Scaling Limits of Interacting Particle Systems

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11th Cornell Probability Summer School
Cornell, Ithaca
Jun 17–20, 2019

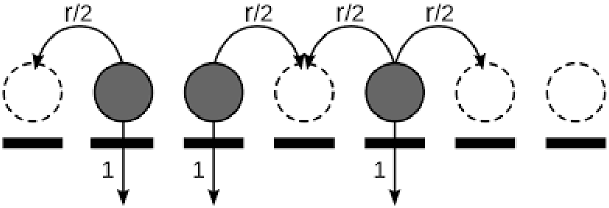
Lecture 1

Motivation and Introduction

- Suppose you want to study how infections (virus in computers or diseases in the real world) spread in the real world.
- Some questions of interest:
  - How many times does a person/computer catch a particular disease/virus in a given time period?
  - What proportion of time is the person sick in a given period?
  - What proportion of time are more than half of the population sick?

Example 1: Discrete-time Contact Process

- Identify a graph (a collection of nodes, with edges between them representing connections) that describes the contact network of people in a population
- Each node has two states: 1 if infected and 0 if sick
- Estimate probability of catching an infection given the state of the population in your local contact neighborhood



- Various multi-state generalizations: SIS, SIR

Example 1: Discrete-time Contact Process

- State space $\mathcal{S} = \{0, 1\} = \{\mathrm{healthy}, \mathrm{infected}\}$.
- Parameters $p, q \in [0, 1]$.
- $X_v(t) \in S$, state of process at time t
- The processes evolve as a discrete-time Markov chain

Transition rule: At time t, evolution of state of particle at node v depends on state of particle at v and the neighbors' empirical distribution at that time:

$$\mu_v(t) = \frac{1}{d_v} \sum_{u \sim v} \delta_{X_u(t)}$$

- if state $X_v(t) = 1$, it switches to $X_v(t+1) = 0$ w.p. q,
- if state $X_v(t) = 0$, it switches to $X_v(t+1) = 1$ w.p.

$$\frac{p}{d_v} \sum_{u \sim v} X_u(t) = p \int y \mu_v(t)(dy)$$

where recall $d_v =$ degree of vertex v.

Ising model

Probability distribution on $\{-1,1\}^V$: for $\sigma \in \{-1,1\}^V$,

$$\mathbb{P}(X = \sigma) = \frac{1}{Z_\beta} e^{-\beta H(\sigma)},$$

where $\beta > 0$ is a parameter referred to as the inverse temperature, and H is the "Hamiltonian":

$$H(\sigma) = -J \sum_{i \sim j} \sigma_i \sigma_j - h \sum_j \sigma_j$$

for suitable parameters $J, h \in \mathbb{R}$, and Z_β is the normalizing constant

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$$Z_\beta = \sum_{\sigma \in \{-1,1\}^V} e^{-\beta H(\sigma)}.$$

Although there is an explicit expression for the probability, it is typically computationally infeasible to calculate Z_β.

Instead, to (approximately) sample from the distribution

$$\mathbb{P}(X = \sigma) = \frac{1}{Z_\beta} e^{-\beta H(\sigma)},$$

one constructs a reversible Markov chain (the so-called Glauber dynamics) that has the target distribution as its stationary distribution.

Note: The transition probability matrix of the Markov chain only depends on ratios of the probabilities, and thus does not require knowledge of Z_β.

Brownian Motion

Brownian motion $\{W_t, t \geq 0\}$ is an $\mathbb{R}$-valued stochastic process such that

- $t \mapsto W_t$ is (almost surely) continuous
- for every $0 < s < t < \infty$, $W_t - W_s$ is independent of W_s and is distributed according to $\mathcal{N}(0, t - s)$.
- $W_0 = 0$ (standard Brownian motion)

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d-dimensional Brownian motion $\{B_t, t \geq 0\}$ is an $\mathbb{R}^d$-valued stochastic process such that its components $B^i, i = 1, \ldots, d$, are independent and identically distributed standard Brownian motions.

- Systemic risk is the risk that in an interconnected system of agents that can fail individually, a large number of them fails simultaneously, or nearly so.
- The interconnectivity of the agents, and its form of evolution, plays an essential role in systemic risk assessment.

X_t^i represents the state of risk of agent/component j
Given independent Brownian motions $W^j, j = 1, \dots, n$,

$$dX_t^j = -hU(X_t^j)dt + \theta(\bar{X}_t - X_t^j)dt + \sigma dW_t^j,$$

for some restoring potential $U : \mathbb{R} \mapsto \mathbb{R}$, $\theta, \sigma > 0$, and with some given initial conditions, where $\bar{X}$ is the empirical mean:

$$\bar{X}_t := \frac{1}{n} \sum_{i=1}^{n} X_t^i.$$

- An (undirected) graph $G = (V, E)$ consists of a countable collection V of vertices/nodes and a set E of edges, or (unordered) pairs in $V \times V$
- The neighboring relation is often denoted $u \sim v$ if $(u, v) \in E$.
- A graph G is said to be finite if $|V| < \infty$.
- The degree d_v of a vertex v is the cardinality of its neighborhood:

$$d_v = |\mathcal{N}_v| := |\{u \in V : u \sim v\}|$$

- An infinite graph G is said to be locally finite if $d_v < \infty$ for every $v \in V$.
- A graph G is said to be simple if it contains no loops (i.e., for every $v \in V$, $(v, v) \notin E$).
- the graph distance $d(u, v) = d_G(u, v)$ between two vertices u and v is the length of the shortest path between the vertices.

2. General Problem Formulation

Given a finite connected graph $G = (V, E)$, we are interested in a stochastic process

$$\{X_t^v, v \in G, t \in \mathbb{T}\},$$

with $\mathbb{T} = \mathbb{N}_0$ or $\mathbb{T} = [0, \infty)$ and the property that:

the evolution of the stochastic process is such that the state X_t^v of each node $v \in V$ at time t evolves stochastically depending only on its own state and those of its neighbors at that time

For example, evolving as a discrete-time Markov chain:

$$X_{t+1}^{v} = F\left(X_t^v, (X_t^u)_{u\sim v}, \xi_{t+1}^v\right), v \in V,$$

for a suitable function

$$F : \mathcal{S} \times \mathcal{S}^J \times U \mapsto \mathcal{S}.$$

- state space $\mathcal{S}$
- continuous transition function F
- independent noises $\xi_t^v, v \in V, \ t = 0, 1, \ldots,$ taking values in some space U.

Networks of interacting Markov chains

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Probabilistic cellular automata, synchronous Markov chains

Networks of interacting Markov chains: a particular form

Symmetric Dependence on Neighbors

The state of each $v \in V$ evolves stochastically depending

only on its own state and the empirical distribution of its neighbors

Networks of interacting Markov chains: a particular form

Symmetric Dependence on Neighbors

The state of each $v \in V$ evolves stochastically depending only on its own state and the empirical distribution of its neighbors

Then the discrete-time Markov chain takes the form:

$$X_{t+1}^{v} = \bar{F}\left((X_t^v), \mu_t^v, \xi_{t+1}^v\right), \quad v \in V,$$

where

$$\bar{F} : \mathcal{S} \times \mathcal{P}(\mathcal{S}) \times U \mapsto \mathcal{S}$$

and $\mu_t^v \in \mathcal{P}(\mathcal{S})$ is the local empirical measure at v:

$$\mu_t^v = \frac{1}{d_v} \sum_{u \sim v} \delta_{X_t^u}.$$

Brownian Motion

Brownian motion $\{W_t, \mathcal{F}_t, t \geq 0\}$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ is an $\mathbb{R}$-valued stochastic process such that W_t is $\mathcal{F}_t$-measurable for every $t \geq 0$,

- $t \mapsto W_t$ is (almost surely) continuous
- for every $0 < s < t < \infty$, $W_t - W_s$ is independent of $\mathcal{F}_s$ and is distributed according to $\mathcal{N}(0, t - s)$.

Networks of interacting diffusions

Or as a diffusion:

$$dX_t^v = \frac{1}{d_v} \sum_{u \sim v} b(X_t^v, X_t^u) dt + dW_t^v,$$

where $(W^v)_{v \in V}$ are independent d-dimensional Brownian motions.

For concreteness, assume standard conditions:
b is Lipschitz and has linear growth, σ is non-degenerate and has an inverse,
∘ initial states $(X_0^v)_{v \in V}$ are i.i.d. and square-integrable.

Networks of interacting stochastic processes

We will focus on
discrete-time Markov chains and diffusions

but one can consider different interacting stochastic evolutions,
including
continuous time Markov chains (A. Ganguly),
jump diffusions, etc. ...

Networks of interacting stochastic processes

Key questions

Given a sequence of graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$, how can we describe the limiting behavior of...

- the dynamics of a fixed or "typical particle" $X_t^v, t \in [0, T]$?
- the empirical distribution of particles $\frac{1}{|V_n|} \sum_{v \in V_n} \delta_{X_t^v}$?

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A much-studied special case

$G_n = K_n$, the complete graph on n vertices

3. The mean field case (McKean-Vlasov 1966)

$G_n = K_n$ complete graph; wlog $V_n = \{1, \ldots, n\}$
Particles $i = 1, \ldots, n$ interact according to

$$dX_t^i = \frac{1}{n} \sum_{k=1}^{n} b(X_t^i, X_t^k)dt + dW_t^i, \quad i = 1, \ldots, n$$

where $W^1, \ldots, W^n$ are independent Brownian motions, with iid initial conditions $(X_0^1, \ldots, X_0^n)$ with common law λ

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This can be reformulated as

$$dX_t^i = B(X_t^i, \bar{\mu}_t^n)dt + dW_t^i, \qquad \bar{\mu}_t^n = \frac{1}{n}\sum_{k=1}^{n} \delta_{X_t^k},$$

where, for a probability measure m on $\mathbb{R}^d$,

$$B(x, m) := \int_{\mathbb{R}^d} b(x, y)\, m(dy),$$

Mean field systems, law of large numbers

Theorem (McKean '67, Oelschlager '84, Sznitman '91, etc.)

$(\bar{\mu}_t^n)_{t\in[0,T]}$ *converges in probability to the unique solution* $(\mu_t)_{t\in[0,T]}$ *of the McKean-Vlasov equation*

$$dX_t = B(X_t, \mu_t)dt + dW_t, \qquad \mu_t = \mathrm{Law}(X_t),$$

with $X_0 \sim \lambda$.
Moreover, the particles become asymptotically independent.
Precisely, for fixed k,

$$(X^1, \ldots, X^k) \Rightarrow \mu^{\otimes k}, \qquad \text{as } n \to \infty.$$

A Slightly Different Perspective

Kurtz and Kotelenez ('10)

- The existence of a limit follows from general results on exchangeable processes:
  As $n \to \infty$, $X^{(n)} \Rightarrow X^{(\infty)}$, where
  $$dX_t^{(\infty),i} = b(X_t^{(\infty),i}, \mu_t)dt + dW^i(t), \qquad i = 1, 2, \dots,$$
  with
  $$\mu_t = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{X_t^{\infty,i}}.$$
- One then characterizes the marginal dynamics of a single partile in this infinite system.

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The McKean-Vlasov equation
provides an autonomous description of this marginal dynamics

- Consider random elements $(Z^n)_{n\in\mathbb{N}}$, Z, taking values in a Polish (complete, separable, metrizable) space $\mathcal{X}$.

Definition of Weak Convergence

Then Z^n is said to converge weakly to Z, as $n \to \infty$, denoted $Z^n \Rightarrow Z$, if for each bounded, continuous function $f : \mathcal{X} \mapsto \mathbb{R}$,

$$\mathbb{E}[f(Z^n)] \to \mathbb{E}[f(Z)] \quad \text{as } n \to \infty.$$

Exercise

Suppose $S = \mathbb{R}$, and P_n, P lie in $\mathcal{P}(\mathbb{R})$ and let F_n and F denote their corresponding cumulative distribution functions. Then show that P_n converges weakly to P if and only if

$$F_n(x) \to F(x), \qquad \forall x : F(x) = F(x-).$$

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- There is a metric (the Prohorov metric) on the space $\mathcal{P}(\mathcal{X})$ of probability measures on $\mathcal{X}$ that metrizes this notion of convergence. The resulting space is again Polish.

Background on Weak Converence, contd.

How does one prove convergence in general?

1. Show relative compactness of the sequence $\{Z_n\}$.
2. Uniquely characterize any subsequential limit.

Background on Weak Convergence, contd.

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2. Uniquely characterize any subsequential limit.

For weak convergence

Prohorov's Theorem provides a criterion for relative compactness

Definition of Tightness

The sequence of random elements $\{Z^n\}$ is said to be tight if for every $\varepsilon > 0$, there exists a compact space $K_\varepsilon \subset \mathcal{X}$ such that

$$\mathbb{P}(Z^n \notin K_\varepsilon) < \varepsilon.$$

Prohorov's Theorem (1956)

A sequence $\{Z^n\}$ is relatively compact if and only if it is tight.

- Recall

$$dX_t^{i,n} = B(X_t^{i,n}, \bar{\mu}_t^n)dt + dW_t^{i,n}, \qquad \bar{\mu}_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{X_t^{k,n}},$$

for $i = 1, \ldots, n$, where $\{W^{i,n}, i = 1, \ldots, n\}$ are independent d-dimensional Brownian motions and where, for a probability measure $m \in \mathcal{P}(\mathbb{R}^d)$,

$$B(x, m) = \int_{\mathbb{R}^d} b(x, y)\, m(dy),$$

- We want to show weak convergence of $\{\bar{\mu}_t^n, t \geq 0\}$
- Note that $\bar{\mu}^n$ is a random element taking values in the (Polish) space $\mathcal{X} := \mathcal{C}([0, \infty) : \mathcal{P}(\mathbb{R}^d))$ of continuous (probability) measure-valued trajectories.

Infinitesimal Generators

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- The infinitesimal generator of a continuous-time stochastic process $\{X_t, t \geq 0\}$ is a differential operator of the form:

$$\mathcal{A}f(x) = \lim_{t \to 0} \frac{\mathbb{E}[f(X_t) - f(x)]}{t},$$

  which acts on the set of functions $f : \mathbb{R}^d \mapsto \mathbb{R}$ for which the above limit is well defined, called the domain of $\mathcal{A}$.
- $\mathcal{A}$ should be viewed as an "average derivative" along the flow of a stochastic process $\{X_t, t \geq 0\}$

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- $\mathcal{A}$ should be viewed as an "average derivative" along the flow of a stochastic process $\{X_t, t \geq 0\}$
- For an SDE of the form:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x.$$

  where $a(x) = \sigma(x)\sigma^T(x)$, $\mathcal{A}$ takes the form

$$\mathcal{A}f(x) = \sum_i b_i(x)\partial_i f(x) + \frac{1}{2}\sum_{ij} a_{ij}(x)\partial_i\partial_j f(x)$$

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- The domain of $\mathcal{A}$ includes $\mathcal{C}_b(\mathbb{R}^d)$, the space of bounded continuous real-valued functions on $\mathbb{R}^d$.

Outline of the Proof of the Mean-Field limit

- Step 1: Show tightness of $\{\bar{\mu}^n\}_{n\in\mathbb{N}}$
- For this, use the "infinitesimal generator" of the original process

$$dX_t^{i,n} = B(X_t^{i,n}, \bar{\mu}_t^n)dt + \sigma(X_t^{i,n}, \bar{\mu}_t^n)dW_t^{i,n}, \qquad \bar{\mu}_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{X_t^{k,n}},$$

Set $a(x,\mu) := \sigma(x,\mu)\sigma^T(x,\mu)$, where $\mu = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$, and fix

$$g \in C_b^2(\mathbb{R}), \quad h \in C_b^2(\mathbb{R}^d), \quad \langle \mu, h\rangle = \int_{\mathbb{R}} h(x)\mu(dx)$$

For $f(\mu) = g(\langle \mu, h\rangle)$, define

$$\begin{aligned}\mathcal{A}_h^n(\mu) &= g'(\langle \mu, h\rangle)\left\{\langle \mu, b(\cdot,\mu)\cdot\nabla h\rangle + \frac{1}{2}\langle \mu, a_{ij}(\cdot,\mu)\partial_{ij}h\rangle\right\} \\ &\quad + \frac{1}{2}g''(\langle \mu, h\rangle)\frac{1}{n}\langle \mu, a_{ij}(\cdot,\mu)\partial_i h\partial_j h\rangle.\end{aligned}$$

- Use the generator, martingale estimates and stochastic calculus (Itô's formula) to show that $\{\bar{\mu}^n\}_{n\in\mathbb{N}}$ is tight;

Outline of Proof of the Mean-Field limit (contd.)

- Recall that the evolution of $\bar{\mu}^n$ is characterized by

$$\mathcal{A}_h^n(\mu) = g'(\langle \mu, h \rangle) \left\{ \langle \mu, b(\cdot, \mu) \cdot \nabla h \rangle + \frac{1}{2} \langle \mu, a_{ij}(\cdot, \mu) \partial_{ij} h \rangle \right\} + \frac{1}{2} g''(\langle \mu, h \rangle) \frac{1}{n} \langle \mu, a_{ij}(\cdot, \mu) \partial_i h \partial_j h \rangle.$$

Outline of Proof of the Mean-Field limit (contd.)

- Recall that the evolution of $\bar{\mu}^n$ is characterized by

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- Step 2: Show that any susbsequential limit $\bar{\mu}$ of $\{\mu^n\}_{n\in\mathbb{N}}$ satisfies a certain martingale problem: or more precisely, satisfies

$$\mathcal{A}^\infty_h(\bar{\mu}) = 0, \quad \forall h \in \mathcal{C}_b(\mathbb{R}^d),$$

where

$$\mathcal{A}^\infty_h(\mu) := g'(\langle \mu, h\rangle)\left\{\langle \mu, b(\cdot,\mu)\cdot\nabla h\rangle + \frac{1}{2}\langle \mu, a_{ij}(\cdot,\mu)\partial_{ij}h\rangle\right\}$$

Note that $\mathcal{A}^\infty(\mu)$ is a **nonlinear** differential operator

Outline of Proof of Mean-Field Limit (contd.)

Recall

- Step 1: Show $\{\bar{\mu}^n\}$ is tight, and
  Step 2: Every subsequential limit $\bar{\mu}$ of $\{\bar{\mu}^n\}$ satisfies

$$\mathcal{A}_h^{\infty}(\bar{\mu}) = 0, \quad \forall h \in \mathcal{C}_b(\mathbb{R}^d), \tag{1}$$

  where

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- Step 3: Show uniqueness of solutions to the (weak) PDE (29) to conclude that $\{\bar{\mu}^n\}$ converges to the unique (weak) solution $\bar{\mu}$ of the PDE (29).

Outline of Proof of Mean-Field Limit (contd.)

$$dX_t^{i,n} = B(X_t^{i,n}, \bar{\mu}_t^n)dt + \sigma(X_t^{i,n}, \bar{\mu}_t^n)dW_t^{i,n}, \qquad \bar{\mu}_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{X_t^{k,n}},$$

- Steps 1–3: We have shown that $\bar{\mu}^n \to \bar{\mu}$, the unique (weak) solution to

$$\mathcal{A}_h^\infty(\bar{\mu}) = 0, \quad \forall h \in \mathcal{C}_b(\mathbb{R}^d),$$

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- Steps 1–3: We have shown that $\bar{\mu}^n \to \bar{\mu}$, the unique (weak) solution to
$$\mathcal{A}_h^\infty(\bar{\mu}) = 0, \quad \forall h \in \mathcal{C}_b(\mathbb{R}^d),$$
- Step 4: We combine this with the above SDE for $X^{i,n}$ and the continuity of the map $m \mapsto B(x,m)$ to conclude that as $n \to \infty$, for any i, $X^{i,n}$ converges weakly to X, which is the unique solution to the (nonlinear or McKean-Vlasov) SDE:
$$dX_t = B(X_t, \bar{\mu}_t)dt + \sigma(X_t, \bar{\mu}_t)dW_t,$$
for some Brownian motion $(W_t)_{t\geq 0}$.
- Step 5: Noting that the PDE for μ is the forward Kolmogorov equation for X, we conclude that $\mathrm{Law}(X_t) = \mu_t$.

Nonlinear Markov Processes

Theorem (McKean '67, Oelschlager '84, Sznitman '91, etc.)

$(\bar{\mu}^n(t))_{t \in [0,T]}$ *converges in probability to the unique solution* $(\mu(t))_{t \in [0,T]}$ *of the McKean-Vlasov equation*

$$dX_t = B(X_t, \mu_t)dt + dW_t, \qquad \mu_t = \mathrm{Law}(X_t),$$

$X_0 \sim \lambda$. *Here,* μ_t *satisfies a nonlinear PDE that is the forward Kolmogorov equation for this inhomogeneous Markov process.*

An alternative approach: asymptotic independence

Recall

$$dX_t^{n,i} = B(X_t^{n,i}, \bar{\mu}_t^n)dt + dW_t^i, \qquad \bar{\mu}_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{X_t^{n,k}},$$

for $i = 1, \ldots, n$, where, for a probability measure m on $\mathbb{R}^d$,

$$B(x, m) = \int_{\mathbb{R}^d} b(x, y)\, m(dy),$$

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Theorem

$(\bar{\mu}^n(t))_{t\in[0,T]}$ *converges in probability to the unique solution* $(\mu_t)_{t\in[0,T]}$ *of the McKean-Vlasov equation*

$$dX_t = B(X_t, \mu_t)dt + dW_t, \qquad \mu_t = \mathrm{Law}(X_t).$$

5. Mean-Field Limits for Sequences of Dense Graphs

Key questions

Given a sequence of graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$, how can we describe the limiting behavior of a "typical" particle X_t^v?

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Theorem (Delattre-Giacomin-Luçon '16; Bhamidi-Budhiraja-Wu '19)

Under suitable conditions on the coefficients, suppose $G_n = G(n, p_n)$ is Erdős-Rényi, with $np_n \to \infty$. Then everything behaves like in the mean field case.

See also Delarue '17, Coppini, Dietert and Giacomin '18, Reis and Oliveira '18

Observation: $np_n \approx$ average degree, so $np_n \to \infty$ means the graphs are suitably dense.

The Main Focus of this Lecture Series

the sparse graph regime

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In this regime, there was not even an existing conjecture as to:

(i) whether it is possible to characterize the limiting dynamics of a typical particle
(ii) what form this characterization would take

6. Beyond Mean-Field Limits ...

$$X_{t+1}^v = F\left(X_t^v, (X_t^u)_{u\sim v}, \xi_{t+1}^v\right), v \in V,$$

$$dX_t^{G_n,v} = \frac{1}{d_v}\sum_{u\sim v} b(X_t^{G_n,v}, X_t^{G_n,u})dt + dW^v(t)$$

Key questions

Given a sequence of graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$, how can we describe the limiting behavior of...

- the state of a "typical" or fixed particle $X_t^{G_n,v}$?
- the empirical measure of particles $\mu_t^{G_n} := \frac{1}{|V_n|}\sum_{v\in V_n}\delta_{X_t^{G_n,v}}$?

Our focus: The sparse regime, where degrees do not diverge. How does the $n \to \infty$ limit reflect the graph structure?

Example: Erdős-Rényi $G(n, p_n)$ with $np_n \to p \in (0, \infty)$.
Open Question: in Delattre-Giacomin-Luçon

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Specific Questions:
(1) Does the whole system admit a scaling limit?
(2) Is there an autonomous description of the limiting dynamics of a typical particle?
(3) Does the (global) empirical measure

$$\bar{\mu}^{G_n} = \frac{1}{|V_n|}\sum_{v\in V_n} \delta_{X^{G_n,v}(\cdot)}$$

have a scaling limit?

Sequence of sparse graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$,

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(Q1) Does the whole system admit a scaling limit?
(A1) Yes, wrt a generalized notion of local weak convergence
In fact, for this, we can allow more general heterogeneous dynamics: e.g.,

$$X^v_{t+1} = F_v\left(t, X^v_t, (X^u_t)_{u\sim v}, \xi^v_{t+1}\right), \quad v \in V,$$

or

$$dX^{G_n,v}_t = \frac{1}{d_v}\sum_{u\sim v} b_v(t, X^{G_n,v}_t, X^{G_n,u}_t)dt + dW^v(t), \quad v \in V.$$