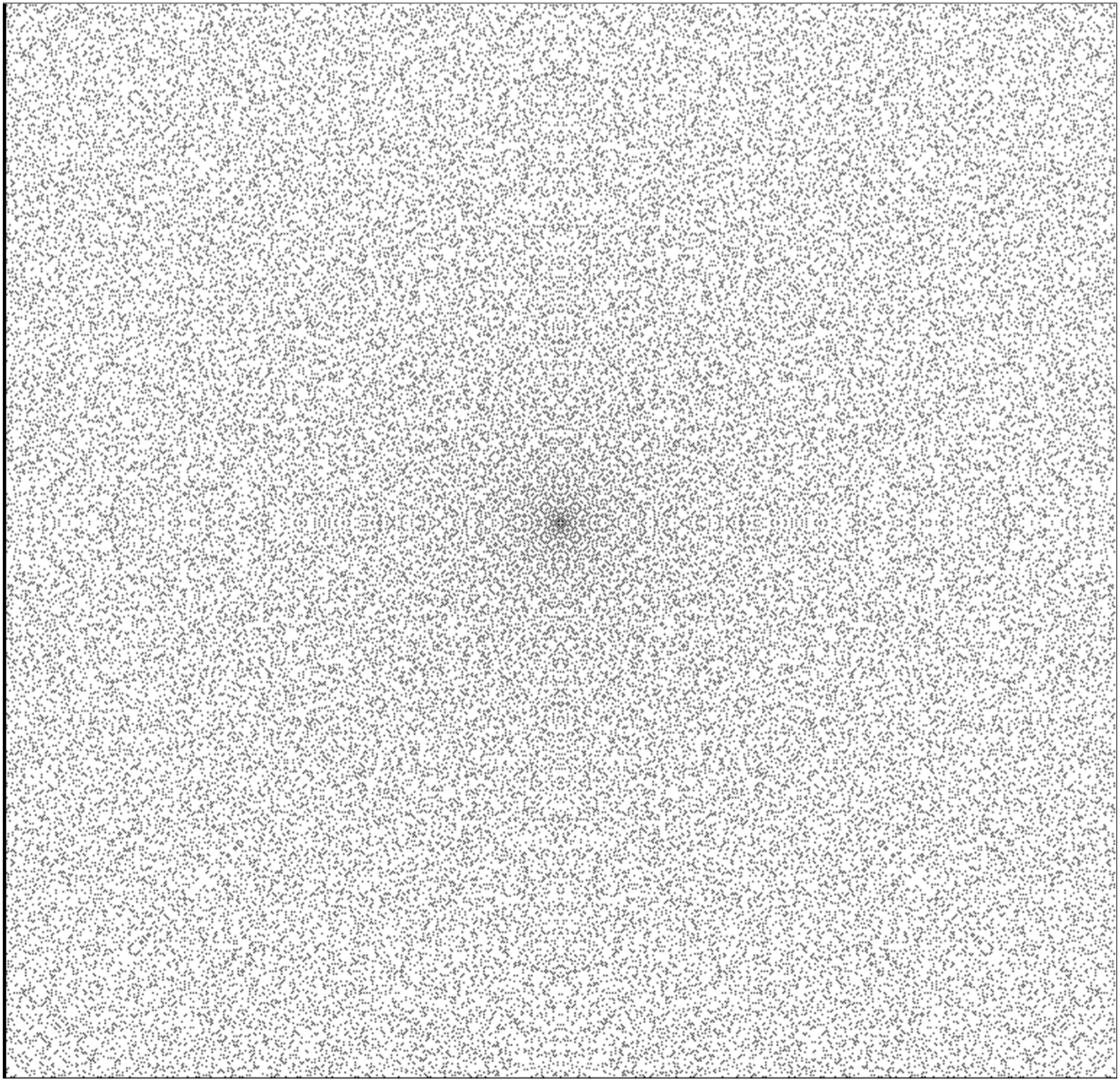
by Gethner, Wagon, and Wick (AmerMathMonthly April 1998)



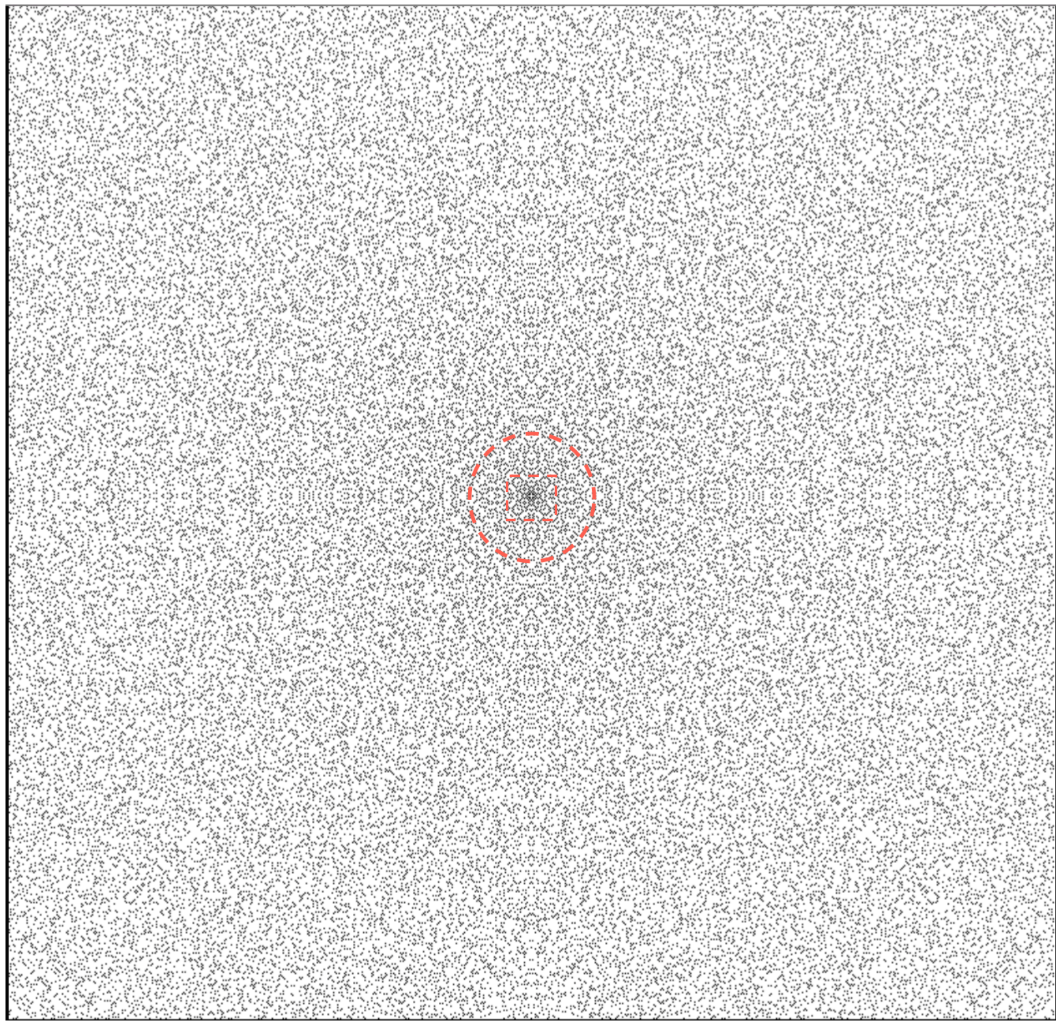
Zooming out further

by Truejackster - Own work Public Domain

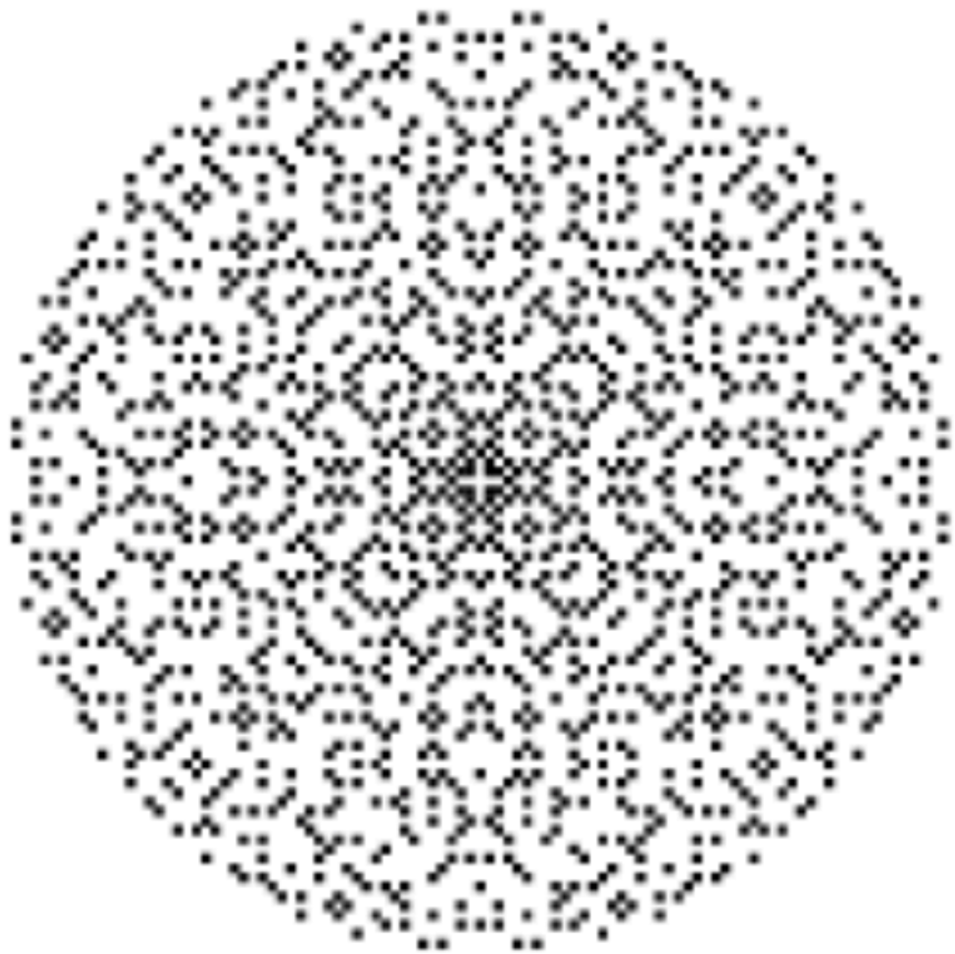




Comparison of tablecloth with primes out to 1000 and final zoom out



One final question:
How many symmetries?



Thank you!

I'll give slides to Mary Ann to post on Math 5080 website.