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Random data wave equations

(lecture 1)

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Multiple Fourier series

- Let $\mathbb{T}^d = (\mathbb{R}/(2\pi\mathbb{Z}))^d$ be the torus of dimension d.
- If $f : \mathbb{T}^d \to \mathbb{C}$ is a C^∞ function then for every $x \in \mathbb{T}^d$,

$$f(x) = \sum_{n \in \mathbb{Z}^d} \widehat{f}(n)\, e^{in\cdot x},$$

where $\widehat{f}(n)$ are the Fourier coefficients of f, defined by

$$\widehat{f}(n) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(x) e^{-in\cdot x} dx\,.$$

Sobolev spaces on the torus

- For $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, we set

$$\langle x \rangle := (1 + x_1^2 + \cdots + x_d^2)^{\frac{1}{2}} .$$

- For $s \in \mathbb{R}$, we define the Sobolev norm of f by

$$\|f\|^2_{H^s(\mathbb{T}^d)} = \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} |\widehat{f}(n)|^2 . \tag{1}$$

- For $s \geq 0$ an integer, we have the norm equivalence

$$\|f\|^2_{H^s(\mathbb{T}^d)} \approx \sum_{|\alpha| \leq s} \|\partial^\alpha f\|^2_{L^2(\mathbb{T}^d)} . \tag{2}$$

In (2), ∂^α denotes a partial derivative of order at most s.

- For $s = 0$, we recover the Lebesgue space $L^2(\mathbb{T}^d)$.

- The Sobolev space $H^s(\mathbb{T}^d)$ is defined as the closure of $C^\infty(\mathbb{T}^d)$ with respect to the norm (1).

The Sobolev embeddings

• It is clear from the definition that if $s_1 \leq s_2$ then

$$\|u\|_{H^{s_1}(\mathbb{T}^d)} \leq \|u\|_{H^{s_2}(\mathbb{T}^d)}$$

and therefore $H^{s_2}(\mathbb{T}^d) \subset H^{s_1}(\mathbb{T}^d)$ (with a continuous embedding).

• How are the $H^s(\mathbb{T}^d)$ spaces situated with respect to the $L^p(\mathbb{T}^d)$ spaces ?

Theorem 1

Fix s such that $0 \leq s < \frac{d}{2}$. Let p be such that $2 \leq p \leq \frac{2d}{d-2s}$. Then $H^s(\mathbb{T}^d) \subset L^p(\mathbb{T}^d)$ with a continuous embedding, i.e. there is C such that for every $f \in H^s(\mathbb{T}^d)$,

$$\|f\|_{L^p(\mathbb{T}^d)} \leq C\|f\|_{H^s(\mathbb{T}^d)} \cdot$$

Remark. If $p_1 \leq p_2$, by the Hölder inequality

$$\|f\|_{L^{p_1}(\mathbb{T}^d)} \leq C\|f\|_{L^{p_2}(\mathbb{T}^d)}$$

and therefore the important case is $p = \frac{2d}{d-2s}$.

The Sobolev embeddings (sequel)

- An important particular case of the Sobolev embedding is :

$$\|f\|_{L^6(\mathbb{T}^3)} \leq C\|f\|_{H^1(\mathbb{T}^3)} \cdot$$

- As a consequence of the Sobolev embedding, we have that if $f \in H^{\frac{d}{2}}(\mathbb{T}^d)$ then $f \in L^p(\mathbb{T}^d)$ for all $p < \infty$.

- We also (trivially) have that :

Theorem 2

For $s > d/2$, $H^s(\mathbb{T}^d) \subset L^\infty(\mathbb{T}^d)$ and

$$\|f\|_{L^\infty(\mathbb{T}^d)} \leq C\|f\|_{H^s(\mathbb{T}^d)} \cdot$$

Almost sure improvements of the Sobolev embedding

• We now discuss *an almost sure improvement* of the Sobolev embedding.

• We say that a random variable g belongs to $\mathcal{N}_{\mathbb{C}}(0,\sigma^2)$, if $g = h + il$, where $h \in \mathcal{N}(0,\sigma^2)$ and $l \in \mathcal{N}(0,\sigma^2)$ are independent.

• Let $u \in L^2(\mathbb{T})$ be a deterministic function. There is a sequence $(c_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$ (the Fourier coefficients of u) such that

$$u(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx} .$$

• Consider now a randomised version of u given by the expression

$$u_\omega(x) = \sum_{n\in\mathbb{Z}} c_n \, g_n(\omega) \, e^{inx} ,$$

where $(g_n(\omega))_{n\in\mathbb{Z}}$ are independent from $\mathcal{N}_{\mathbb{C}}(0,1)$.

Almost sure improvements of the Sobolev embedding (sequel)

- We have $g_n(\omega)\, e^{inx} \in \mathcal{N}_{\mathbb{C}}(0,1)$ (invariance under rotations of $\mathcal{N}_{\mathbb{C}}(0,1)$).
- Next, using the independence of g_n we get that for a fixed $x \in \mathbb{T}$,

$$u_\omega(x) \in \mathcal{N}_{\mathbb{C}}\Big(0, \sum_{n\in\mathbb{Z}} |c_n|^2\Big).$$

- Consequently $u_\omega(x) \in L^p(\Omega \times \mathbb{T}),\ \forall\, p < \infty$ which gives :

Proposition 3

For every $p < \infty$, $u_\omega(x) \in L^p(\mathbb{T})$, almost surely.

- The last statement is to compare with the Sobolev embedding :

$H^{\frac{1}{2}}(\mathbb{T})$ *is continuously embedded in* $L^p(\mathbb{T})$ *for every* $p < \infty$. *The statement is false, if we replace* $H^{\frac{1}{2}}(\mathbb{T})$ *with* $H^s(\mathbb{T})$ *for some* $s < 1/2$.

Almost sure improvements of the Sobolev embedding (sequel)

• Informally : **the randomisation gains a** $1/2$ **derivative.**

• We can replace the gaussians with much more general random variables. For instance, let

$$v_\omega(x) = \sum_{n\in\mathbb{Z}} c_n\, h_n(\omega)\, e^{inx}\,,$$

where $(h_n(\omega))_{n\in\mathbb{Z}}$ are independent standard Bernoulli random variables. Then we have :

Proposition 4

For every $p < \infty$, $v_\omega(x) \in L^p(\mathbb{T})$, *almost surely.*

Remark. The random function $v_\omega(x)$ and the deterministic function

$$v(x) = \sum_{n\in\mathbb{Z}} c_n\, e^{inx}$$

have Fourier coefficients of the same "size".

Almost sure improvements of the Sobolev embedding (sequel)

• The key point in the case of Bernoulli random variables is the large deviation bound

$$p\Big(\omega \,:\, \Big|\sum_{n\in\mathbb{Z}} c_n\, h_n(\omega)\, e^{inx}\Big| > \lambda\Big) \le C \exp^{-\frac{c\lambda^2}{\sum_n |c_n|^2}} ,$$

where C and c are positive constants independent of $x \in \mathbb{T}$.

• The large deviation bound in turn implies

$$\Big\| \sum_{n\in\mathbb{Z}} c_n\, h_n(\omega)\, e^{inx}\Big\|_{L^p(\Omega)} \le C\sqrt{p}\Big(\sum_{n\in\mathbb{Z}} |c_n|^2\Big)^{\frac{1}{2}} ,$$

with a constant C independent of x and ω.

• With the last bound in hand, we can proceed as in the gaussian case.

Products in Sobolev spaces of negative indexes

- Consider the random series :

$$u_\omega(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n\rangle^\alpha}\, e^{inx}, \qquad \frac{1}{4} < \alpha < \frac{1}{2},$$

with g_n as in the previous discussion.

- We have that a.s. $u_\omega \in H^\sigma(\mathbb{T})$, $\sigma < \alpha - \frac{1}{2}$ but a.s. $u_\omega \notin H^{\alpha-\frac{1}{2}}(\mathbb{T})$.
- Fix $\sigma < \alpha - \frac{1}{2}$ (close to $\alpha - \frac{1}{2}$).
- u_ω belongs to a Sobolev space of negative regularity and therefore it is hard to define an object like $|u_\omega|^2$.
- It is however possible, *after a renormalisation*, to define $|u_\omega|^2$ and even compute its Sobolev regularity.

Products in Sobolev spaces of negative indexes

• Consider the partial sums

$$u_{\omega,N}(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle^{\alpha}} e^{inx} \in C^{\infty}(\mathbb{T})$$

and write

$$|u_{\omega,N}(x)|^2 = \sum_{|n|\leq N} \frac{|g_n(\omega)|^2}{\langle n\rangle^{2\alpha}} + \sum_{\substack{n_1\neq n_2\\ |n_1|,|n_2|\leq N}} \frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}}{\langle n_1\rangle^{\alpha}\langle n_2\rangle^{\alpha}} e^{i(n_1-n_2)x}.$$

• The first term (the zero Fourier coefficient) contains all the singularity while the second has an a.s. limit in $H^{2\sigma}(\mathbb{T})$.

Products in Sobolev spaces of negative indexes

- Consequently, we set

$$c_N := \mathbb{E}\Big(\sum_{|n|\le N} \frac{|g_n(\omega)|^2}{\langle n\rangle^{2\alpha}} \Big) = \sum_{|n|\le N} \frac{2}{\langle n\rangle^{2\alpha}} \sim N^{1-2\alpha},$$

and we define the renormalised partial sums

$$|u_{\omega,N}(x)|^2 - c_N = \sum_{|n|\le N} \frac{|g_n(\omega)|^2 - 2}{\langle n\rangle^{2\alpha}} + \sum_{\substack{n_1\neq n_2\\ |n_1|,|n_2|\le N}} \frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}}{\langle n_1\rangle^{\alpha}\langle n_2\rangle^{\alpha}}\, e^{i(n_1-n_2)x}.$$

- Thanks to the independence of g_n we have

$$\mathbb{E}\Big(\Big| \sum_{|n|\le N} \frac{|g_n(\omega)|^2 - 2}{\langle n\rangle^{2\alpha}} \Big|^2 \Big) = \sum_{|n|\le N} \frac{4}{\langle n\rangle^{4\alpha}},$$

which has a limit as $N \to \infty$ when $\alpha > 1/4$.

Products in Sobolev spaces of negative indexes

- Another use of the independence yields that

$$\mathbb{E}\Big(\Big\| \sum_{\substack{n_1 \neq n_2 \\ |n_1|,|n_2| \leq N}} \frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}}{\langle n_1 \rangle^{\alpha} \langle n_2 \rangle^{\alpha}} e^{i(n_1-n_2)x} \Big\|_{H^{2\sigma}}^2 \Big)$$

is bounded by

$$C \sum_{n_1,n_2} \frac{\langle n_1 - n_2 \rangle^{4\sigma}}{\langle n_1 \rangle^{2\alpha} \langle n_2 \rangle^{2\alpha}} .$$

The last sum is convergent as far as $-4\sigma + 4\alpha > 2$, which is equivalent to our assumption $\sigma < \alpha - \frac{1}{2}$.

- Hence the sequence

$$\Big(|u_{\omega,N}(x)|^2 - c_N \Big)_{N \geq 1}$$

has a limit in $L^2(\Omega; H^{2\sigma}(\mathbb{T}))$. This limit is by definition the renormalisation of $|u_\omega|^2$.

Remarks

• Using more involved arguments, we can also show the almost sure convergence in the Sobolev space $H^{2\sigma}(\mathbb{T})$ of the sequence

$$\Big(|u_{\omega,N}(x)|^2 - c_N\Big)_{N\geq 1}.$$

• Since $\sigma < 0$ the norm in $H^{2\sigma}(\mathbb{T})$ is weaker than in $H^{\sigma}(\mathbb{T})$ (where $u_\omega(x)$ is defined).

• Informally : **the square of the modulus of an element of H^σ is in $H^{2\sigma}$, after a renormalisation.**

• This is a remarkable probabilistic phenomenon, in the heart of the study of evolution partial differential equations in the presence of randomness in Sobolev spaces of negative indexes.