Scaling Limits of Interacting Particle Systems

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Lecture 3

Characterization of Local Dynamics on Large Sparse Networks

Recall the Conditional Independence Property we Identified

$$X_{t+1}^{G,v} = F\left(X_t^{G,v}, \mu_t^{G,v}, \xi_{t+1}^v\right), \quad v \in V,$$

$$dX_t^{G,v} = b(X_t^{G,v}, \mu_t^{G,v})dt + dW_t^v, \quad v \in V,$$

Question D:

Denoting $X = X^G$, given iid initial conditions $(X^v(0))_{v \in V}$, at each time t, do the particle trajectories $(X_v[t])_{v \in V}$ form a second-order Markov random field?

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Theorem 4: (Lacker, R, Wu '17) YES!

$$(X_v[t])_{v\in A} \perp (X_v[t])_{v\in B} \mid (X_v[t])_{v\in \partial^2 A},$$

for all $A \subset V$ finite and $B \subset V$ with $B \cap (A \cup \partial^2 A) = \emptyset$.

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When $(X_v(0))_{v\in V}$ are iid.

Comments on the Conditional Independence Property

- Relevant to work on Gibbs-non-Gibbs transitions of interacting diffusions – mostly studied when $G = \mathbb{Z}^m$
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- For systems with gradient drifts, a related property was established by Deuschel ('87), but the proof crucially used the gradient structure
- We also need an extension to the case when A is infinite:

Theorem (Lacker, R, Wu '18)

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for all $A \subset V$, $B \subset V$ with $B \cap (A \cup \partial^2 A) = \emptyset$.

In fact, suffices for $(X_v(0))_{v \in V}$ to form a second-order MRF.

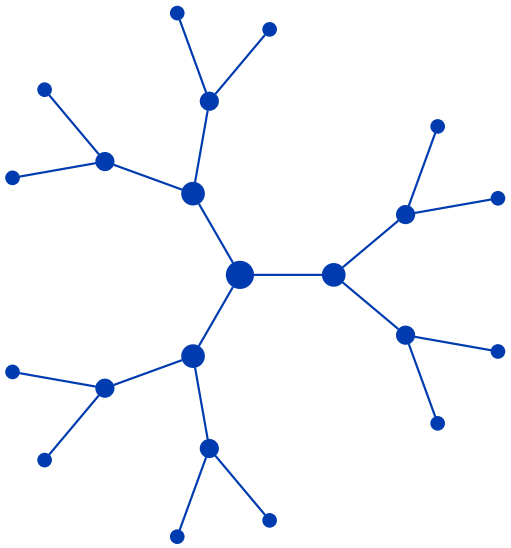
Marginal Dynamics

Is the conditional independence property of any use?

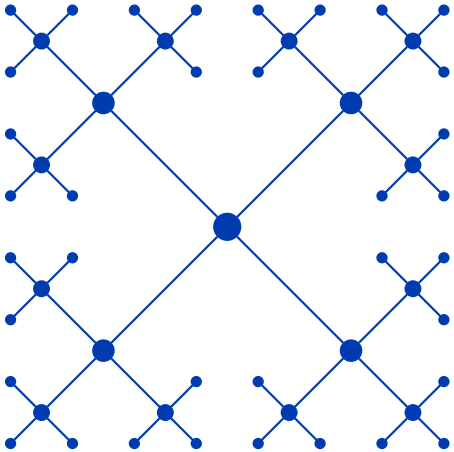
Taking inspiration from the "static" case, let us start by considering trees

Marginal Dynamics on Trees

Suppose the limiting graph G is an infinite d-regular tree.



$d = 3$



$d = 4$

Marginal Dynamics on Trees

- Let $\mathcal{T}_\kappa$ denote the infinite κ-regular tree.
- For simplicity consider the case $\kappa = 2$.
- Note that Theorem 1 implies that for a typical vertex ρ, $\{X_\rho, (X_v)_{v\sim\rho}\}$ can be obtained as the marginal of the infinite coupled system of Markov chains:

$$X_{t+1}^{G,v} = F\left(X_t^{G,v}, \mu_t^{G,v}, \xi_{t+1}^v\right), \quad v \in V_n,$$

  with, as usual, $\mu_t^{G,v} = \frac{1}{d_v^G}\sum_{u\sim v}\delta_{X_t^u}$.
- Identify $\mathcal{T}_2$ with $\mathbb{Z}$, set $\rho = 0$, and denote $X := X^G$.
- Then we are interested in an autonomous characterization of the marginal law of

$$X^{-1,0,1} = (X^{-1}, X^0, X^1).$$

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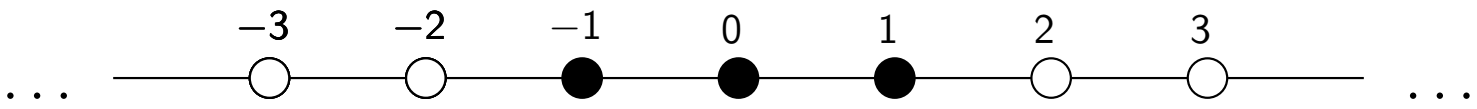
How can we exploit the conditional independence structure?

Marginal dynamics for the infinite 2-regular tree



For convenience, we represent the dependence of the dynamics on the neighboring particles instead of on the empirical measures, but keep in mind the dependence is symmetric and still denoting the transition map by F, we can write

$$X_{t+1}^i = F\left(X_t^i, (X_t^{i-1}, X_t^{i+1}), \xi_{t+1}^i\right), \quad i \in \mathbb{Z}.$$

- We will now describe how to generate another stochastic process $Y^{-1,0,1}$ that has the same law as the marginal process $X^{-1,0,1}$, but whose evolution only depends on the history of its state, and the law of the history of the state, or equivalently, the law of the marginal process $X^{-1,0,1}$.

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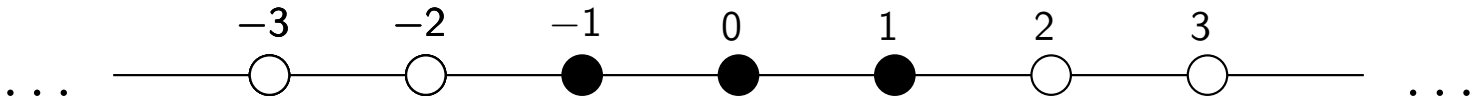

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- First, note that the evolution of the (law of the) middle particle only depends on the states of the neighboring particles, so its evolution would remain as before:

$$Y_{t+1}^0 = F\left(Y_t^0, (Y_t^{-1}, Y_t^1), \xi_{t+1}^i\right).$$

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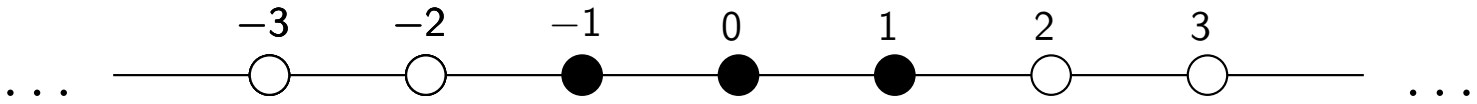

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- However, the evolution of (the law of) the state of one of the other particles, say X^{-1}, depends on (the law of) $X_t^0 \overset{(d)}{=} Y_t^0$, *as well as* on (the joint law with) X_t^{-2}.

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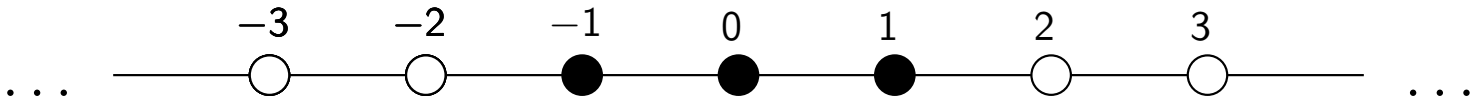

$$X^i_{t+1} = F\left(X^i_t, (X^{i-1}_t, X^{i+1}_t), \xi^i_{t+1}\right), \quad i \in \mathbb{Z}$$

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- But the joint law of $X^{-2,-1,0,1}_t$ cannot be obtained from the $Y^{-1,0,1}$ dynamics ...

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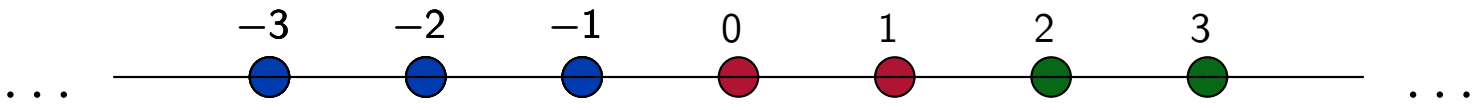

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- **Key observation 1:** We assume we have access to the states and law of $Y^{-1,0,1}$, and so it suffices to know the conditional law of Y^{-2}_t, given the past $Y^{-1,0,1}[t]$.

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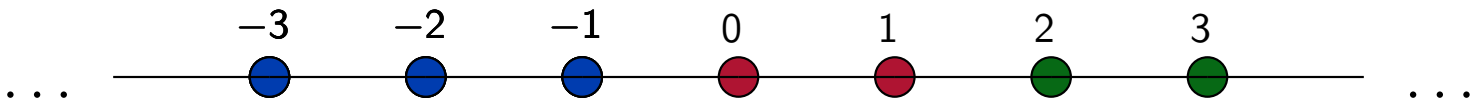

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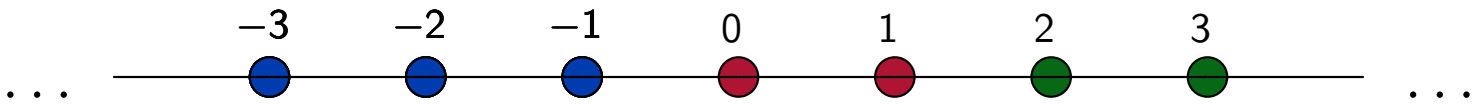

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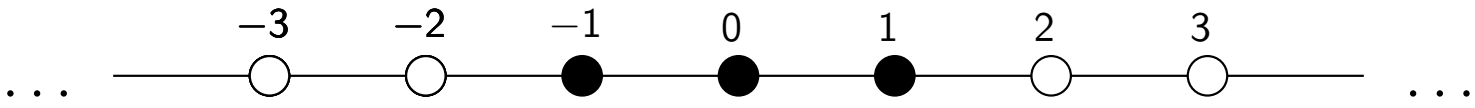
- **Key observation 3:** But by translation symmetry, this is the same as the conditional law of Y^{-1}_t given $Y^{0,1}[t]$
- But this only requires the knowledge of the law of $Y^{-1,0,1}$, (specifically, $Y^{-1,0,1}[t]$), so the evolution is autonomous!

Algorithm to compute marginal law on $\mathcal{T}_2$



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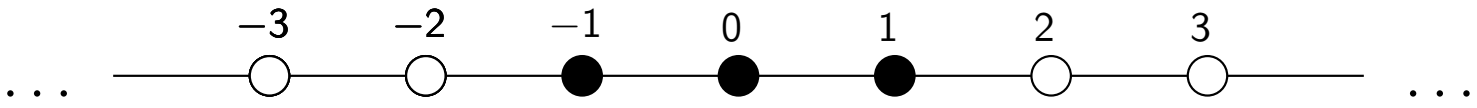

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$$\gamma_t(\cdot \mid y^0, y^1) = \mathrm{Law}\Big(Y_t^{-1} \mid Y^0[t] = y^0[t], Y^1[t] = y^1[t]\Big).$$

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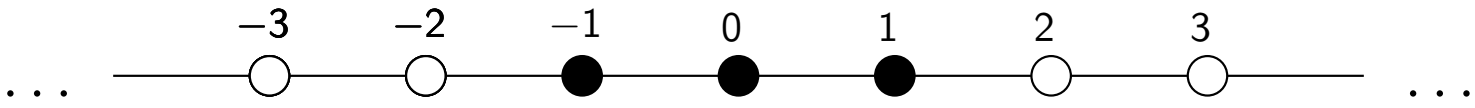

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- Sample ghost particles Y_t^{-2} (and, similarly, Y_t^2) so that

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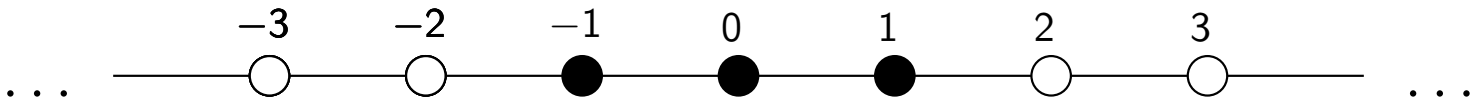

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- Sample new iid noises $(\xi_{t+1}^{-1}, \xi_{t+1}^{0}, \xi_{t+1}^{1})$, and update:

$$Y_{t+1}^i = F\Big(Y_t^i, Y_t^{i-1,i+1}, \xi_{t+1}^i\Big), \quad i = -1, 0, 1$$

Marginal Dynamics on the 2-regular tree: Diffusion

- As before, identify $\mathcal{T}_2 = \mathbb{Z}$, $\rho = 0$.
- Note that our basic convergence result implies that the law of $X^{-1,0,1}$ is the marginal of the infinite system of SDEs: where (for simplicity) we choose to work with a particular form of the dynamics

$$dX_t^i = \frac{1}{2} \sum_{u=i+1,i-1} b(X_t^i, X_t^u)dt + dW_t^i, \quad i \in \mathbb{Z}$$

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$$f_t^i \quad = \quad \frac{1}{2}[b(X_t^i, X_t^{i-1}) + b(X_t^i, X_t^{i+1})]$$

where recall each X^i is a measurable function of $\{W_t^i, s \leq t\}$

A result from stochastic calculus

Lemma (A Markovian mimicking lemma: cf. Gyongy '86)

On some $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$*, let* W *be* $\mathbb{F}$*-Brownian. Let* $\{f_t, t \geq 0\}$ *be a random stochastic process that is square-integrable and* $\mathbb{F}$*-adapted (e.g., for every* t*,* f_t *is a measurable function of* $\{W_s, s \leq t\}$*). Suppose* X *solves the (possibly non-Markovian SDE)*

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Define, for $y \in \mathbb{R}^d$ *and* $t \geq 0$*,*

$$\widetilde{f}(t, z) = \mathbb{E}[f_t \mid X_t = y].$$

Then there is a weak solution Y *of the Markovian SDE*

$$dY_t = \widetilde{f}(t, Y_t)dt + d\widetilde{W}_t$$

such that $Y_t \stackrel{d}{=} X_t$ *for every* $t \geq 0$*.*

An Additional ingredient: A projection/mimicking lemma

Lemma (cf. Brunick-Shreve '13; extension of Gyongy '86)

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Define (by optional projection) for $x \in \mathcal{C}$*,*

$$\widetilde{f}(t, x) = \mathbb{E}[f_t \,|\, X[t] = x[t]].$$

Then there is a weak solution Y *of*

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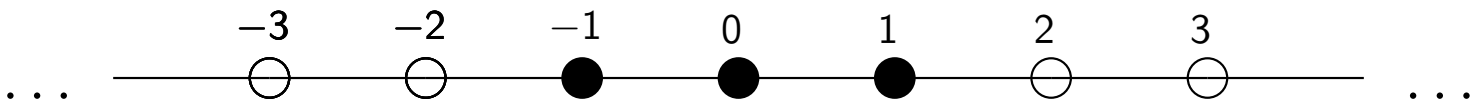
such that $Y \stackrel{d}{=} X$ *as laws on* $\mathcal{C}$*.*

From Conditional Independence to Local Equations



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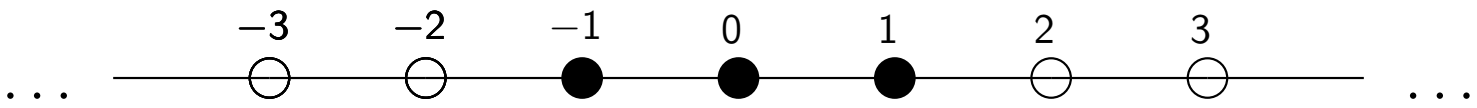

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Note: By the mimicking lemma, there exists a particle system $Y = (Y^{-1}, Y^0, Y^1)$ on a possibly different filtered probability space with independent Brownian motions $(\widetilde{W}^{-1}, \widetilde{W}^0, \widetilde{W}^1)$ such that Y is equal in law to $X^{-1,0,1}$ and

$$Y_t^i = \tilde{b}^i(t, Y)dt + d\widetilde{W}_t^i,$$

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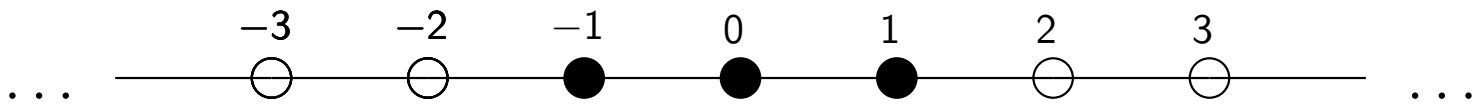

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Note: By the mimicking lemma, there exists a particle system $Y = (Y^{-1}, Y^0, Y^1)$ on a possibly different filtered probability space with independent Brownian motions $(\widetilde{W}^{-1}, \widetilde{W}^0, \widetilde{W}^1)$ such that Y is equal in law to $X^{-1,0,1}$ and

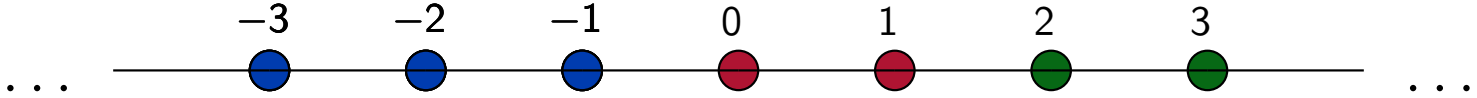
$$Y_t^i = \tilde{b}^i(t, Y)dt + d\widetilde{W}_t^i,$$

where $\tilde{b}^i$ is (a jointly measurable version of)

$$\tilde{b}^i(t,x) = \frac{1}{2}\mathbb{E}\left[(b(X_t^i, X_t^{i-1}) + b(X_t^i, X_t^{i+1}))|X^{-1,0,1}[t] = x[t]\right],$$

where recall $x[t] = (x(s))_{s \le t}$ for any path $x \in \mathcal{C}$.

Marginal Dynamics on Regular Trees: $Y \sim X^{-1,0,1}$



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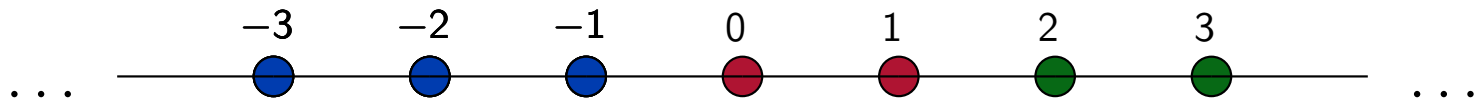

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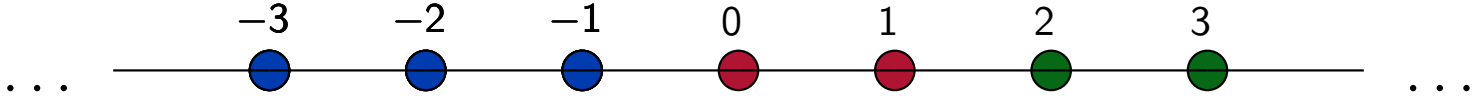
Note that $\tilde{b}^0(t,x) = \frac{1}{2}[b(x_t^0, x_t^{-1}) + b(x_t^0, x_t^1)]$, while

$$\tilde{b}^{-1}(t,x) \quad = \frac{1}{2}\mathbb{E}[b(X_t^{-1}, X_t^{-2})|X^{-1,0,1}[t] = x[t]] + \frac{1}{2}b(x_t^{-1}, x_t^0)$$

Marginal Dynamics on Regular Trees: $Y \sim X^{-1,0,1}$



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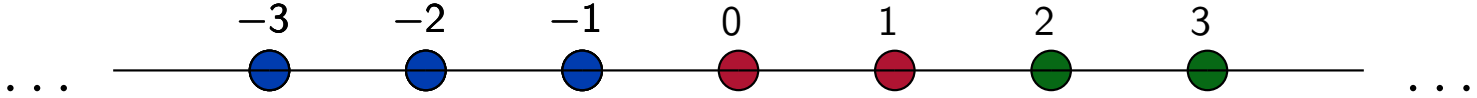
Marginal Dynamics on Regular Trees: $Y \sim X^{-1,0,1}$



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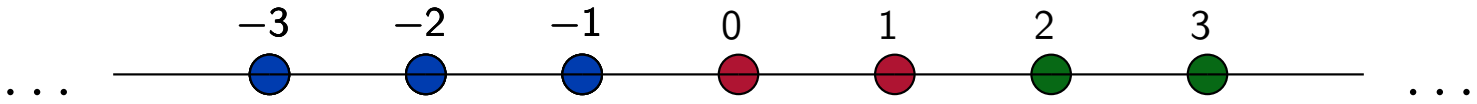
&= \frac{1}{2}\mathbb{E}\left[b(X_t^{-1}, X_t^{-2})|(X^{-1,0})[t] = x^{-1,0}[t]\right] + \frac{1}{2}b(x_t^{-1}, x_t^0) \\
&= \frac{1}{2}\mathbb{E}\left[b(X_t^0, X_t^{-1})|(X^{0,1}[t] = x^{-1,0}[t]\right] + \frac{1}{2}b(x_t^{-1}, x_t^0).
\end{aligned}$$

From Conditional Independence to Local Equations



Particle system on infinite line graph, $i \in \mathbb{Z}$:

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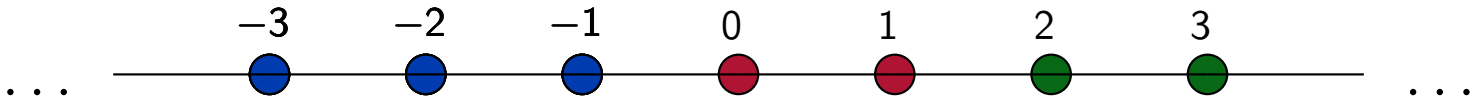
$$\gamma_t(x^0, x^1) := \mathrm{Law}\big(X_t^{-1} \,|\, X^0[t] = x^0[t],\ X^1[t] = x^1[t]\big).$$

From Conditional Independence to Local Equations



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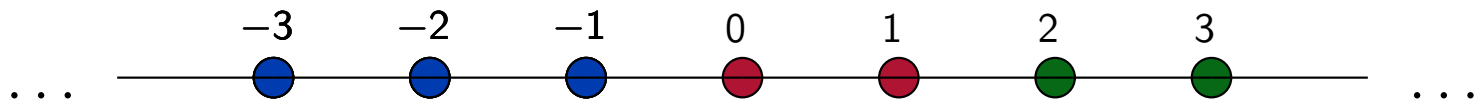
Symmetry: (shift)

$$(X^1, \ldots, X^k) \stackrel{d}{=} (X^{i+1}, \ldots, X^{i+k}), \quad \forall i \in \mathbb{Z}, k \in \mathbb{N}$$

Implication:

$$\gamma_t(X^{-1}, X^0) = \mathrm{Law}\big(X^{-2}(t) \,|\, X^{-1}[t],\ X^0[t]\big)$$

From Conditional Independence to Local Equations



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Summary: $(X^{-1}, X^0, X^1) \stackrel{d}{=} Y = (Y^{-1}, Y^0, Y^1)$ for a weak solution to

$$dY_t^1 = \tfrac{1}{2}\left(b(Y_t^1, Y_t^0) + \langle \gamma_t(Y^1, Y^0), b(Y_t^1, \cdot)\rangle\right) dt + d\widetilde{W}_t^1$$
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Summary

- In answer to (Q1) we showed that when $(G_n, \boldsymbol{x}^n) \to G, \boldsymbol{x})$,

$$dX_t^{G_n,v} = b(X_t^{G,v}, \mu_t^v)dt + dW_t^v, \quad v \in V,$$

converges locally to the infinite graph dynamics

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$$dX_t^{G,v} = b(X_t^{G,v}, \mu_t^v)dt + dW_t^v, \quad v \in V,$$

- Then we showed that when $G = \mathbb{T}_2 = \mathbb{Z}$, the law of $X^{-1,0,1}$ is equal to the law of the solution $Y^{-1,0,1}$ to this non-Markovian "functional" SDE:

$$\begin{aligned}
dY_t^1 &= \tfrac{1}{2}\left(b(Y_t^1, Y_t^0) + \langle \gamma_t(Y^1, Y^0), b(Y_t^1, \cdot)\rangle\right) dt + d\widetilde{W}_t^1 \\
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Summary (contd.)

Recall we showed that when $G = \mathbb{T}_2 = \mathbb{Z}$, the law of $X^{-1,0,1}$ is equal to the law of the solution $Y^{-1,0,1}$ to this non-Markovian "functional" SDE:

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\end{aligned}
$$

Q. But, does this functional SDE have a unique solution in law?

Summary (contd.)

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\end{aligned}
$$

Q. But, does this functional SDE have a unique solution in law?
A. Yes, it does!

Uniqueness proof idea: Build an infinite particle system using the 2MRF property by iterating the transition kernel $\mathrm{Law}(Y^1 \mid Y^0, Y^{-1})$, and show it solves the original SDE system on the line $\mathbb{Z}$. Then invoke weak uniqueness of the IPS on $\mathbb{Z}$.

Summary: Beyond Mean-Field Limits

Mean-Field Dynamics (Dense Sequences)

$$dX(t) = B(X(t), \mu(t))dt + dW(t), \qquad \mu(t) = \mathrm{Law}(X(t)).$$

where $B(x, m) = \int_{\mathbb{R}^d} b(x, y)\, m(dy)$.

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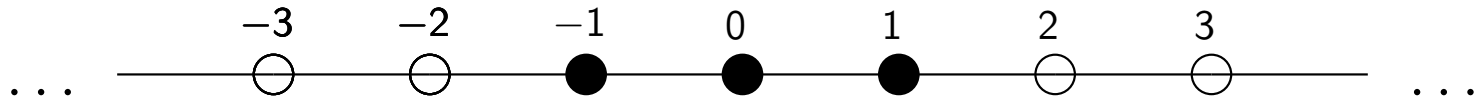
Beyond Mean-Field Dynamics (The Sparse Case of $G = \mathbb{T}_2$)



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dY_t^0 &= \tfrac{1}{2}\left(b(Y_t^0, Y_t^{-1}) + b(Y_t^0, Y_t^1)\right) dt + d\widetilde{W}^0)_t \\
dY_t^{-1} &= \tfrac{1}{2}\left(\langle \gamma_t(Y^{-1}, Y^0), b(Y_t^{-1}, \cdot)\rangle + b(Y_t^{-1}, Y_t^0)\right) dt + d\widetilde{W}_t^{-1}
\end{aligned}$$

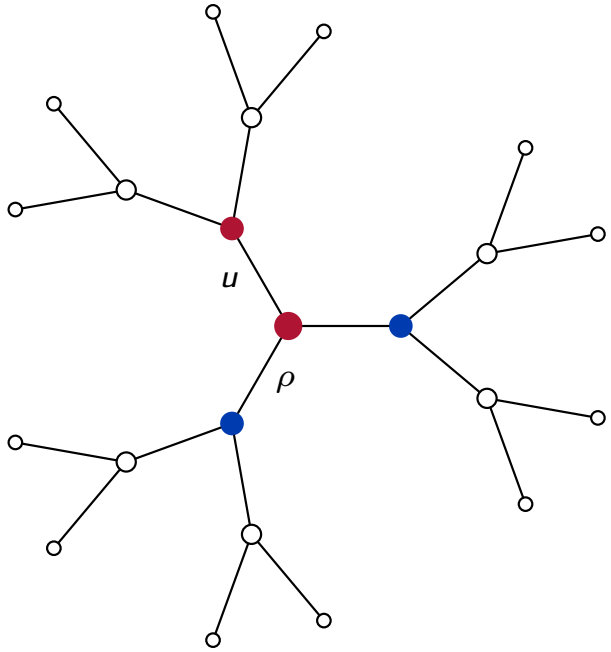
Infinite κ-regular trees $\mathbb{T}_\kappa$

Autonomous SDE system for root particle and its neighbors,

$$X_t^\rho,\ (X_t^v)_{v\sim\rho},$$

involving conditional law of $\kappa - 1$ children given root and one other child u:

$$\mathrm{Law}((X^v)_{v\sim\rho,\, v\neq u} \,|\, X^\rho,\, X^u)$$



$\kappa = 3$

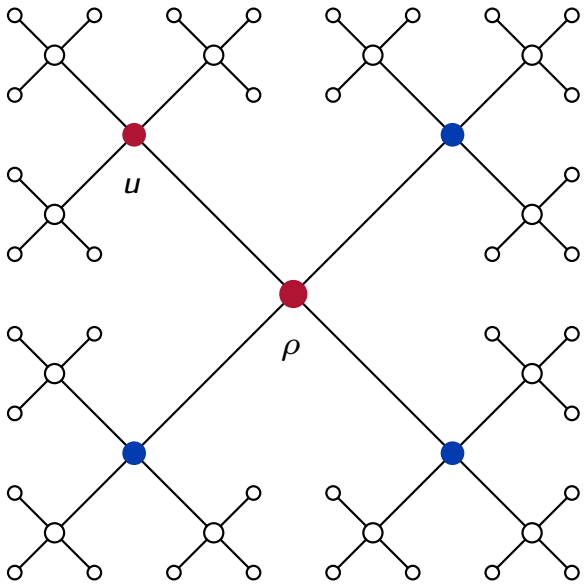
Infinite κ-regular trees

Autonomous SDE system for root particle and its neighbors,

$$X_\rho(t),\ (X_v(t))_{v\sim\rho},$$

involving conditional law of $d-1$ children given root and one other child u:

$$\mathrm{Law}((X_v)_{v\sim\rho,\, v\neq u} \,|\, X_\rho,\, X_u)$$



$\kappa = 4$

Infinite d-regular trees

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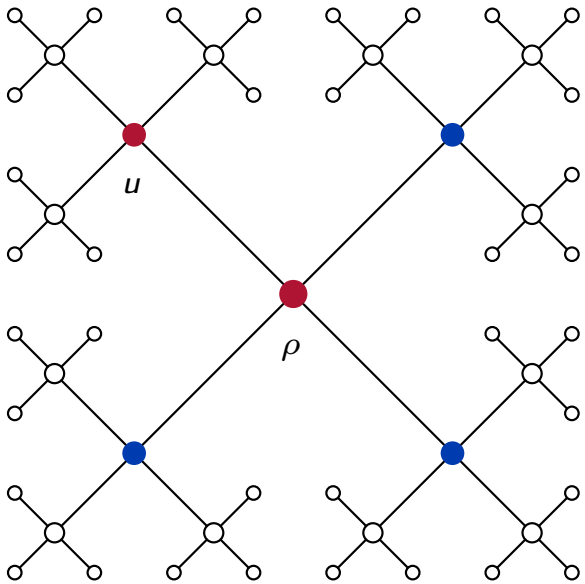

$d = 4$

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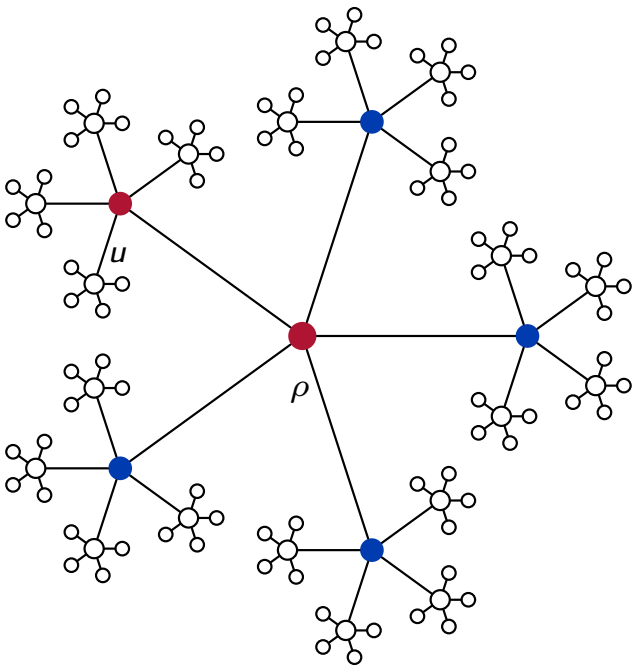

$\kappa = 5$

Local Equations when $\mathbb{T}_\kappa$, $\kappa > 2$

Let $\mathbb{U}_1 = \{\rho, 1, \ldots, \kappa\}$

Theorem

Then $X^{\mathbb{U}_1}$ has the same law as the unique (weak) solution $Y = (Y^\rho, Y^{1,\ldots,\kappa})$ to the following SDE system

$$\begin{aligned} dY_t^\rho &= b(t, Y^\rho, Y^{1,\ldots,\kappa})dt + dB_t^\rho \\ dY_t^i &= \gamma_t(Y^i, Y^\rho)dt + dB_t^i \end{aligned}$$

where

$$\gamma_t(Y^\rho, Y^1) := \mathbb{E}[b(t, Y^\rho, Y^{1,\ldots,\kappa} | Y^\rho[t], Y^1[t]).$$

But what about random graph limits?

For example, can we find the marginal dynamics on a unimodular Galton-Watson tree?

Next lecture ...