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Random data wave equations

(lecture 5)

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The nonlinear wave equation

• Recall that we study the Cauchy problem

$$\begin{cases} \partial_t^2 u - \Delta u + u^3 = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3), \end{cases}$$

where $u : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$ and $\mathcal{H}^s(\mathbb{T}^3) = H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$.

• The initial data is chosen among the realisations of the random series

$$u_0^\omega = \sum_{n \in \mathbb{Z}^3} \frac{g_n(\omega)}{\langle n \rangle^\alpha} e^{in \cdot x} \qquad \text{and} \qquad u_1^\omega = \sum_{n \in \mathbb{Z}^3} \frac{h_n(\omega)}{\langle n \rangle^{\alpha - 1}} e^{in \cdot x},$$

where the scalar random variables g_n and h_n satisfy suitable assumptions.

• Recall that μ_α is the non degenerate measure on $\mathcal{H}^s(\mathbb{T}^3)$, $0 < s < \alpha - 3/2$, induced by the map

$$\omega \longmapsto (u_0^\omega, u_1^\omega)\,.$$

Theorem 1 (Burq-Tz. (2010))
Let $\alpha > \frac{3}{2}$ *and* $s < \alpha - \frac{3}{2}$. *Let* $\{u_N\}_{N\in\mathbb{N}}$ *be a sequence of the smooth global solutions to*

$$\partial_t^2 u - \Delta u + u^3 = 0 \tag{1}$$

with the following random C^∞*-initial data:*

$$u_{0,N}^\omega(x) = \sum_{|n|\le N} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x} \qquad \text{and} \qquad u_{1,N}^\omega(x) = \sum_{|n|\le N} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x}\,.$$

Then, as $N \to \infty$, u_N *converges almost surely to a (unique) limit* u *in* $C(\mathbb{R}; H^s(\mathbb{T}^3))$, *satisfying* (1) *in a distributional sense.*

Theorem 2 (Oh-Pocovnicu-Tz. (2018))
Let $\frac{5}{4} < \alpha \leq \frac{3}{2}$ and $s < \alpha - \frac{3}{2}$. There exists a divergent sequence $\{\beta_N\}_{N\in\mathbb{N}}$ of positive numbers such that the following holds true. There exist small $T_0 > 0$ and positive constants C, c, κ such that for every $T \in (0, T_0]$, there exists a set Ω_T of complemental probability smaller than $C\exp(-c/T^\kappa)$ such that if we denote by $\{u_N\}_{N\in\mathbb{N}}$ the smooth global solutions to

$$\begin{cases} \partial_t^2 u_N - \Delta u_N + u_N^3 - \beta_N u_N = 0 \\ (u_N, \partial_t u_N)|_{t=0} = (u_{0,N}^\omega, u_{1,N}^\omega), \end{cases}$$

where the random initial data $(u_{0,N}^\omega, u_{1,N}^\omega)$ is given by the truncated Fourier series

$$u_{0,N}^\omega(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x}\ , \qquad u_{1,N}^\omega(x) = \sum_{|n|\leq N} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x}$$

then for every $\omega \in \Omega_T$, the sequence $\{u_N\}_{N\in\mathbb{N}}$ converges to some (unique) limiting distribution u in $C([-T,T]; H^s(\mathbb{T}^3))$ as $N \to \infty$.

Deterministic well-posedness in $\mathcal{H}^s(\mathbb{T}^3)$, $s \geq 1/2$

• As mentioned, the main tool are the Strichartz estimates that we discuss below.

• Let $\mathcal{L} = \partial_t^2 - \Delta + 1$. We write

$$\mathcal{L}^{-1} = (\partial_t^2 - \Delta + 1)^{-1}$$

to denote the Duhamel integral operator

$$\mathcal{L}^{-1}F(t) := \int_0^t \frac{\sin((t-t')\langle D\rangle)}{\langle D\rangle} F(t')dt', \quad \langle D\rangle = \sqrt{-\Delta + 1}.$$

• In other words, $u := \mathcal{L}^{-1}(F)$ is the solution to

$$\mathcal{L}u = F, \quad (u, \partial_t u)|_{t=0} = (0,0).$$

The Strichartz estimates

• The most basic regularity property of $\mathcal{L}^{-1}$ is the "wave regularity" estimate:

$$\|\mathcal{L}^{-1}(F)\|_{L^\infty([-T,T];H^s(\mathbb{T}^3))} \lesssim \|F\|_{L^1([-T,T];H^{s-1}(\mathbb{T}^3))}. \tag{2}$$

• The Strichartz estimates are important extensions of (2).

Theorem 3

• *Let* $0 < T \leq 1$. *Then, the following estimate holds:*

$$\|\mathcal{L}^{-1}(F)\|_{L^4([-T,T]\times\mathbb{T}^3)} + \|\mathcal{L}^{-1}(F)\|_{L^\infty([-T,T];H^{\frac{1}{2}}(\mathbb{T}^3))}$$
$$\lesssim \min\Big(\|F\|_{L^1([-T,T];H^{-\frac{1}{2}}(\mathbb{T}^3))}, \|F\|_{L^{\frac{4}{3}}([-T,T]\times\mathbb{T}^3)}\Big).$$

• *As a consequence (by duality) we also have that the solution of*

$$\mathcal{L}u = 0, \quad u(0) = u_0,\ \partial_t u(0) = u_1$$

satisfies

$$\|u\|_{L^4([-T,T]\times\mathbb{T}^3)} + \|u\|_{L^\infty([-T,T];H^{\frac{1}{2}}(\mathbb{T}^3))} \lesssim \Big(\|u_0\|_{H^{\frac{1}{2}}} + \|u_1\|_{H^{-\frac{1}{2}}}\Big).$$

Closing the circle with a cubic nonlinearity in $\mathcal{H}^{1/2}$

• For $T > 0$, we denote by X_T the closed subspace of $C([-T,T]; H^{\frac{1}{2}}(\mathbb{T}^3))$ endowed with the norm

$$\|u\|_{X_T} = \|u\|_{L^\infty([-T,T];H^{\frac{1}{2}}(\mathbb{T}^3))} + \|u\|_{L^4([-T,T]\times\mathbb{T}^3)}\,.$$

• Then the Strichartz estimates imply that

$$\|\mathcal{L}^{-1}(u^3)\|_{X_T} \leq C\|u\|_{X_T}^3$$

• By a fixed point argument in X_T, the last estimate together with the corresponding estimate for the free evolution can be easily transformed into small data local well-posedness in $\mathcal{H}^{1/2}(\mathbb{T}^3)$ of

$$\mathcal{L}u + u^3 = 0\,.$$

• A very small variation yields large data local well-posedness in $\mathcal{H}^s(\mathbb{T}^3)$, $s > 1/2$ (we need to lose a bit of regularity in order to remove the smallness condition).

The free random evolution

- Fix $\alpha \leq \frac{3}{2}$. Recall that

$$\mathcal{L} = \partial_t^2 - \Delta + 1\,.$$

- Denote by $z_{1,N} = z_{1,N}(t,x,\omega)$ the solution to

$$\mathcal{L} z_{1,N}(t,x,\omega) = 0$$

with the random initial data $(u^\omega_{0,N}, u^\omega_{1,N})$ given by the truncated Fourier series

$$u^\omega_{0,N}(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x}\,, \qquad u^\omega_{1,N}(x) = \sum_{|n|\leq N} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x}\,.$$

The free random evolution (sequel)

- Given $t \in \mathbb{R}$, define $g_n^t(\omega)$ by

$$g_n^t(\omega) := \cos(t\langle n\rangle)\, g_n(\omega) + \sin(t\langle n\rangle)\, h_n(\omega). \tag{3}$$

- Then, we have

$$\begin{aligned} z_{1,N}(t,x,\omega) &= \cos(t\langle\nabla\rangle)\Big(z_{1,N}(0,x,\omega)\Big) + \frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle}\Big(\partial_t z_{1,N}(0,x,\omega)\Big) \\ &= \sum_{|n|\le N} \frac{g_n^t(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x}. \end{aligned}$$

- Using the definitions of the Gaussian random variables $\{g_n\}_{n\in\mathbb{Z}^3}$ and $\{h_n\}_{n\in\mathbb{Z}^3}$, we see that $\{g_n^t\}_{n\in\mathbb{Z}^3}$ defined in (3) forms a family of independent standard complex-valued Gaussian random variables conditioned that $g_n^t = \overline{g_{-n}^t}$ (in particular, g_0^t is real-valued).

The free random evolution (sequel)

• In the following, we discuss spatial regularities of various stochastic terms for fixed $t \in \mathbb{R}$. For simplicity of notation, we suppress the t-dependence and discuss spatial regularities.

• It is easy to see that $z_{1,N}$ converges almost surely to some limit z_1 in $H^{s_1}(\mathbb{T}^3)$ as $N \to \infty$, provided that

$$s_1 < \alpha - \frac{3}{2}.$$

• In particular, when $\alpha \leq \frac{3}{2}$, $z_{1,N}$ has negative Sobolev regularity (in the limiting sense) and thus $(z_{1,N})^2$ and $(z_{1,N})^3$ do not have well defined limits (in any topology) as $N \to \infty$ since it involves products of two distributions of negative regularities.

Giving a sense of the square and the cube in negative Sobolev spaces

- Let u_N be the solution to the renormalized nonlinear wave equation

$$\partial_t^2 u_N - \Delta u_N + u_N^3 - \beta_N u_N = 0$$

with the same truncated random initial data $(u_{0,N}^\omega, u_{1,N}^\omega)$. By writing u as

$$u_N = z_{1,N} + v_N,$$

we see that the residual term $v_N = u_N - z_{1,N}$ satisfies the following equation:

$$\mathcal{L} v_N + v_N^3 + 3 z_{1,N} v_N^2 + 3((z_{1,N})^2 - \sigma_N) v_N + ((z_{1,N})^3 - 3\sigma_N z_{1,N}) = 0$$

with zero initial data, where the parameter σ_N is defined by

$$\sigma_N := \frac{\beta_N + 1}{3}.$$

Giving a sense of the square and the cube in negative Sobolev spaces

• The key point is that the terms

$$Z_{2,N} := (z_{1,N})^2 - \sigma_N \qquad \text{and} \qquad Z_{3,N} := (z_{1,N})^3 - 3\sigma_N z_{1,N}$$

are "renormalizations" of $(z_{1,N})^2$ and $(z_{1,N})^3$.

• Here, by "renormalizations", we mean that by choosing a suitable renormalization constant σ_N, the terms $Z_{2,N}$ and $Z_{3,N}$ converge almost surely in suitable negative Sobolev spaces as $N \to \infty$.

• The regularity $s_1 < \alpha - \frac{3}{2}$ of $z_{1,N}$ (in the limit) and a basic computation (as in the first lecture) show that if the expressions $Z_{2,N} = (z_{1,N})^2 - \sigma_N$ and $Z_{3,N} = (z_{1,N})^3 - 3\sigma_N z_{1,N}$ have any well defined limits as $N \to \infty$, then their regularities in the limit are expected to be

$$s_2 < 2\Big(\alpha - \frac{3}{2}\Big) \qquad \text{and} \qquad s_3 < 3\Big(\alpha - \frac{3}{2}\Big),$$

respectively.

Giving a sense of the square and the cube in negative Sobolev spaces

• In fact, by choosing the renormalization constant σ_N as

$$\sigma_N := \mathbb{E}\Big[\Big(z_{1,N}(t,x,\omega)\Big)^2\Big],$$

we show that $Z_{j,N}$, $j = 2,3$ converge in $H^{s_j}(\mathbb{T}^3)$ almost surely.

• The renormalization constant σ_N a priori depends on t, x but it turns out to be independent of t and x. This fact can be seen by a direct computation. It also follows from the stationarity (in both t and x) of the stochastic process $\{z_{1,N}(t,x)\}_{(t,x)\in\mathbb{R}\times\mathbb{T}^3}$.

• We will also see that, for $N \gg 1$, σ_N behaves like $\sim N^{3-2\alpha}$ when $\alpha < \frac{3}{2}$.

The first Picard iteration ansatz

- We know that the deterministic Cauchy problem for

$$\mathcal{L}v + v^3 = 0$$

is locally well-posed in $\mathcal{H}^s(\mathbb{T}^3)$ for $s \geq \frac{1}{2}$.

- Therefore, we may hope to solve the equation

$$\mathcal{L}v_N + v_N^3 + 3z_{1,N}v_N^2 + 3((z_{1,N})^2 - \sigma_N)v_N + ((z_{1,N})^3 - 3\sigma_N z_{1,N}) = 0$$

uniformly in $N \in \mathbb{N}$ if we can ensure that the solution v_N to the following linear problem:

$$\mathcal{L}v_N + ((z_{1,N})^3 - 3\sigma_N z_{1,N}) = 0 \tag{4}$$

with the zero initial data $(v_N, \partial_t v_N)|_{t=0} = (0,0)$ remains bounded in $H^{\frac{1}{2}}(\mathbb{T}^3)$ as $N \to \infty$.

- Using the wave-regularity, we see that the solution to (4) is almost surely bounded in $H^{\frac{1}{2}}(\mathbb{T}^3)$ uniformly in $N \in \mathbb{N}$, provided

$$3\left(\alpha - \frac{3}{2}\right) + 1 > \frac{1}{2} \quad \Longrightarrow \quad \alpha > \frac{4}{3}.$$

Therefore, $\alpha = \frac{4}{3}$ is the limit of the first Picard iteration ansatz.

The second Picard iteration ansatz

• In order to go below the $\alpha = \frac{4}{3}$ threshold, a new argument is needed. More precisely, we further decompose v_N as

$$v_N = z_{2,N} + w_N,$$

where $z_{2,N}$ is the solution to the following equation:

$$\begin{cases} \mathcal{L} z_{2,N} + ((z_{1,N})^3 - 3\sigma_N z_{1,N}) = 0 \\ (z_{2,N}, \partial_t z_{2,N})|_{t=0} = (0,0). \end{cases}$$

• Thanks to the one degree of smoothing, we see that $z_{2,N}$ converges to some limit in $H^s(\mathbb{T}^3)$, provided that

$$s = s_3 + 1 < 3\Big(\alpha - \frac{3}{2}\Big) + 1$$

The second Picard iteration ansatz (sequel)

• In terms of the original solution u_N we have

$$u_N = z_{1,N} + z_{2,N} + w_N.$$

Note that $z_{1,N} + z_{2,N}$ corresponds to the Picard second iterate for the truncated renormalized equation.

• The equation for w_N can now be written as

$$\mathcal{L}w_N + (w_N + z_{2,N})^3 + 3z_{1,N}(w_N + z_{2,N})^2 + 3((z_{1,N})^2 - \sigma_N)(w_N + z_{2,N}) = 0,$$

with zero initial data.

• By using the second order expansion, we have eliminated the most singular term $Z_{3,N} = (z_{1,N})^3 - 3\sigma_N z_{1,N}$ in the equation for v_N.

• In the equation for w_N, there are several source terms (namely, purely stochastic terms independent of the unknown w_N) and they are precisely the quintic, septic, and nonic (i.e. degree nine) terms added in considering the Picard third iterate.

The second Picard iteration ansatz (sequel)

• It turns out that the most singular source term is the following quintic term:

$$Z_{5,N} := 3((z_{1,N})^2 - \sigma_N) z_{2,N},$$

where we recall that $z_{2,N}$ is the solution to

$$\begin{cases} \mathcal{L} z_{2,N} + ((z_{1,N})^3 - 3\sigma_N z_{1,N}) = 0, \\ (z_{2,N}, \partial_t z_{2,N})|_{t=0} = (0,0). \end{cases}$$

• As we already mentioned, the term $Z_{2,N} = (z_{1,N})^2 - \sigma_N$ and the second order term $z_{2,N}$ pass to the limits in $H^s(\mathbb{T}^3)$ for $s < 2(\alpha - \frac{3}{2})$ and $s < 3(\alpha - \frac{3}{2}) + 1$, respectively.

• In order to make sense of the product of $Z_{2,N}$ and $z_{2,N}$ by deterministic paradifferential calculus, we need the sum of the two regularities to be positive, namely

$$2\left(\alpha - \frac{3}{2}\right) + 3\left(\alpha - \frac{3}{2}\right) + 1 > 0 \quad \Longrightarrow \quad \alpha > \frac{13}{10}.$$

The second Picard iteration ansatz (sequel)

• Otherwise, i.e. for $\alpha \leq \frac{13}{10}$, we will need to make sense of the product of $Z_{2,N}$ and $z_{2,N}$ using stochastic analysis (this is the mosts intricate point in the argument).

• In either case, when the second term in

$$Z_{5,N} := 3((z_{1,N})^2 - \sigma_N) z_{2,N},$$

has positive regularity $3(\alpha - \frac{3}{2}) + 1 > 0$, i.e. $\alpha > \frac{7}{6}$, we show that the product

$$3((z_{1,N})^2 - \sigma_N) z_{2,N}$$

(in the limit) inherits the regularity from $Z_{2,N} = (z_{1,N})^2 - \sigma_N$, allowing us to pass to a limit in $H^s(\mathbb{T}^3)$ for

$$s < 2\left(\alpha - \frac{3}{2}\right).$$

The equation for w_N

- Once we are able to pass the term

$$Z_{5,N} = 3((z_{1,N})^2 - \sigma_N)z_{2,N},$$

in the limit $N \to \infty$, the main issue in solving the equation for w_N by the deterministic Strichartz theory is to ensure that the solution of

$$\mathcal{L}w + 3((z_{1,N})^2 - \sigma_N)z_{2,N} = 0 \tag{5}$$

with zero initial data remains bounded in $H^{\frac{1}{2}}(\mathbb{T}^3)$ as $N \to \infty$.

- Using again one degree of smoothing under the wave Duhamel operator, we see that the solution to (5) is almost surely bounded in $H^{\frac{1}{2}}(\mathbb{T}^3)$, provided

$$2\left(\alpha - \frac{3}{2}\right) + 1 > \frac{1}{2} \quad \Longrightarrow \quad \alpha > \frac{5}{4}.$$

- This explains the restriction $\alpha > \frac{5}{4}$ in our result.

The limit equation

• In the proof of our result, we apply the deterministic Strichartz theory and show that w_N converges almost surely to some limit w.
• Along with the almost sure convergence of $z_{1,N}$ and $z_{2,N}$ to some limits z_1 and z_2, respectively, we conclude from the decomposition

$$u_N = z_{1,N} + z_{2,N} + w_N$$

that u_N converges almost surely to

$$u := z_1 + z_2 + w\,.$$

• By taking a limit in the equation for w_N, we see that w is almost surely the solution to

$$\begin{cases} \mathcal{L}w + (w + z_2)^3 + 3z_1(w + z_2)^2 + 3Z_2 w + 3Z_5 = 0 \\ (w, \partial_t w)|_{t=0} = (0, 0), \end{cases}$$

where Z_2 and Z_5 are the limits of $Z_{2,N}$ and $Z_{5,N}$, respectively.
• This equation for w may be seen as the limit equation for $u - z_1 - z_2$.