Scaling Limits of Interacting Particle Systems

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11th Cornell Probability Summer School
Cornell, Ithaca
Jun 17–20, 2019

Lecture 2

Convergence of Interacting Diffusions on Sparse Graph Sequences

Recap: Networks of locally interacting stochastic processes

We want to study locally interacting stochastic processes on a given finite connected graph $G = (V, E)$:

that evolve as a discrete-time Markov chain of the form:

$$X_{t+1}^{G,v} = F\left(X_t^{G,v}, \mu_t^v, \xi_{t+1}^v\right), \quad v \in V,$$

where $F : \mathcal{S} \times \mathcal{P}(\mathcal{S}) \times \mathcal{S}' \mapsto \mathcal{S}$,
or as a diffusion of the form

$$dX_t^{G,v} = b(X_t^{G,v}, \mu_t^v)dt + dW_t^v, \quad v \in V,$$

with $b : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \mapsto \mathbb{R}^d$,
where in both cases μ_t^v is the local empirical measure at v:

$$\mu_t^v = \frac{1}{d_v} \sum_{u \sim v} \delta_{X_t^u}, \quad v \in V.$$

Key Questions

Sequence of sparse (possibly random) graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$ with bounded (average) degree

$$X_{t+1}^{G_n,v} = F\left(X_t^{G_n,v}, \mu_t^v, \xi_{t+1}^v\right), \quad v \in V_n,$$

or

$$dX_t^{G_n,v} = b(X_t^{G_n,v}, \mu_t^v)dt + dW_t^v, \quad v \in V_n,$$

Specific Questions:

(1) Does the whole system admit a scaling limit?

(2) Is there an autonomous description of the limiting dynamics of a typical particle?

(3) Does the (global) empirical measure

$$\bar{\mu}^{G_n} = \frac{1}{|V_n|} \sum_{v \in V_n} \delta_{X^{G_n,v}(\cdot)}$$

have a scaling limit?

Networks of interacting stochastic processes

We will focus on
discrete-time Markov chains and diffusions
and consider the simplest setting to better illustrate the main ideas

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Lecture 2 focuses on the convergence question

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(Q1) Does the whole system admit a scaling limit?
(A1) Yes, wrt a small generalization of local weak convergence
In fact, for this, we can allow more general heterogeneous, time-dependent, dynamics: e.g.,

$$X_{t+1}^{G_n,v} = F_v^t\left(X_t^{G_n,v}, \mu_t^v, \xi_{t+1}^v\right), \quad v \in V_n$$

or

$$dX_t^{G_n,v} = b_v(t, X_t^{G_n,v}, \mu_t^v)dt + dW_t^v, \quad v \in V_n$$

Outline of Lecture 2

Part A: Question of Convergence

- Background on local weak convergence of graphs
- Examples of local weak convergence of graphs
- Background on local weak convergence of marked graphs
- Statement of convergence theorem
- Proof of convergence theorem

Part B: Question of Characterizing Marginals

History and References for local weak convergence

- Introduced by Benjamini and Schramm (2001)
- Further developed by Aldous-Steele in "The Objective Method: Probabilistic Combinatorial Optimization and Local Weak Convergence"
- Also, Aldous-Lyons "Processes on Unimodular Random Networks", EJP (2007)
- Charles Bordenave lecture notes:
- Also (especially for the marked case) see the Lacker-R-Wu paper "Large sparse networks of interacting diffusions'paper (especially the Appendix) https://arxiv.org/abs/1904.02585

1. Background on local weak convergence of graphs

◦ A connected rooted graph $G = (V, E,)$ is a graph (V, E) (assumed as usual to be locally finite with either finite or countable vertex set) with a distinguished vertex $\rho \in V$.

Definition: Isomorphisms of rooted graphs

We say two connected rooted graphs $G_i = (V_i, E_i, \rho_i)$ are *isomorphic* if there exists a bijection $\varphi : V_1 \to V_2$ such that $\varphi(\rho_1) = \rho_2$ and $(\varphi(u), \varphi(v)) \in E_2$ if and only if $(u, v) \in E_1$, for each $u, v \in V_1$. We denote this by $G_1 \cong G_2$. We refer to the map φ as an *isomorphism* from G_1 to G_2.

Let $I(G, G')$ denote the set of isomorphisms between two graphs $G, G' \in \mathcal{G}_*$

Let $\mathcal{G}_*$ denote the set of isomorphism classes of connected rooted graphs.

1. Background on local weak convergence of graphs

◦ Given $k \in \mathbb{N}$ and $G = (V, E, \rho) \in \mathcal{G}_*$, let $B_k(G)$ denote the induced subgraph of G consisting of those vertices whose graph distance from ρ is no more than k.

Definition of local weak convergence

We say that a sequence $(G_n \in \mathcal{G}_*)_{n\in\mathbb{N}}$ converges locally to $G \in \mathcal{G}_*$ if

$$\forall k \ \exists n_k \text{ s.t. } B_k(G) \cong B_k(G_n) \text{ for all } n \geq n_k,$$

where $\cong$ denotes isomorphism.

◦ There is a metric compatible with this notion of convergence that renders $\mathcal{G}_*$ a complete and separable space

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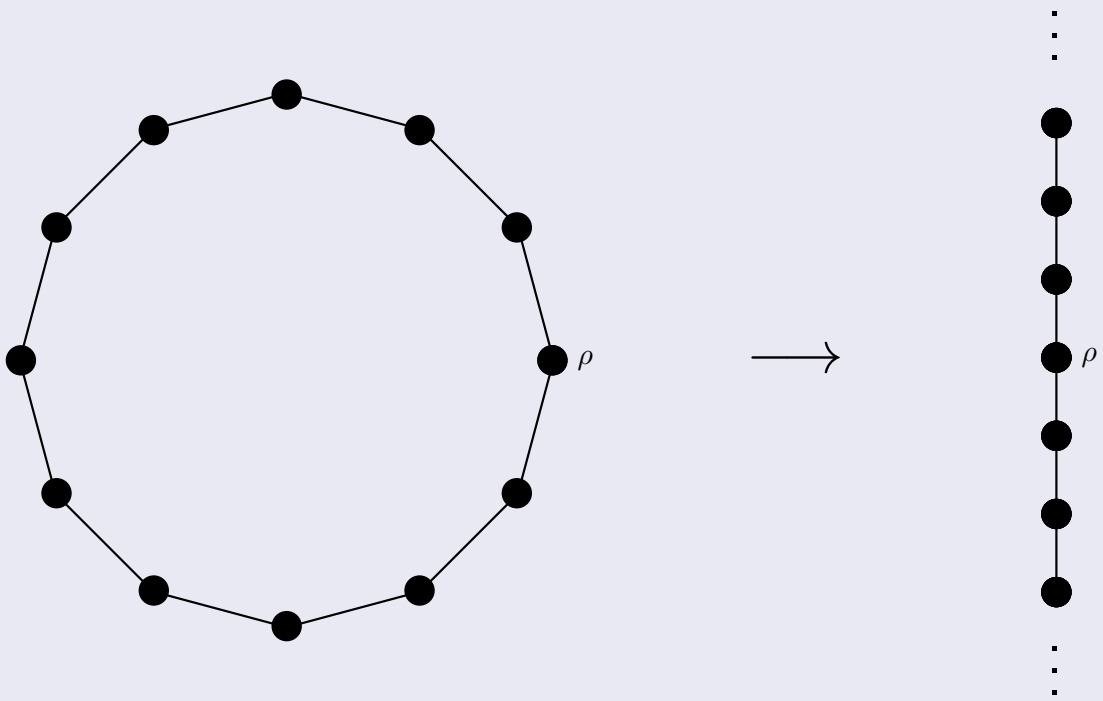
◦ There is a metric compatible with this notion of convergence that renders $\mathcal{G}_*$ a complete and separable space

◦ Thus we can talk of random $\mathcal{G}_*$-valued (rooted) graphs (or rather their isomorphism classes) and (usual) weak convergence of $G_n \in \mathcal{G}_*$ to a (possibly random) limit $G \in \mathcal{G}_*$.

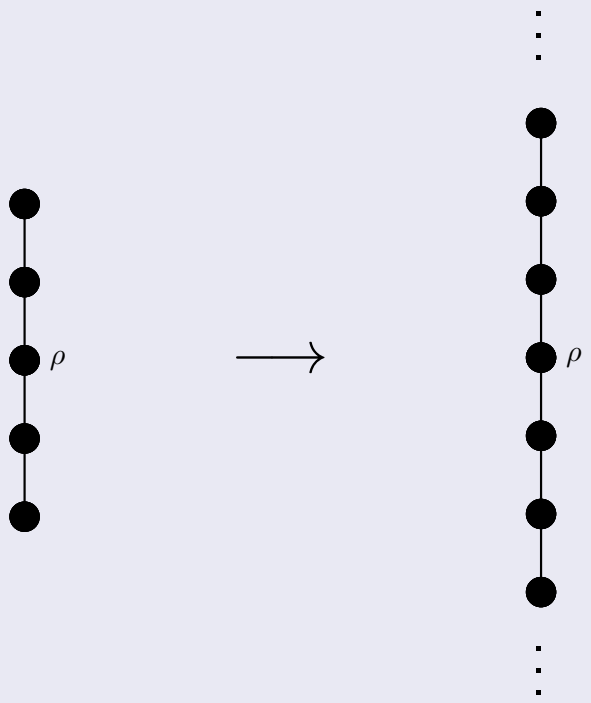
2. Examples of local weak convergence of graphs

1. Cycle graph converges to infinite line



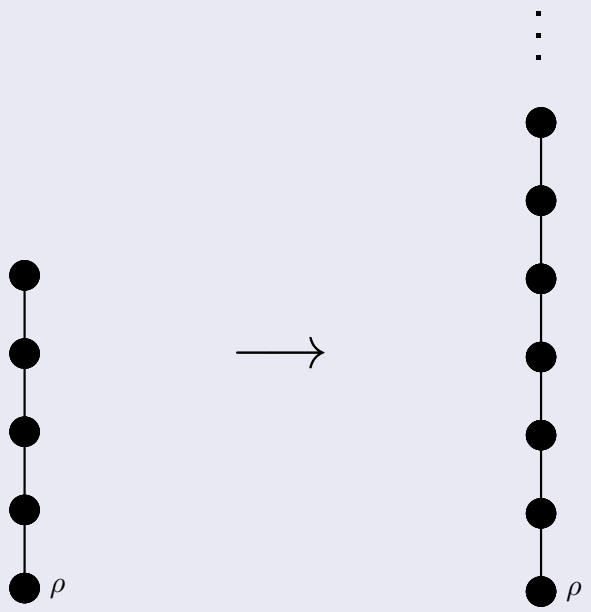
Examples of local weak convergence

2. Line graph converges to infinite line



Examples of local weak convergence

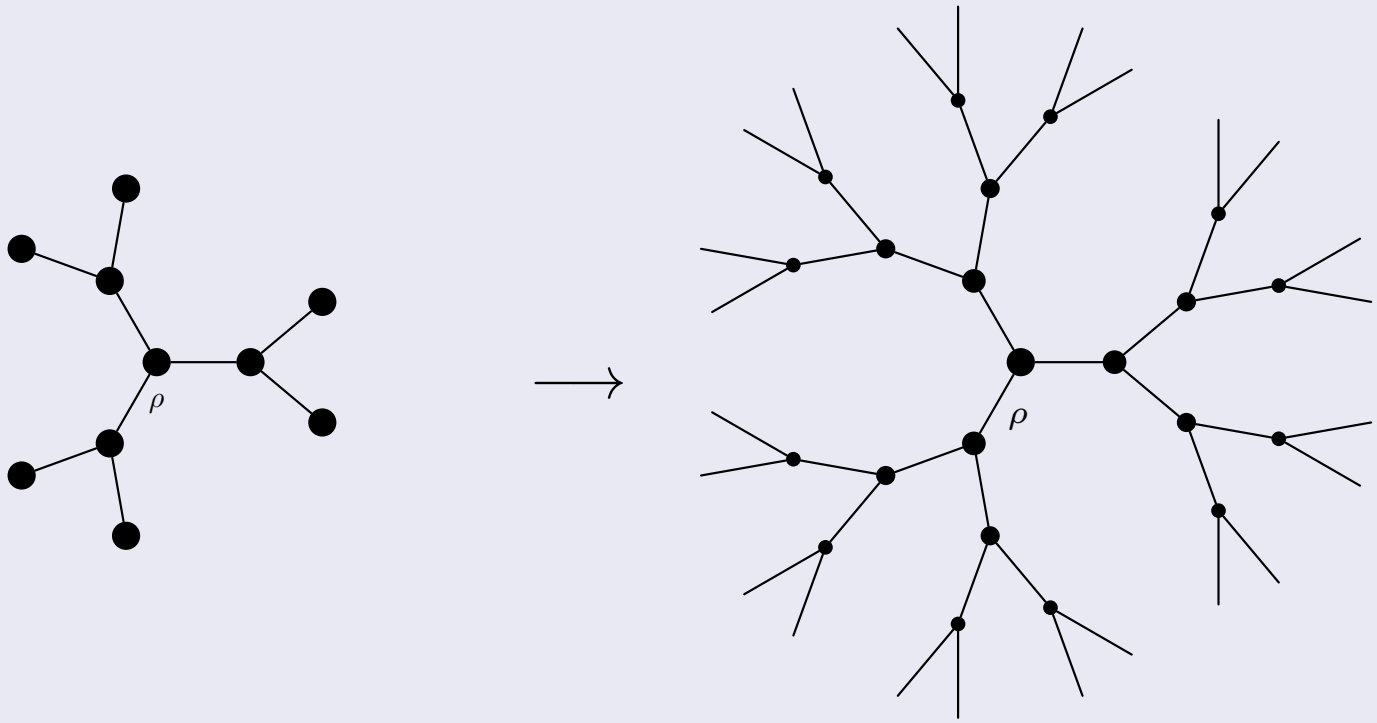
3. Line graph rooted at end converges to semi-infinite line



Examples of local weak convergence

4. Finite to infinite d-regular trees

(A graph is d-regular if ever vertex has degree d.)

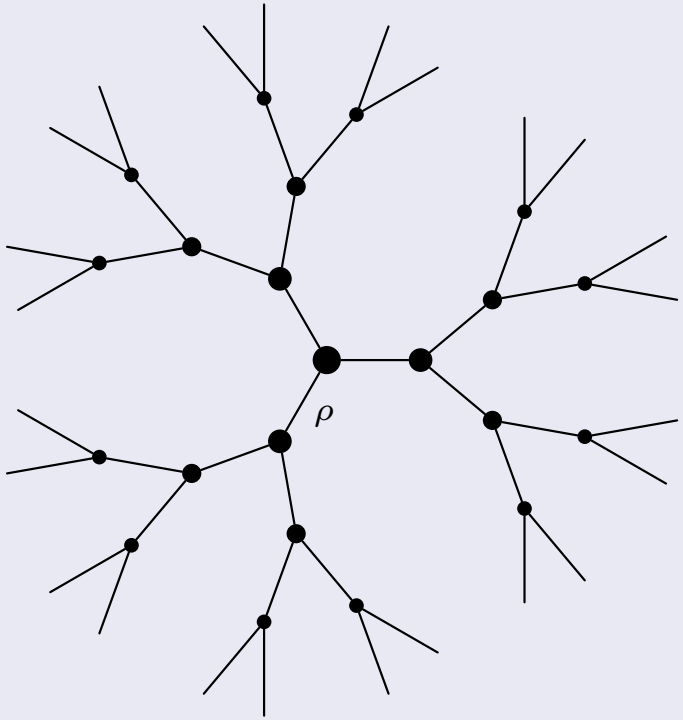


Examples of local weak convergence

5. Uniformly random regular graph to infinite regular tree

Fix d. Among all d-regular graphs on n vertices, select one uniformly at random. Place the root at a (uniformly) random vertex. When $n \to \infty$, this converges (in law) to the infinite d-regular tree. (McKay '81)

$\longrightarrow$



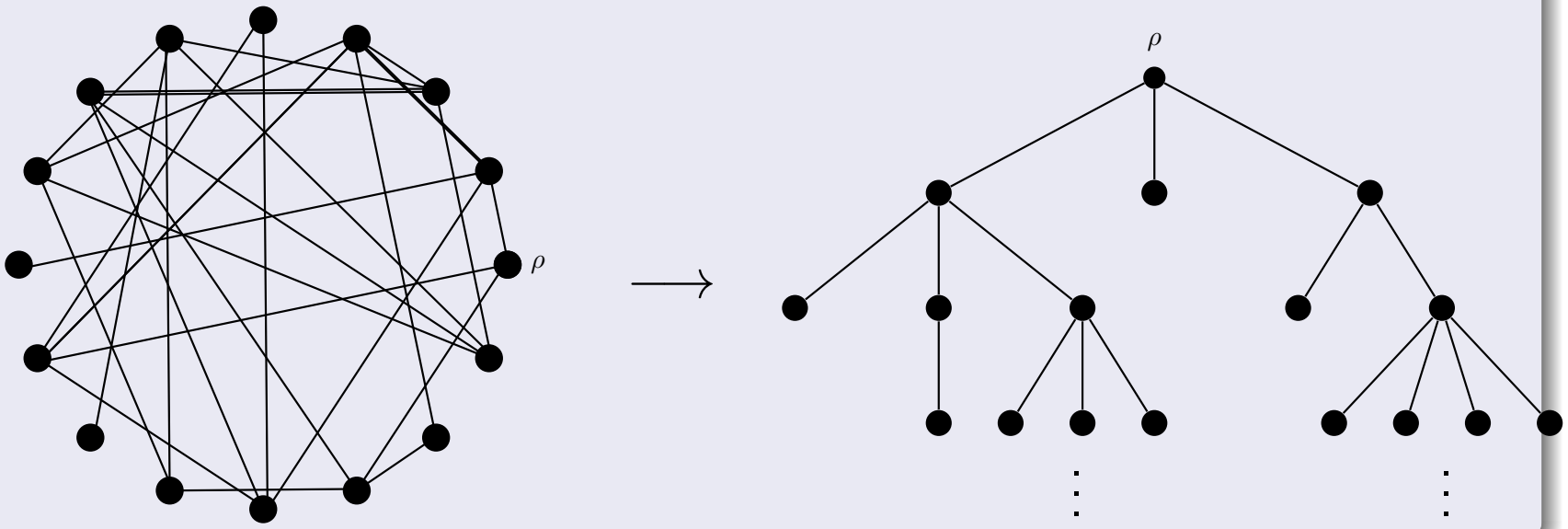
The random rooted graph $U(G)$

Given a finite (possibly disconnected graph) G we write $U(G)$ for the random connected rooted graph obtained by assigning a root uniformly at random and then isolating the connected component containing the root.

Examples of local weak convergence

6. Erdős-Rényi to Galton-Watson

If $G_n = G(n, p_n)$ with $np_n \to p \in (0, \infty)$, then both G_n (with a root assigned uniformly at random) and $U(G_n)$ converge in law to GW(Poisson (p)), the Galton-Watson tree with offspring distribution $\mathrm{Poisson}(p)$.



Examples of Local weak convergence

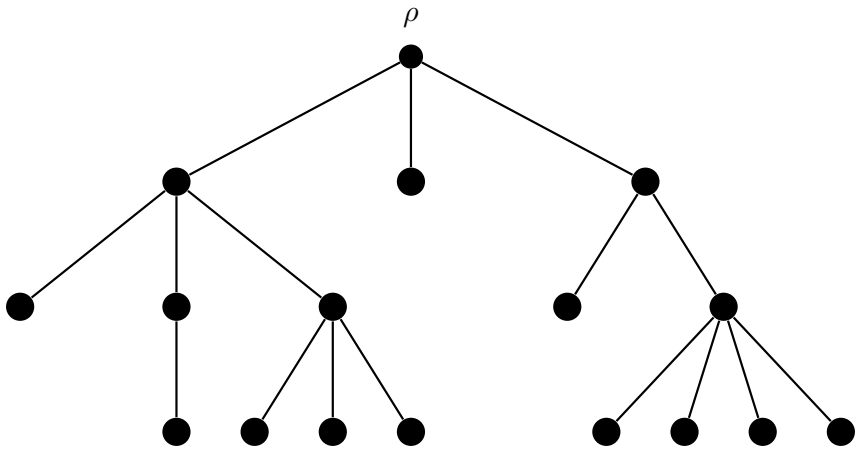
7. Configuration Model

Let G_n be drawn from the configuration model with (empirical) degree distribution converging to some distribution π on $\mathbb{N}_0$, with finite nonzero first moment. Then $U(G_n)$ converges locally in law to the *augmented* or unimodular Galton-Watson tree with offspring distribution π, denoted UGW (π).

Unimodular Galton Watson Tree

The unimodular Galton-Watson tree UGW (π). is a random tree defined so that the root has offspring distribution π and each of its children has offspring distribution $\widehat{\pi}$, where

$$\widehat{\pi}(k) = \frac{(k+1)\pi(k+1)}{\sum_{n\in\mathbb{N}} n\pi(n)}, \quad k \in \mathbb{N}_0.$$



Examples of Local weak convergence

8. Preferential Attachment Graphs to a Random Tree

A result by Berger-Borgs-Chayes-Saberi ('14) shows convergence of preferential attachment graphs to a random tree

Key Point

Local limits of many classes of random graphs are often trees

For references for the different limit theorems, see

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Of course, deterministic graphs could have local limits of a different nature

9. Convergence of Finite Lattices

$\mathbb{Z}^\kappa \cap [-n, n]^\kappa$ converges to $\mathbb{Z}^\kappa$

Convergence of Interacting Processes on Sparse Graphs

But we want to understand convergence of dynamics on graphs, i.e., as $|V_n| \to \infty$, understand behavior of

$$X_{t+1}^{G_n,v} = F\left(X_t^{G_n,v}, \mu_t^v, \xi_{t+1}^v\right), \quad v \in V_n$$

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• We can think of the trajectory $X^{G_n,v} = \{X_t^{n,v}, t \in \mathbb{T}\}$ for $\mathbb{T} = \mathbb{N}_0$ or $\mathbb{T} = [0, \infty)$, as a **(random) mark** associated to the vertex v

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- the mark lies in $\mathcal{X}$, where $\mathcal{X}$ is either $\mathcal{S}^{\mathbb{N}_0}$ or $\mathcal{C} := \mathcal{C}([0,\infty) : \mathbb{R}^d)$
- For any graph G, let $\boldsymbol{X}^G := \{X^{G,v}, v \in V\}$

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- For any graph G, let $\boldsymbol{X}^G := \{X^{G,v}, v \in V\}$

Leads us to consider convergence of marked graphs $(G_n, \boldsymbol{X}^{G_n})$

Local weak convergence for marked graphs

Let $(\mathcal{X}, d_{\mathcal{X}})$ be a metric space, then a marked (rooted connected) graph is a pair $(G, \boldsymbol{x})$ such that $G = (V, E, \rho)$ is rooted connected graph and a vector of marks $\boldsymbol{x} = (x_v)_{v \in V} \in \mathcal{X}^V$.

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Definition: Isomorphims of marked Graphs

We say that two marked graphs $(G, \boldsymbol{x})$ and $(G', \boldsymbol{x}')$ are *isomorphic* if there exists an isomorphism φ from G to G' such that $(x_v)_{v \in V} = (x'_{\varphi(v)})_{v \in V}$. We write $(G, \boldsymbol{x}) \cong (G', \boldsymbol{x}')$ to indicate isomorphism. We let $\mathcal{G}_*[\mathcal{X}]$ refer to the set of isomorphism classes of marked graphs.

Local convergence of marked graphs

Definition: Local convergence of marked graphs

We say that a sequence $(G_n, \boldsymbol{x}^n) \in \mathcal{G}_*[\mathcal{X}]$ converges locally to $(G, \boldsymbol{x}) \in \mathcal{G}_*[\mathcal{X}]$ if, for every $k \in \mathbb{N}$ and $\epsilon > 0$, there exist $n_k \in \mathbb{N}$ such that for all $n \geq n_k$, there exists an isomorphism $\varphi : B_k(G_n) \to B_k(G)$ with $\max_{v \in B_k(G_n)} d_{\mathcal{X}}(\boldsymbol{x}_v^n, \boldsymbol{x}_{\varphi(v)}) < \epsilon$, where $B_k(G)$ (respy, $B_k(G_n)$) is the set of vertices in G (respy, G_n) that are at most distance k from the root ρ.

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The space $\mathcal{G}_*[\mathcal{X}]$ can be equipped with the metric

$$d_*((G, \boldsymbol{x}), (G', \boldsymbol{x}')) = \sum_{k=1}^{\infty} \frac{1}{2^k} \left(1 \wedge \inf_{\varphi \in I(B_k(G), B_k(G'))} \max_{v \in B_k(G)} d_{\mathcal{X}}(x_v, x'_{\varphi(v)}) \right)$$

where recall $I(G, G')$ denotes the set of isomorphisms between two graphs $G, G' \in \mathcal{G}_*$.

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where recall $I(G, G')$ denotes the set of isomorphisms between two graphs $G, G' \in \mathcal{G}_*$.

Exercise: Show that if $(\mathcal{X}, d_{\mathcal{X}})$ is complete and separable then so is $(\mathcal{G}_*[\mathcal{X}], d_*)$.

A property of local convergence of marked graphs

Lemma

Suppose $G_n \to G$ in $\mathcal{G}_*$. Suppose, for each n, $\boldsymbol{X}^{G_n} = (X^{v,G_n})_{v \in G_n}$ is a $\mathcal{X}^{G_n}$-valued random variable. Suppose finally that for each k, the family

$$\{X^{v,G_n} : n \in \mathbb{N},\ v \in B_k(G_n)\}$$

of $\mathcal{X}$-valued random variables is tight. Then $(G_n, \boldsymbol{X}^{G_n})$ is tight in $\mathcal{G}_*[\mathcal{X}]$.

Convergence Results: Discrete-time Markov Chains

- Given a marked graph $(G, \boldsymbol{x}) \in \mathcal{G}_*[\mathcal{S}]$, consider the chain

$$X_{t+1}^{G,v} = F_t\left(X_t^{G,v}, \mu_t^{G,v}, \xi_{t+1}^{G,v}\right), \quad v \in V, t \in \mathbb{N}_0,$$

with initial condition $\boldsymbol{X}_0^G = \boldsymbol{x}$. Denote soln. with init cond. as $\boldsymbol{X}^{G,\boldsymbol{x}}$

Assumptions (Lacker-R-Wu '19)

We assume the following conditions are satisfied:

1. For each (possibly disconnected) graph G, $\{\xi_t^v : v \in G, t \in \mathbb{N}\}$ are iid $\mathcal{S}'$-valued with common law $\theta \in \mathcal{P}(\mathcal{S})$.
2. $F_t : \mathcal{S} \times \mathcal{P}(\mathcal{S}) \times \mathcal{S}' \mapsto \mathcal{S}$ is continuous for each $t \in \mathbb{N}_0$

- Let $P^{G,\boldsymbol{x}} = \mathrm{Law}(\boldsymbol{X}^{G,\boldsymbol{x}}) \in \mathcal{P}((\mathcal{S}^\infty)^V)$

Convergence Results: Discrete-time Markov Chains

Now consider the sequence $\boldsymbol{X}^{G_n} = \{X^{G_n,v}, v \in G_n\}$ where

$$X_{t+1}^{G_n,v} = F_t\left(X_t^{G_n,v}, \mu_t^{G_n,v}, \xi_{t+1}^{G_n,v}\right), \quad v \in V_n, t \in \mathbb{N},$$

with initial condition $\boldsymbol{X}_0^{G_n} = \boldsymbol{x}^n$,

Theorem (Lacker-R-Wu '18)

Under the stated assumptions

1. the map $(G, \boldsymbol{x}) \mapsto P^{G,\boldsymbol{x}}$ is continuous
2. as $n \to \infty$, if $(G_n, \boldsymbol{x}_n) \to (G, \boldsymbol{x})$ in $\mathcal{G}_*[\mathcal{S}]$, then $(G_n, \boldsymbol{X}^{G_n}, \boldsymbol{x}^n) \Rightarrow (G, \boldsymbol{X}^{G,\boldsymbol{x}})$.

In particular, the trajectory of the root particle $X^{G_n,\rho}$ converges in law in $\mathcal{S}^\infty$ to $X^{G,\rho}$, the trajectory of the root particle of the limiting graph.

Convergence Results: Discrete-time Markov Chains

$\boldsymbol{X}^{G,\boldsymbol{x}}$ is the unique solution $\boldsymbol{X}^{G} = \{X^{G,v}, v \in V\}$, where

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with initial condition $\boldsymbol{X}_0^{G} = \boldsymbol{x} \in \mathcal{S}^V$.
Note $(G, \boldsymbol{x}) \in \mathcal{G}_*[\mathcal{S}]$

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Note $(G, \boldsymbol{x}) \in \mathcal{G}_*[\mathcal{S}]$ and $(G, \boldsymbol{X}^{G,\boldsymbol{x}}) \in \mathcal{G}_*[\mathcal{S}^\infty]$

Basic Idea of Proof

Showing that the map $(G, \boldsymbol{x}) \mapsto (G, \boldsymbol{X}^{G,\boldsymbol{x}})$ from $\mathcal{G}_*[\mathcal{S}]$ to $\mathcal{G}_*[\mathcal{S}^\infty]$ is continuous is equivalent to showing that for every $t \in \mathbb{N}_0$ and bounded and continuous map φ on $\mathcal{G}_*[\mathcal{S}^{k+1}]$, the function

$$\psi(G, \boldsymbol{x}) = \mathbb{E}[\varphi(G, \boldsymbol{X}_t^{G,\boldsymbol{x}})].$$

is continuous on $\mathcal{G}_*[\mathcal{S}]$.

Outline of Proof of Convergence for Markov Chains

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- Fix $r \in \mathbb{N}$. It suffices to verify that the following function is continuous:

$$\psi_r(G, \boldsymbol{x}) := \varphi(B_r(G), \boldsymbol{X}_t^{B_r(G), \boldsymbol{x}_{B_r(G)}}).$$

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- The conclusion then follows from the fact that $(G_n, \boldsymbol{x}^{\boldsymbol{n}}) \to (G, \boldsymbol{x})$ in $\mathcal{G}_*[\mathcal{X}]$ implies that the projections of the initial conditions converge: $\boldsymbol{x}^{\boldsymbol{n}}_{\varphi(B_{r+t}(G^n))} \to \boldsymbol{x}_{B_{r+t}(G)}$

Wasserstein Metric

Definition

Let $\mathcal{P}_1(\mathbb{R}^d)$ be the subspace of measures $\mu \in \mathcal{P}(\mathbb{R}^d)$ that satisfy

$$\int_{\mathcal{X}} |y| \mu(dy) < \infty.$$

Then a sequence of measures $\mu_n \in \mathcal{P}_1(\mathbb{R}^d)$ is said to converge to $\mu \in \mathcal{P}_1(\mathbb{R}^d)$ in the Wasserstein metric if μ_n converges to μ weakly and the first moments also converge:

$$\int_{\mathbb{R}^d} |y| \mu_n(dy) \to \int_{\mathbb{R}^d} |y| \mu(dy).$$

Note: $\mathcal{P}_1(\mathbb{R}^d)$, equipped with the Wasserstein-1 metric, is Polish.

Convergence Results: Diffusion Setting

$\boldsymbol{X}^G = (\boldsymbol{X}^{G,v})_{v \in V}$, where

$$dX_t^{G_n,v} = b(X_t^{G_n,v}, \mu_t^{G_n,v})dt + dW_t^v, \quad v \in V,$$

with initial condition $\boldsymbol{X}_0^{G_n} = \boldsymbol{x}^n$, and

$$\mu_t^{G_n,v} = \frac{1}{d_v} \sum_{u \sim v} \delta_{X_t^{G_n,u}}, \quad v \in V.$$

Assumptions in the Diffusion Setting

Suppose the following properties hold:

1. The initial conditions are iid with common distribution θ that has a finite second moment.
2. The drift function $b : \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \mapsto \mathbb{R}^d$ is Lipschitz continuous and has linear growth:

$$b(x, m) \leq C(1 + |x| + \int |y| m(dy).$$

Convergence Results: Interacting Diffusions

$\boldsymbol{X}^G = (X^{G,v})_{v \in V}$, where

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with initial condition $\boldsymbol{X}_0^{G_n} = \boldsymbol{x}^n$, where $b : \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \mapsto \mathbb{R}^d$, and

$$\mu_t^{G_n,v} = \frac{1}{d_v} \sum_{u \sim v} \delta_{X_t^{G_n,u}}, \quad v \in V.$$

Theorem (Lacker-R-Wu '19)

Under the stated assumptions, there is a unique strong solution to the SDE and and as $n \to \infty$, if $(G_n, \boldsymbol{x}^n) \to (G, \boldsymbol{x})$ in $\mathcal{G}_*[\mathcal{S}]$, then $(G_n, \boldsymbol{X}^{G_n,\boldsymbol{x}^n}) \Rightarrow (G, \boldsymbol{X}^{G,\boldsymbol{x}})$.

Diffusion Convergence Results: Outline of Proof

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1. Use stochastic calculus to show that under the assumption, for every $T < \infty$,

$$\sup_{G \in \mathcal{G}_*} \sup_{v \in G} \mathbb{E}\left[\sup_{t \in [0,T]} |X_t^{G,v}|^2\right] < \infty.$$

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2. Apply Aldous' tightness criterion for measures in $\mathcal{P}(\mathcal{C})$, where $\mathcal{C}$ is the space of $\mathbb{R}^d$-valued continuous functions on $[0, \infty)$, to show that the family of $\mathcal{C}$-valued random variables $\{X^{G,v}, G \in \mathcal{G}_*, v \in G\}$ is tight.

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3. Use above lemma on local convergence of marked graphs to conclude that (G_n, X^{G_n}) is tight as a sequence of $\mathcal{G}_*[\mathcal{C}]$ random elements.
4. Use stochastic calculus to show that the limit is a (weak) solution of the limit SDE (with G_n replaced with G), and

Part B of Lecture 2: Local Characterization of Dynamics

We have shown that $X^{G_n,v}$ can be approximated by a dynamical system on an infinite graph G:

$$X_{t+1}^{G,v} = F\left(X_t^{G,v}, \mu_t^v, \xi_{t+1}^v\right), \quad v \in V,$$

where $F : \mathcal{S} \times \mathcal{P}(\mathcal{S}) \times \mathcal{S}' \mapsto \mathcal{S}$,
or as a diffusion of the form

$$dX_t^{G,v} = b(X_t^{G,v}, \mu_t^v)dt + dW_t^v, \quad v \in V,$$

with $b : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \mapsto \mathbb{R}^d$,
where in both cases μ_t^v is the local empirical measure at v:

$$\mu_t^v = \frac{1}{d_v} \sum_{u \sim v} \delta_{X_t^u}, \quad v \in V.$$

Specific Questions:

(2) Is there an autonomous description of the limiting dynamics of a typical particle?

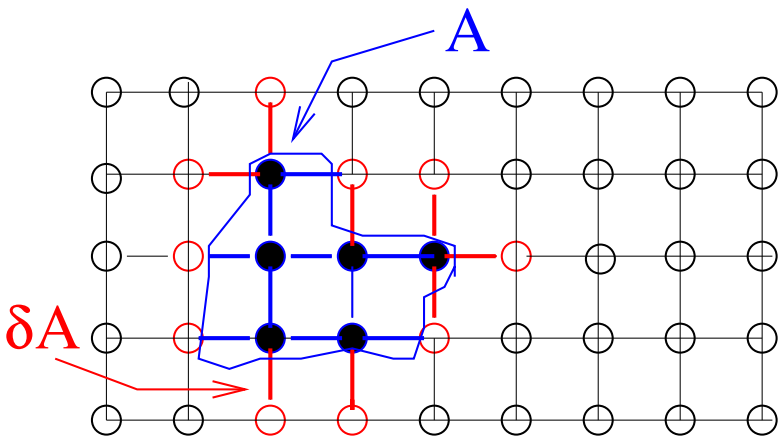
A Well-Studied Static Analog: Markov Random Fields

Notation: For a set A of vertices in a graph $G = (V, E)$, define

$$\text{Boundary: } \partial A = \{u \in V \backslash A : \exists u \in A \text{ s.t. } u \sim v\},$$

State space $\mathbb{S}$; $\{Y^v, v \in V\}$ canonical variables acting on $\mathbb{S}^V$
Defn. A probability measure π on $\mathbb{S}^V$ is said to be a Markov Random Field if for π a.e. η_A,

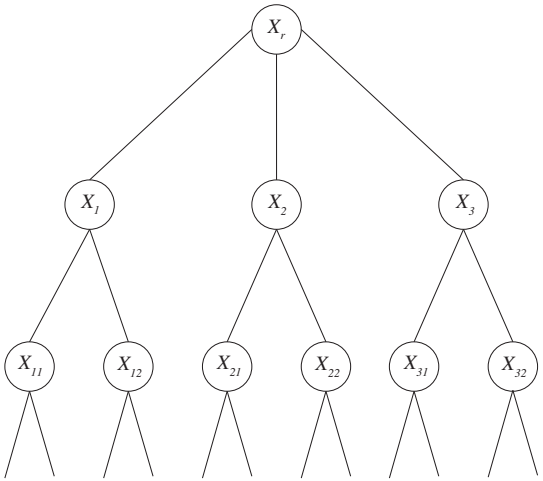
$$\pi\left(Y^A = \eta^A | Y^{V\backslash A} = \eta^{V\backslash A}\right) = \pi\left(Y^A = \eta^A | Y^{\partial A} = \eta^{\partial A}\right)$$



Markov Random Fields on Trees

An Equivalent Formulation: $(Y^v)_{v\in V}$ is a Markov random field wrt $G=(V,E)$ if for finite $A\subset V$, $B\subset V\setminus[A\cup\partial A]$,

$$(Y^v)_{v\in A}\perp(Y^v)_{v\in B}\ \mid\ (Y^v)_{v\in\partial A},$$

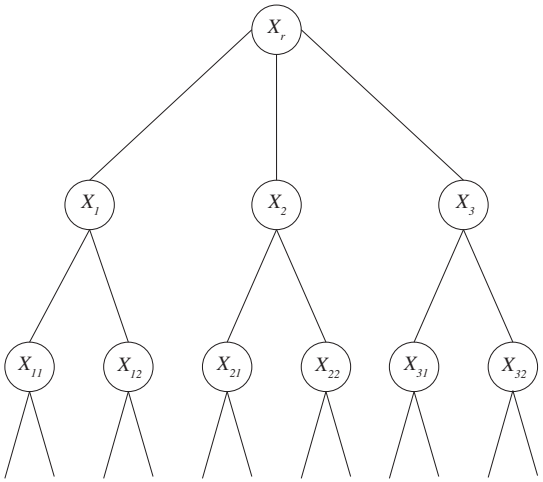


• Tree structure allows one to analyze the marginal distribution of MRF at a node.

Markov Random Fields

Markov Random Fields or Gibbs measures on Trees are much easier to analyze on trees
e.g. Ising model on the tree

$$(Y_v)_{v\in A} \perp (Y_v)_{v\in B} \mid (Y_v)_{v\in \partial A},$$



Use the implied MRF conditional independence structure to create a recursion, and analyze that recursion to compute the marginal at

In Search of a Conditional Independence Property

$$X_{t+1}^{G,v} = F\left(X_t^{G,v}, \mu_t^v, \xi_{t+1}^v\right), \quad v \in V,$$

$$dX_t^{G,v} = b(X_t^{G,v}, \mu_t^v)dt + dW_t^v, \quad v \in V,$$

Question A:

Is the MRF property retained: that is, if $(X_0^v)_{v\in V}$ are iid, at each time t will $(X_t^v)_{v\in V}$ form a Markov random field?

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Answer A

No!

In Search of a Conditional Independence Property

$$dX_t^{G,v} = b(X_t^{G,v}, \mu_t^v)dt + dW_t^v, \quad v \in V,$$

Question B:

Does the following space-time Markov-random field property hold:
For each time t, if $(X_0^v)_{v \in V}$ are iid, do the particle trajectories $(X^v[t])_{v \in V}$ form a Markov random field?
Here, $x[t] = (x(s), s \in [0, t])$.

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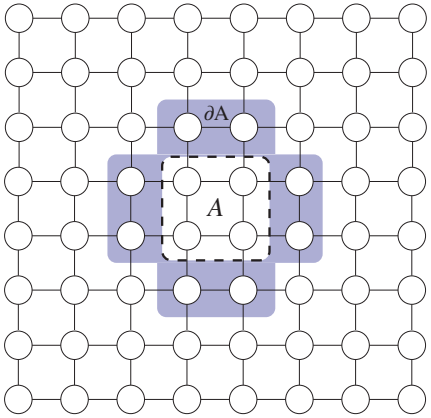

Reformulation of Question B:

Given $t > 0$, if $(X_0^v)_{v \in V}$ are iid, for any finite $A \subset V$, is

$$\mathbf{X^A[t] \perp X^{V \setminus [A \cup \partial A]}[t] | X^{\partial A}[t]}?$$

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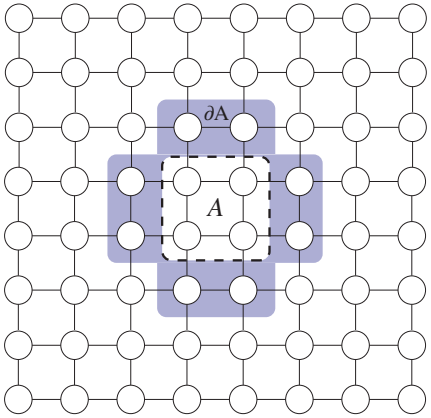

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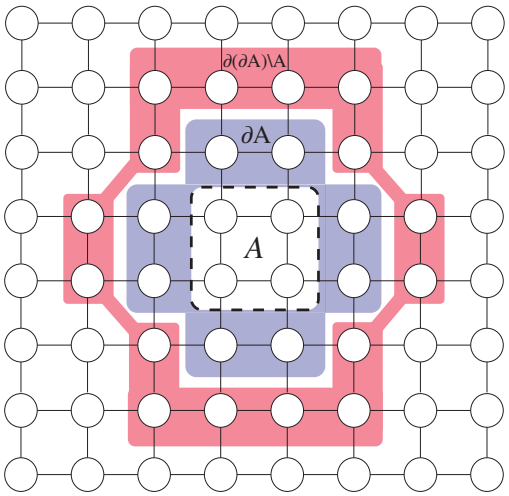
Answer B

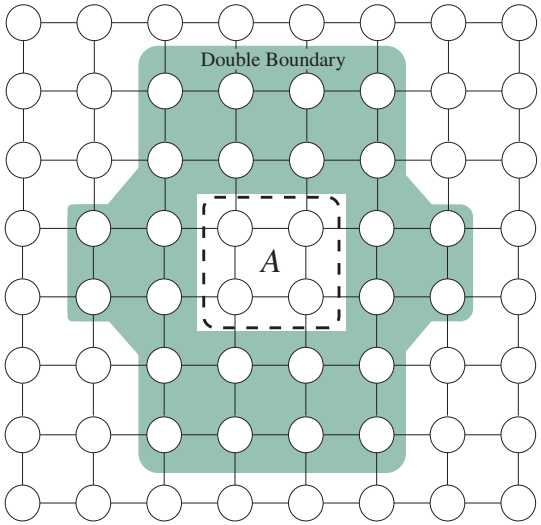
No!

Second-order Markov Random Fields

Double Boundary

$$\partial^2 A = \partial A \cup [\partial(\partial A) \setminus A]$$





Definition: A family of random variables $(Y^v)_{v \in V}$ is a 2nd-order Markov random field if

$$(Y^v)_{v \in A} \perp (Y^v)_{v \in B} \mid (Y^v)_{v \in \partial^2 A},$$

for all finite sets $A, B \subset V$ with $B \cap (A \cup \partial^2 A) = \emptyset$.

In Search of a Conditional Independence Property

$$dX_t^{G,v} = b(X_t^{G,v}, \mu_t^v)dt + dW_t^v, \quad v \in V,$$

Question C:

If $(X_0^v)_{v\in V}$ are iid, at each time t, do the states $(X^v(t))_{v\in V}$ form a second-order Markov random field?

$$(X_t^v)_{v\in A} \perp (X_t^v)_{v\in B} \mid (X_t^v)_{v\in \partial^2 A},$$

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for all finite sets $A, B \subset V$ with $B \cap (A \cup \partial^2 A) = \emptyset$.

Answer C:

NO

At Last A Conditional Independence Property

$$dX^v(t) = \frac{1}{d_v} \sum_{u \sim v} b(X_t^v, X_t^u)dt + dW_t^v, \quad (X_0^v)_{v \in V} \text{ i.i.d.}$$

Question D:

If $(X_0^v)_{v \in V}$ are iid, at each time t, do the particle trajectories $(X^v[t])_{v \in V}$ form a second-order Markov random field?

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Theorem 4: (Lacker, R, Wu '17) YES!

$$(X^v[t])_{v\in A} \perp (X^v[t])_{v\in B} \mid (X^v[t])_{v\in \partial^2 A},$$

for all $A \subset V$ finite and $B \subset V$ with $B \cap (A \cup \partial^2 A) = \emptyset$.

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In fact, suffices for $(X_0^v)_{v\in V}$ to form a second-order MRF.

Comments on the Conditional Independence Property

- Relevant to work on Gibbs-non-Gibbs transitions – mostly studied when $G = \mathbb{Z}^m$ (e.g., Dereudre-Roelly '05; Redig-Roelly-Ruszel '10)

Comments on the Conditional Independence Property

- Relevant to work on Gibbs-non-Gibbs transitions – mostly studied when $G = \mathbb{Z}^m$ (e.g., Dereudre-Roelly '05; Redig-Roelly-Ruszel '10)
- We also need an extension to the case when A is infinite:

Theorem (Lacker, R, Wu '18)

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for all $A \subset V$, $B \subset V$ with $B \cap (A \cup \partial^2 A) = \emptyset$.

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