Random data wave equations
(lecture 5)

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The nonlinear wave equation

- Recall that we study the Cauchy problem

\[
\begin{aligned}
\frac{\partial^2}{\partial t^2} u - \Delta u + u^3 &= 0 \\
(u, \partial_t u)|_{t=0} &= (u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3),
\end{aligned}
\]

where \( u : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R} \) and \( \mathcal{H}^s(\mathbb{T}^3) = H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3) \).

- The initial data is chosen among the realisations of the random series

\[
u_0^\omega = \sum_{n \in \mathbb{Z}^3} \frac{g_n(\omega)}{\langle n \rangle^\alpha} e^{in \cdot x} \quad \text{and} \quad u_1^\omega = \sum_{n \in \mathbb{Z}^3} \frac{h_n(\omega)}{\langle n \rangle^{\alpha-1}} e^{in \cdot x},
\]

where the scalar random variables \( g_n \) and \( h_n \) satisfy suitable assumptions.

- Recall that \( \mu_\alpha \) is the non degenerate measure on \( \mathcal{H}^s(\mathbb{T}^3) \), \( 0 < s < \alpha - 3/2 \), induced by the map

\[
\omega \mapsto (u_0^\omega, u_1^\omega).
\]
Theorem 1 (Burq-Tz. (2010))

Let $\alpha > \frac{3}{2}$ and $s < \alpha - \frac{3}{2}$. Let $\{u_N\}_{N \in \mathbb{N}}$ be a sequence of the smooth global solutions to

$$\partial^2_t u - \Delta u + u^3 = 0$$

with the following random $C^\infty$-initial data:

$$u^\omega_{0,N}(x) = \sum_{|n| \leq N} \frac{g_n(\omega)}{\langle n \rangle^\alpha} e^{in \cdot x} \quad \text{and} \quad u^\omega_{1,N}(x) = \sum_{|n| \leq N} \frac{h_n(\omega)}{\langle n \rangle^{\alpha-1}} e^{in \cdot x}.$$

Then, as $N \to \infty$, $u_N$ converges almost surely to a (unique) limit $u$ in $C(\mathbb{R}; H^s(\mathbb{T}^3))$, satisfying (1) in a distributional sense.
Theorem 2 (Oh-Pocovnicu-Tz. (2018))
Let $\frac{5}{4} < \alpha \leq \frac{3}{2}$ and $s < \alpha - \frac{3}{2}$. There exists a divergent sequence $\{\beta_N\}_{N \in \mathbb{N}}$ of positive numbers such that the following holds true. There exist small $T_0 > 0$ and positive constants $C$, $c$, $\kappa$ such that for every $T \in (0, T_0]$, there exists a set $\Omega_T$ of complementary probability smaller than $C \exp(-c/T^\kappa)$ such that if we denote by $\{u_N\}_{N \in \mathbb{N}}$ the smooth global solutions to

$$\begin{cases}
\partial_t^2 u_N - \Delta u_N + u_N^3 - \beta_N u_N = 0 \\
(u_N, \partial_t u_N)|_{t=0} = (u_{0,N}^\omega, u_{1,N}^\omega),
\end{cases}$$

where the random initial data $(u_{0,N}^\omega, u_{1,N}^\omega)$ is given by the truncated Fourier series

$$u_{0,N}^\omega(x) = \sum_{|n| \leq N} \frac{g_n(\omega)}{\langle n \rangle^\alpha} e^{i n \cdot x}, \quad u_{1,N}^\omega(x) = \sum_{|n| \leq N} \frac{h_n(\omega)}{\langle n \rangle^\alpha-1} e^{i n \cdot x}$$

then for every $\omega \in \Omega_T$, the sequence $\{u_N\}_{N \in \mathbb{N}}$ converges to some (unique) limiting distribution $u$ in $C([-T, T]; H^s(\mathbb{T}^3))$ as $N \to \infty$. 
Deterministic well-posedness in $\mathcal{H}^s(\mathbb{T}^3)$, $s \geq 1/2$

- As mentioned, the main tool are the Strichartz estimates that we discuss below.
- Let $\mathcal{L} = \partial_t^2 - \Delta + 1$. We write
  \[
  \mathcal{L}^{-1} = (\partial_t^2 - \Delta + 1)^{-1}
  \]
  to denote the Duhamel integral operator
  \[
  \mathcal{L}^{-1}F(t) := \int_0^t \frac{\sin((t - t')\langle D \rangle)}{\langle D \rangle} F(t') dt', \quad \langle D \rangle = \sqrt{-\Delta + 1}.
  \]
- In other words, $u := \mathcal{L}^{-1}(F)$ is the solution to
  \[
  \mathcal{L}u = F, \quad (u, \partial_t u)|_{t=0} = (0, 0).
  \]
The Strichartz estimates

- The most basic regularity property of $\mathcal{L}^{-1}$ is the "wave regularity" estimate:

$$\|\mathcal{L}^{-1}(F)\|_{L^\infty([-T,T];H^s(\mathbb{T}^3))} \lesssim \|F\|_{L^1([-T,T];H^{s-1}(\mathbb{T}^3))}.$$ (2)

- The Strichartz estimates are important extensions of (2).

**Theorem 3**

- Let $0 < T \leq 1$. Then, the following estimate holds:

$$\|\mathcal{L}^{-1}(F)\|_{L^4([-T,T]\times\mathbb{T}^3)} + \|\mathcal{L}^{-1}(F)\|_{L^\infty([-T,T];H^{\frac{1}{2}}(\mathbb{T}^3))}$$

$$\lesssim \min \left( \|F\|_{L^1([-T,T];H^{-\frac{1}{2}}(\mathbb{T}^3))}, \|F\|_{L^\frac{4}{3}([-T,T]\times\mathbb{T}^3)} \right).$$

- As a consequence (by duality) we also have that the solution of

$$\mathcal{L}u = 0, \quad u(0) = u_0, \quad \partial_t u(0) = u_1$$

satisfies

$$\|u\|_{L^4([-T,T]\times\mathbb{T}^3)} + \|u\|_{L^\infty([-T,T];H^{\frac{1}{2}}(\mathbb{T}^3))} \lesssim \left( \|u_0\|_{H^{\frac{1}{2}}} + \|u_1\|_{H^{-\frac{1}{2}}} \right).$$
For $T > 0$, we denote by $X_T$ the closed subspace of $C([-T,T]; H^{1/2}(\mathbb{T}^3))$ endowed with the norm
\[ \|u\|_{X_T} = \|u\|_{L^\infty([-T,T]; H^{1/2}(\mathbb{T}^3))} + \|u\|_{L^4([-T,T] \times \mathbb{T}^3)}. \]

Then the Strichartz estimates imply that
\[ \|\mathcal{L}^{-1}(u^3)\|_{X_T} \leq C\|u\|_{X_T}^3. \]

By a fixed point argument in $X_T$, the last estimate together with the corresponding estimate for the free evolution can be easily transformed into small data local well-posedness in $H^{1/2}(\mathbb{T}^3)$ of
\[ \mathcal{L}u + u^3 = 0. \]

A very small variation yields large data local well-posedness in $H^s(\mathbb{T}^3)$, $s > 1/2$ (we need to lose a bit of regularity in order to remove the smallness condition).
The free random evolution

• Fix $\alpha \leq \frac{3}{2}$. Recall that

$$\mathcal{L} = \partial_t^2 - \Delta + 1.$$ 

• Denote by $z_{1,N} = z_{1,N}(t, x, \omega)$ the solution to

$$\mathcal{L}z_{1,N}(t, x, \omega) = 0$$

with the random initial data $(u_{0,N}^\omega, u_{1,N}^\omega)$ given by the truncated Fourier series

$$u_{0,N}^\omega(x) = \sum_{|n| \leq N} \frac{g_n(\omega)}{\langle n \rangle^\alpha} e^{in \cdot x}, \quad u_{1,N}^\omega(x) = \sum_{|n| \leq N} \frac{h_n(\omega)}{\langle n \rangle^{\alpha-1}} e^{in \cdot x}.$$
• Given $t \in \mathbb{R}$, define $g_n^t(\omega)$ by

$$g_n^t(\omega) := \cos(t\langle n \rangle) g_n(\omega) + \sin(t\langle n \rangle) h_n(\omega).$$

(3)

• Then, we have

$$z_{1,N}(t, x, \omega) = \cos(t\langle \nabla \rangle) \left( z_{1,N}(0, x, \omega) \right) + \frac{\sin(t\langle \nabla \rangle)}{\langle \nabla \rangle} \left( \partial_t z_{1,N}(0, x, \omega) \right)$$

$$= \sum_{|n| \leq N} \frac{g_n^t(\omega)}{\langle n \rangle^\alpha} e^{i n \cdot x}.$$

• Using the definitions of the Gaussian random variables $\{g_n\}_{n \in \mathbb{Z}^3}$ and $\{h_n\}_{n \in \mathbb{Z}^3}$, we see that $\{g_n^t\}_{n \in \mathbb{Z}^3}$ defined in (3) forms a family of independent standard complex-valued Gaussian random variables conditioned that $g_n^t = \overline{g_{-n}^t}$ (in particular, $g_0^t$ is real-valued).
In the following, we discuss spatial regularities of various stochastic terms for fixed $t \in \mathbb{R}$. For simplicity of notation, we suppress the $t$-dependence and discuss spatial regularities.

It is easy to see that $z_{1,N}$ converges almost surely to some limit $z_1$ in $H^{s_1}(\mathbb{T}^3)$ as $N \to \infty$, provided that

$$s_1 < \alpha - \frac{3}{2}.$$

In particular, when $\alpha \leq \frac{3}{2}$, $z_{1,N}$ has negative Sobolev regularity (in the limiting sense) and thus $(z_{1,N})^2$ and $(z_{1,N})^3$ do not have well defined limits (in any topology) as $N \to \infty$ since it involves products of two distributions of negative regularities.
Let $u_N$ be the solution to the renormalized nonlinear wave equation

$$\partial_t^2 u_N - \Delta u_N + u_N^3 - \beta_N u_N = 0$$

with the same truncated random initial data $(u_{0,N}^\omega, u_{1,N}^\omega)$. By writing $u$ as

$$u_N = z_{1,N} + v_N,$$

we see that the residual term $v_N = u_N - z_{1,N}$ satisfies the following equation:

$$\mathcal{L}v_N + v_N^3 + 3z_{1,N}v_N^2 + 3((z_{1,N})^2 - \sigma_N)v_N + ((z_{1,N})^3 - 3\sigma_N z_{1,N}) = 0$$

with zero initial data, where the parameter $\sigma_N$ is defined by

$$\sigma_N := \frac{\beta_N + 1}{3}.$$
Giving a sense of the square and the cube in negative Sobolev spaces

• The key point is that the terms

\[ Z_{2,N} := (z_{1,N})^2 - \sigma_N \quad \text{and} \quad Z_{3,N} := (z_{1,N})^3 - 3\sigma_N z_{1,N} \]

are "renormalizations" of \((z_{1,N})^2\) and \((z_{1,N})^3\).

• Here, by "renormalizations", we mean that by choosing a suitable renormalization constant \(\sigma_N\), the terms \(Z_{2,N}\) and \(Z_{3,N}\) converge almost surely in suitable negative Sobolev spaces as \(N \to \infty\).

• The regularity \(s_1 < \alpha - \frac{3}{2}\) of \(z_{1,N}\) (in the limit) and a basic computation (as in the first lecture) show that if the expressions \(Z_{2,N} = (z_{1,N})^2 - \sigma_N\) and \(Z_{3,N} = (z_{1,N})^3 - 3\sigma_N z_{1,N}\) have any well defined limits as \(N \to \infty\), then their regularities in the limit are expected to be

\[ s_2 < 2\left(\alpha - \frac{3}{2}\right) \quad \text{and} \quad s_3 < 3\left(\alpha - \frac{3}{2}\right), \]

respectively.
• In fact, by choosing the renormalization constant $\sigma_N$ as
\[
\sigma_N := \mathbb{E}\left[\left(z_{1,N}(t,x,\omega)\right)^2\right],
\]
we show that $Z_{j,N}$, $j = 2, 3$ converge in $H^{8j}(\mathbb{T}^3)$ almost surely.

• The renormalization constant $\sigma_N$ a priori depends on $t, x$ but it turns out to be independent of $t$ and $x$. This fact can be seen by a direct computation. It also follows from the stationarity (in both $t$ and $x$) of the stochastic process $\{z_{1,N}(t,x)\}_{(t,x)\in\mathbb{R}\times\mathbb{T}^3}$.

• We will also see that, for $N \gg 1$, $\sigma_N$ behaves like $\sim N^{3-2\alpha}$ when $\alpha < \frac{3}{2}$. 
The first Picard iteration ansatz

- We know that the deterministic Cauchy problem for
  \[ \mathcal{L}v + v^3 = 0 \]
is locally well-posed in \( \mathcal{H}^s(\mathbb{T}^3) \) for \( s \geq \frac{1}{2} \).
- Therefore, we may hope to solve the equation
  \[ \mathcal{L}v_N + v_N^3 + 3z_{1,N}v_N^2 + 3((z_{1,N})^2 - \sigma_N)v_N + ((z_{1,N})^3 - 3\sigma_Nz_{1,N}) = 0 \]
uniformly in \( N \in \mathbb{N} \) if we can ensure that the solution \( v_N \) to the following linear problem:
  \[ \mathcal{L}v_N + ((z_{1,N})^3 - 3\sigma_Nz_{1,N}) = 0 \] (4)
with the zero initial data \( (v_N, \partial_t v_N)|_{t=0} = (0,0) \) remains bounded in \( H^{\frac{1}{2}}(\mathbb{T}^3) \) as \( N \to \infty \).
- Using the wave-regularity, we see that the solution to (4) is almost surely bounded in \( H^{\frac{1}{2}}(\mathbb{T}^3) \) uniformly in \( N \in \mathbb{N} \), provided
  \[ 3\left(\alpha - \frac{3}{2}\right) + 1 > \frac{1}{2} \quad \implies \quad \alpha > \frac{4}{3}. \]
Therefore, \( \alpha = \frac{4}{3} \) is the limit of the first Picard iteration ansatz.
The second Picard iteration ansatz

• In order to go below the $\alpha = \frac{4}{3}$ threshold, a new argument is needed. More precisely, we further decompose $v_N$ as

$$v_N = z_{2,N} + w_N,$$

where $z_{2,N}$ is the solution to the following equation:

$$\begin{cases}
\mathcal{L}z_{2,N} + ((z_{1,N})^3 - 3\sigma_N z_{1,N}) = 0 \\
(z_{2,N}, \partial_t z_{2,N})|_{t=0} = (0,0).
\end{cases}$$

• Thanks to the one degree of smoothing, we see that $z_{2,N}$ converges to some limit in $H^s(\mathbb{T}^3)$, provided that

$$s = s_3 + 1 < 3\left(\alpha - \frac{3}{2}\right) + 1.$$
In terms of the original solution $u_N$ we have

$$u_N = z_{1,N} + z_{2,N} + w_N.$$ 

Note that $z_{1,N} + z_{2,N}$ corresponds to the Picard second iterate for the truncated renormalized equation.

The equation for $w_N$ can now be written as

$$\mathcal{L}w_N + (w_N + z_{2,N})^3 + 3z_{1,N}(w_N + z_{2,N})^2 + 3((z_{1,N})^2 - \sigma_N)(w_N + z_{2,N}) = 0,$$

with zero initial data.

By using the second order expansion, we have eliminated the most singular term $Z_{3,N} = (z_{1,N})^3 - 3\sigma_N z_{1,N}$ in the equation for $u_N$.

In the equation for $w_N$, there are several source terms (namely, purely stochastic terms independent of the unknown $w_N$) and they are precisely the quintic, septic, and nonic (i.e. degree nine) terms added in considering the Picard third iterate.
• It turns out that the most singular source term is the following quintic term:

\[ Z_{5,N} := 3((z_{1,N})^2 - \sigma_N)z_{2,N}, \]

where we recall that \( z_{2,N} \) is the solution to

\[
\begin{cases}
Lz_{2,N} + ((z_{1,N})^3 - 3\sigma_N z_{1,N}) = 0, \\
(z_{2,N}, \partial_t z_{2,N})|_{t=0} = (0, 0).
\end{cases}
\]

• As we already mentioned, the term \( Z_{2,N} = (z_{1,N})^2 - \sigma_N \) and the second order term \( z_{2,N} \) pass to the limits in \( H^s(\mathbb{T}^3) \) for \( s < 2(\alpha - \frac{3}{2}) \) and \( s < 3(\alpha - \frac{3}{2}) + 1 \), respectively.

• In order to make sense of the product of \( Z_{2,N} \) and \( z_{2,N} \) by deterministic paradifferential calculus, we need the sum of the two regularities to be positive, namely

\[
2\left(\alpha - \frac{3}{2}\right) + 3\left(\alpha - \frac{3}{2}\right) + 1 > 0 \quad \implies \quad \alpha > \frac{13}{10}.
\]
Otherwise, i.e. for $\alpha \leq \frac{13}{10}$, we will need to make sense of the product of $Z_{2,N}$ and $z_{2,N}$ using stochastic analysis (this is the most intricate point in the argument).

In either case, when the second term in

$$Z_{5,N} := 3((z_{1,N})^2 - \sigma_N) z_{2,N},$$

has positive regularity $3(\alpha - \frac{3}{2}) + 1 > 0$, i.e. $\alpha > \frac{7}{6}$, we show that the product

$$3((z_{1,N})^2 - \sigma_N) z_{2,N}$$

(in the limit) inherits the regularity from $Z_{2,N} = (z_{1,N})^2 - \sigma_N$, allowing us to pass to a limit in $H^s(\mathbb{T}^3)$ for

$$s < 2\left(\alpha - \frac{3}{2}\right).$$
The equation for $w_N$

- Once we are able to pass the term
  \[ Z_{5, N} = 3((z_{1, N})^2 - \sigma_N)z_{2, N}, \]
in the limit $N \to \infty$, the main issue in solving the equation for $w_N$ by the deterministic Strichartz theory is to ensure that the solution of
  \[ Lw + 3((z_{1, N})^2 - \sigma_N)z_{2, N} = 0 \tag{5} \]
with zero initial data remains bounded in $H^\frac{1}{2}(\mathbb{T}^3)$ as $N \to \infty$.
- Using again one degree of smoothing under the wave Duhamel operator, we see that the solution to (5) is almost surely bounded in $H^\frac{1}{2}(\mathbb{T}^3)$, provided
  \[ 2\left(\alpha - \frac{3}{2}\right) + 1 > \frac{1}{2} \implies \alpha > \frac{5}{4}. \]
- This explains the restriction $\alpha > \frac{5}{4}$ in our result.
In the proof of our result, we apply the deterministic Strichartz theory and show that $w_N$ converges almost surely to some limit $w$.

Along with the almost sure convergence of $z_{1,N}$ and $z_{2,N}$ to some limits $z_1$ and $z_2$, respectively, we conclude from the decomposition

$$u_N = z_{1,N} + z_{2,N} + w_N$$

that $u_N$ converges almost surely to

$$u := z_1 + z_2 + w.$$ 

By taking a limit in the equation for $w_N$, we see that $w$ is almost surely the solution to

$$\begin{cases}
    Lw + (w + z_2)^3 + 3z_1(w + z_2)^2 + 3Z_2w + 3Z_5 = 0 \\
    (w, \partial_t w)|_{t=0} = (0, 0),
\end{cases}$$

where $Z_2$ and $Z_5$ are the limits of $Z_{2,N}$ and $Z_{5,N}$, respectively.

This equation for $w$ may be seen as the limit equation for $u - z_1 - z_2$. 

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