Random data wave equations
(lecture 1)

Nikolay Tzvetkov
Université de Cergy-Pontoise, France
Multiple Fourier series

- Let $\mathbb{T}^d = (\mathbb{R}/(2\pi\mathbb{Z}))^d$ be the torus of dimension $d$.
- If $f : \mathbb{T}^d \to \mathbb{C}$ is a $C^\infty$ function then for every $x \in \mathbb{T}^d$,
  \[ f(x) = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{in \cdot x}, \]
  where $\hat{f}(n)$ are the Fourier coefficients of $f$, defined by
  \[ \hat{f}(n) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(x) e^{-in \cdot x} \, dx. \]
Sobolev spaces on the torus

- For \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), we set
  \[
  \langle x \rangle := (1 + x_1^2 + \cdots + x_d^2)^{\frac{1}{2}}.
  \]

- For \( s \in \mathbb{R} \), we define the Sobolev norm of \( f \) by
  \[
  \|f\|_{H^s(\mathbb{T}^d)}^2 = \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} |\widehat{f}(n)|^2. \tag{1}
  \]

- For \( s \geq 0 \) an integer, we have the norm equivalence
  \[
  \|f\|_{H^s(\mathbb{T}^d)}^2 \approx \sum_{|\alpha| \leq s} \|\partial^\alpha f\|_{L^2(\mathbb{T}^d)}^2. \tag{2}
  \]

In (2), \( \partial^\alpha \) denotes a partial derivative of order at most \( s \).

- For \( s = 0 \), we recover the Lebesgue space \( L^2(\mathbb{T}^d) \).

- The Sobolev space \( H^s(\mathbb{T}^d) \) is defined as the closure of \( C^\infty(\mathbb{T}^d) \) with respect to the norm (1).
The Sobolev embeddings

• It is clear from the definition that if $s_1 \leq s_2$ then
  $$\|u\|_{H^{s_1}(\mathbb{T}^d)} \leq \|u\|_{H^{s_2}(\mathbb{T}^d)}$$
and therefore $H^{s_2}(\mathbb{T}^d) \subset H^{s_1}(\mathbb{T}^d)$ (with a continuous embedding).

• How are the $H^s(\mathbb{T}^d)$ spaces situated with respect to the $L^p(\mathbb{T}^d)$ spaces?

**Theorem 1**

Fix $s$ such that $0 \leq s < \frac{d}{2}$. Let $p$ be such that $2 \leq p \leq \frac{2d}{d-2s}$. Then $H^s(\mathbb{T}^d) \subset L^p(\mathbb{T}^d)$ with a continuous embedding, i.e. there is $C$ such that for every $f \in H^s(\mathbb{T}^d)$,
  $$\|f\|_{L^p(\mathbb{T}^d)} \leq C\|f\|_{H^s(\mathbb{T}^d)}.$$

**Remark.** If $p_1 \leq p_2$, by the Hölder inequality
  $$\|f\|_{L^{p_1}(\mathbb{T}^d)} \leq C\|f\|_{L^{p_2}(\mathbb{T}^d)}$$
and therefore the important case is $p = \frac{2d}{d-2s}$.  

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The Sobolev embeddings (sequel)

- An important particular case of the Sobolev embedding is:
  \[ \|f\|_{L^6(T^3)} \leq C \|f\|_{H^1(T^3)}. \]

- As a consequence of the Sobolev embedding, we have that if
  \( f \in H^2(T^d) \) then \( f \in L^p(T^d) \) for all \( p < \infty \).

- We also (trivially) have that:

  **Theorem 2**
  For \( s > d/2 \), \( H^s(T^d) \subset L^\infty(T^d) \) and
  \[ \|f\|_{L^\infty(T^d)} \leq C \|f\|_{H^s(T^d)}. \]
Almost sure improvements of the Sobolev embedding

- We now discuss *an almost sure improvement* of the Sobolev embedding.
- We say that a random variable \( g \) belongs to \( \mathcal{N}_\mathbb{C}(0, \sigma^2) \), if \( g = h + il \), where \( h \in \mathcal{N}(0, \sigma^2) \) and \( l \in \mathcal{N}(0, \sigma^2) \) are independent.
- Let \( u \in L^2(\mathbb{T}) \) be a deterministic function. There is a sequence \((c_n)_{n \in \mathbb{Z}} \in l^2(\mathbb{Z})\) (the Fourier coefficients of \( u \)) such that
  \[
  u(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}.
  \]
- Consider now a randomised version of \( u \) given by the expression
  \[
  u_\omega(x) = \sum_{n \in \mathbb{Z}} c_n g_n(\omega) e^{inx},
  \]
  where \((g_n(\omega))_{n \in \mathbb{Z}}\) are independent from \( \mathcal{N}_\mathbb{C}(0, 1) \).
• We have $g_n(\omega) e^{inx} \in \mathcal{N}_\mathbb{C}(0, 1)$ (invariance under rotations of $\mathcal{N}_\mathbb{C}(0, 1)$).

• Next, using the independence of $g_n$ we get that for a fixed $x \in \mathbb{T}$,

$$u_\omega(x) \in \mathcal{N}_\mathbb{C}(0, \sum_{n \in \mathbb{Z}} |c_n|^2).$$

• Consequently $u_\omega(x) \in L^p(\Omega \times \mathbb{T})$, $\forall p < \infty$ which gives:

**Proposition 3**

*For every $p < \infty$, $u_\omega(x) \in L^p(\mathbb{T})$, almost surely.*

• The last statement is to compare with the Sobolev embedding:

$H^{1/2}(\mathbb{T})$ is continuously embedded in $L^p(\mathbb{T})$ for every $p < \infty$. The statement is false, if we replace $H^{1/2}(\mathbb{T})$ with $H^s(\mathbb{T})$ for some $s < 1/2$.  

Almost sure improvements of the Sobolev embedding (sequel)

- Informally: the randomisation gains a 1/2 derivative.
- We can replace the gaussians with much more general random variables. For instance, let

\[ v_\omega(x) = \sum_{n \in \mathbb{Z}} c_n h_n(\omega) e^{inx}, \]

where \((h_n(\omega))_{n \in \mathbb{Z}}\) are independent standard Bernoulli random variables. Then we have:

**Proposition 4**
For every \( p < \infty \), \( v_\omega(x) \in L^p(\mathbb{T}) \), almost surely.

**Remark.** The random function \( v_\omega(x) \) and the deterministic function

\[ v(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx} \]

have Fourier coefficients of the same "size".
The key point in the case of Bernoulli random variables is the large deviation bound
\[
p\left( \omega : \left| \sum_{n \in \mathbb{Z}} c_n h_n(\omega) e^{inx} \right| > \lambda \right) \leq C \exp \left( -\frac{c\lambda^2}{\sum_{n \in \mathbb{Z}} |c_n|^2} \right),
\]
where \( C \) and \( c \) are positive constants independent of \( x \in \mathbb{T} \).

The large deviation bound in turn implies
\[
\left\| \sum_{n \in \mathbb{Z}} c_n h_n(\omega) e^{inx} \right\|_{L^p(\Omega)} \leq C \sqrt{p} \left( \sum_{n \in \mathbb{Z}} |c_n|^2 \right)^{\frac{1}{2}},
\]
with a constant \( C \) independent of \( x \) and \( \omega \).

With the last bound in hand, we can proceed as in the gaussian case.
Consider the random series:

\[ u_\omega(x) = \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{\langle n \rangle^\alpha} e^{inx}, \quad \frac{1}{4} < \alpha < \frac{1}{2}, \]

with \( g_n \) as in the previous discussion.

- We have that a.s. \( u_\omega \in H^\sigma(\mathbb{T}) \), \( \sigma < \alpha - \frac{1}{2} \) but a.s. \( u_\omega \notin H^{\alpha - \frac{1}{2}}(\mathbb{T}) \).
- Fix \( \sigma < \alpha - \frac{1}{2} \) (close to \( \alpha - \frac{1}{2} \)).
- \( u_\omega \) belongs to a Sobolev space of negative regularity and therefore it is hard to define an object like \( |u_\omega|^2 \).
- It is however possible, after a renormalisation, to define \( |u_\omega|^2 \) and even compute its Sobolev regularity.
Products in Sobolev spaces of negative indexes

- Consider the partial sums

\[ u_{\omega,N}(x) = \sum_{|n| \leq N} \frac{g_n(\omega)}{\langle n \rangle^\alpha} e^{inx} \in C^\infty(\mathbb{T}) \]

and write

\[ |u_{\omega,N}(x)|^2 = \sum_{|n| \leq N} \frac{|g_n(\omega)|^2}{\langle n \rangle^{2\alpha}} + \sum_{n_1 \neq n_2, |n_1|,|n_2| \leq N} \frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}}{\langle n_1 \rangle^\alpha \langle n_2 \rangle^\alpha} e^{i(n_1-n_2)x}. \]

- The first term (the zero Fourier coefficient) contains all the singularity while the second has an a.s. limit in \( H^{2\sigma}(\mathbb{T}) \).
Consequently, we set
\[ c_N := \mathbb{E}\left( \sum_{|n| \leq N} \frac{|g_n(\omega)|^2}{\langle n \rangle^{2\alpha}} \right) = \sum_{|n| \leq N} \frac{2}{\langle n \rangle^{2\alpha}} \sim N^{1-2\alpha}, \]
and we define the renormalised partial sums
\[ |u_{\omega,N}(x)|^2 - c_N = \sum_{|n| \leq N} \frac{|g_n(\omega)|^2 - 2}{\langle n \rangle^{2\alpha}} + \sum_{n_1 \neq n_2, |n_1|,|n_2| \leq N} \frac{g_{n_1}(\omega)g_{n_2}(\omega)}{\langle n_1 \rangle^{\alpha}\langle n_2 \rangle^{\alpha}} e^{i(n_1-n_2)x}. \]

Thanks to the independence of \( g_n \) we have
\[ \mathbb{E}\left( \left| \sum_{|n| \leq N} \frac{|g_n(\omega)|^2 - 2}{\langle n \rangle^{2\alpha}} \right|^2 \right) = \sum_{|n| \leq N} \frac{4}{\langle n \rangle^{4\alpha}}, \]
which has a limit as \( N \to \infty \) when \( \alpha > 1/4 \).
Another use of the independence yields that

\[ E\left( \left\| \sum_{n_1 \neq n_2, |n_1|, |n_2| \leq N} \frac{g_{n_1}(\omega)g_{n_2}(\omega)}{\langle n_1 \rangle^\alpha \langle n_2 \rangle^\alpha} e^{i(n_1-n_2)x} \right\|^2_{H^{2\sigma}} \right) \]

is bounded by

\[ C \sum_{n_1,n_2} \frac{\langle n_1 - n_2 \rangle^{4\sigma}}{\langle n_1 \rangle^{2\alpha} \langle n_2 \rangle^{2\alpha}}. \]

The last sum is convergent as far as \(-4\sigma + 4\alpha > 2\), which is equivalent to our assumption \(\sigma < \alpha - \frac{1}{2}\).

Hence the sequence

\[ \left( |u_{\omega,N}(x)|^2 - c_N \right)_{N \geq 1} \]

has a limit in \(L^2(\Omega; H^{2\sigma}(\mathbb{T}))\). This limit is by definition the renormalisation of \(|u_{\omega}|^2\).
Remarks

• Using more involved arguments, we can also show the almost sure convergence in the Sobolev space $H^{2\sigma}(\mathbb{T})$ of the sequence

$\left( |u_{\omega,N}(x)|^2 - c_N \right)_{N \geq 1}$.

• Since $\sigma < 0$ the norm in $H^{2\sigma}(\mathbb{T})$ is weaker than in $H^{\sigma}(\mathbb{T})$ (where $u_{\omega}(x)$ is defined).

• Informally: the square of the modulus of an element of $H^{\sigma}$ is in $H^{2\sigma}$, after a renormalisation.

• This is a remarkable probabilistic phenomenon, in the heart of the study of evolution partial differential equations in the presence of randomness in Sobolev spaces of negative indexes.