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GENERALIZED COMPLEX GEOMETRY, SASAKIAN MANIFOLDS AND ADS/CFT

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The goal of this thesis is to illustrate the AdS/CFT Conjecture of High-Energy Physics in a mathematical light. This thesis will begin with an exposition of the analytical nature and background of AdS/CFT in terms of a specific type of Yamabe problem and Sasaki-Einstein manifolds. This will conclude with a brief discussion of the $Y^{p,q}$ manifolds of String Theory, which disprove a conjecture of G. Tian and J. Cheeger. The thesis will then discuss a (hopefully) balanced view of Generalized Complex Geometry, oscillating between the coordinate-focused exposition of Koerber and the G-Structure and Gerbe focused viewpoints of Gualtieri and Hitchin. A (hopefully) cleaner presentation of Sparks' *Generalized Sasaki Manifolds* will conclude the expository portion of this thesis. At the end, a new and explicit example of a Generalized Sasakian Metric will be presented.

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Part I

Physical Motivation

1 Brief History

Much of the early history of the close relationship between Physics and Mathematics is well-known, for one often hears of Isaac Newton, Joseph Louis Lagrange and Pierre Simon Laplace in early calculus and mechanics classes. The close relationship continued through the 19th century with the outstanding contributions of Gauss, Maxwell and Hamilton that led to drastic improvements in understanding the then-novel force of electromagnetism. At the dawn of the 20th century, the intuitive and geometric wizardry of Einstein helped formulate the equations of General Relativity and subsequently led to an explosion of research into Lorentzian Geometry. Modern Math Historians often claim that the apex of this relationship occurred in the 1920s and 1930s with the works of John von Neumann and Paul Dirac on Quantum Mechanics. This relationship is exemplified in the following quote of Paul Dirac:

> The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected. What, however, was not expected by the scientific workers of the last century was the particular form that the line of advancement of the mathematics would take, namely, it was expected that the mathematics would get more and more complicated, but would rest on a permanent basis of axioms and definitions, while actually the modem physical developments have required a mathematics that continually shifts its foundations and gets more abstract. Non-euclidean geometry and non-commutative algebra, which were at one time considered to be purely fictions of the mind and pastimes for logical thinkers, have now been found to be very necessary for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of the mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation.
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> — Paul A.M. Dirac, **Quantised Singularities in the Electromagnetic Field**, Proceedings of the Royal Society A, 1931.

However, Dirac's prediction of theoretical physics requiring increasing mathematical sophistication was for the most part left unfulfilled from the 1930s to the 1970s as the interest in General Relativity waned (perhaps due to the war) and the great practitioners of Quantum Field Theory found it far easier to employ techniques that were ill-defined mathematically, but agreed well with accelerator experiments. However, many of the problems that arose in describing Quantum Chromodynamics stymied physicists in the 1960s and 1970s and were resolved when Murray Gell-Mann discovered the eightfold path and its relation to the representation theory of semi-simple Lie Groups. This realization that mathematicians may have "inadvertently" discovered some of theory behind modern physical conundrums led to an increased level of cooperation between the mathematics and physics communities. A hallmark moment in this relationship of mathematics and physics took place in 1975 when C.N. Yang invited mathematician James Simons to Stony Brook to give a series of lectures explaining the mathematics behind the "non-integrable phase factors" that Dirac introduced in *Quantised Singularities in the Electromagnetic Field*. During these lectures, Yang learned about Fiber Bundles, Differential Forms, Characteristic Classes and in particular Chern-Weil Theory, which relates the Lie Algebra $\mathfrak{g}$ associated to a principal-G bundle to certain characteristic classes. By 1979, Yang had realized that mathematicians had invented a theoretical framework for the non-abelian gauge theories that he had stumbled upon while formulating what became Yang-Mills gauge theory.

The above two descriptions point out examples of mathematics furthering an understanding of physics and in the late 1970s and early 1980s, examples of the converse began to occur; that is, physical concepts played an important role in the proofs of various important theorems. The most prominent example of such an occurrence is the following theorem of Simon Donaldson:

Theorem 1.1. *(Donaldson, 1983) A simply connected, smooth 4-manifold with positive definite intersection form*[1] q *has the property that* q *is always diagonalizable over* $\mathbb{Z}$ *to* $q = diag(1, \ldots, 1)$

[1]M. Freedman defined the intersection form for a 4-dimensional topological manifold M as a map $q : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \to \mathbb{Z}, q(\alpha, \beta) = (\alpha \smile \beta)[M]$, where $[M]$ is the fundamental class if M is orientable.

The significance of this theorem in mathematics is that it gives a sufficient condition for a manifold to be smooth as opposed to piecewise linear or simply a topological manifold. As such one can begin to classify the smooth structures on a 4-manifold via this theorem, a feat that surprised the mathematical community at the time of its publication.The proof heavily relies on the use of the Yang-Mills action, which concerns the finite-dimensional moduli space of connections on line bundles, to determine the eigenvalues of a given intersection form.

The main subject of this thesis, Generalized Sasakian Geometry, is closely related to **String Theory** and indeed its roots lie in both Mathematics and Physics. Just as in the proceeding examples, String Theory has had an enormous amount of interaction with the mathematical community and for better or for worse, it is often the "poster-child" of the close relationship between Mathematics and Theoretical Physics.

2 What is a Field Theory?

From a mathematical perspective, one is used to hearing that "quantum field theories are ill-defined." However, some of the mathematical structures involved can be described precisely and can provide a mathematician with some intuition into what a "field theory" might be. In the case of String Theory, one cares about **effective field theories**. Before going any further, let's establish the definitions for effective field theories:

- An **effective field theory** is a field theory that has an energy scale E, which can roughly be thought of as an approximation to the "maximum" allowable energy, such that all observed physical phenomena can be produced if the systems energy is less than E. Note that this also means that one need not know anything about the degrees of freedom at higher energies; this is a key point for the existence of String Theory. String Theory purports to reduce in certain limits to an effective field theory with energy scale of $E \approx 1.22 \times 10^{19}$ GeV (the Planck Energy) that reproduces known phenomena that are at far lower energy scales. The mathematical objects required to specify an effective field theory are:
  - A smooth n-manifold M that serves as a model of spacetime
  - The configuration space $\mathcal{C}$ is a smooth manifold with metric g that serves as the space that the **scalar fields** take values in. The scalar fields are simply smooth maps $M \to \mathcal{C}$ and represent a coordinate-independent, scalar observable, such as temperature or pressure at a point. Note that we can define the time-derivative of a scalar field $\dot{\phi}$ as $\dot{\phi} = \phi_*(\partial_t)$ where ∂_t is the tangent vector conjugate to the distinguished time coordinate of M. In this paper, we will see that $\mathcal{C} = \wedge^\bullet M$.
  - A function $V : \mathcal{C} \to \mathbb{R}$ is known as the **potential** and is used to determine the equations of motion. In particular, if $\phi \in \mathcal{S} = \{f : M \to \mathcal{C}\}$ is a scalar field then we construct a Lagrangian $\mathcal{L} : \mathcal{S} \to \mathbb{R}$ by

$$\mathcal{L}(\phi) = \frac{1}{2} g(\dot{\phi}, \dot{\phi}) + V \circ \phi \tag{1}$$

  - A complex vector bundle $\mathcal{F}$ with metric $\hat{g}$ over $\mathcal{C}$ in which the fermions take values. In the case of interest for this paper, $\mathcal{F}$ will be $\wedge^\bullet \mathcal{C}$; the reasons for this will be elucidated in part III. Note that the Lagrangian (1) only includes scalar fields. It is possible (and physically necessary) to include fermionic contributions so that in general one defines the potential as a map $V : \mathcal{S} \times \mathcal{F}^M \to \mathbb{R}$, where $\mathcal{F}^M = \{f : M \to \mathcal{F} : f \text{ smooth}\}$. We can naturally define the Lagrangian for $\alpha \in \mathcal{F}^M$,

$$\mathcal{L}(\phi, \alpha) = \frac{1}{2} \left(g(\dot{\phi}, \dot{\phi}) + \hat{g}(\dot{\alpha}, \dot{\alpha}) \right) + V \circ \phi + V \circ \alpha \tag{2}$$

  - A Lie group G with a free action on $\mathcal{C}$ which serves as the set of symmetries of the scalar fields and the Fermions via a free, transitive and often proper action on them.
  - A bilinear form $Y : \mathcal{F} \otimes \mathcal{F}|_\phi \to \mathbb{R}$ known as the **coupling** determines the experimental constants that V depends on.
- An $\mathcal{N} = 1$ **supersymmetric effective field theory** is an effective field theory with the following additional pieces of data:

- The configuration space $\mathcal{C}$ is now a complex manifold with Hermitian metric h and Kähler form ω
- The Lie group G now acts via holomorphic isometries; that is the action of $g \in G$ preserves h, ω
- A **moment map** $\mu : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ for that action of G is now specified.
- The potential V is replaced by a holomorphic **superpotential** $W : \mathcal{C} \to \mathbb{C}$ which is a G-invariant. In fact we define the real potential by $V = g(\partial W, \partial W) + |\mu|^2$, where ∂ is the holomorphic gradient

The equations of motion are constructed by extremizing the functional $S : \mathcal{S} \times \mathcal{F} \to \mathbb{R}$ defined in (2). Finally note that if $\mathcal{F}$ is a principal G-bundle, there are often additional terms in this functional that descend from a $\mathfrak{g}$-valued connection A with curvature $F = dA + A \wedge A$ on M. Note that G is often $\mathsf{Spin}(n)$. Terms of this form include the Yang-Mills action,

$$S_{YM} = \frac{1}{4g^2} \int_M \mathbf{Tr}\,(F \wedge *F) \tag{3}$$

and also Chern-Simons terms[2],

$$S_{CS} = \frac{k}{4\pi} \int_M \mathbf{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \tag{4}$$

where **Tr** is a G-invariant bilinear form on $\mathfrak{g}$. An example of how this form is constructed and used in physics will be defined explicitly in the section on supersymmetry.

3 String Theory

To many philosophers, the main problem of physics is the question, "How can we describe the natural universe in a quantitatively consistent manner?" Some theoretical physicists believe that String Theory provides an answer to this question; however, the theory has much humbler roots. String Theory generally finds its roots in the 1970s with the ambiguity over the discovery of the strong nuclear force. In 1970, the existence of smaller particles that made up protons and neutrons in the nucleus was well-known with the discovery of the quark. However, the mystery of why these particles "stuck" together and created coherent objects, such as a proton, was still unsolved. Quantum Chromodynamics (QCD) was invented later in the 1970s to describe this mysterious force that assembled quarks giving rise to hadrons.[3] However until the introduction of QCD, theorists proposed alternative models for the strong interaction, such as String Theory.

Development of the String

The model that later became String Theory was known as the dual-resonance model, proposed by Gabriele Veneziano in 1968. In the 1960s, theories attempted to use scattering amplitudes, which correspond to the energy of a scattered particle, to demystify the unexplainable experimental evidence for the strong force. One of the fundamental problems of the time was the issue of how the Strong Force and the particles found during scattering experiments were related. Veneziano's model proposed a relationship between the incoming particles and outgoing particles in a scattering experiment that treated the intermediate particle formed during the reaction as a simple harmonic oscillator. By 1970, Nombu and Susskind generalized this scattering reaction to an ensemble of N

[2] Let us define the Chern-Simons Form, since this is often overlooked in the String Theory literature. Let $\pi : P \to M$ be a principal G-bundle. Recall that the Chern-Weil Homomorphism is a map from $\mathbb{R}[\mathfrak{g}]^{\mathrm{Ad}(G)}$, the polynomials of $\mathfrak{g}$ with coefficients in $\mathbb{R}$ invariant under the Adjoint action to $H^\bullet(B^G, \mathbb{R})$ where B^G is the classifying space for principal G-bundles. The **Chern-Simons form** is a form $\theta_{\mathcal{P}} \in \wedge^\bullet P$ such that $d\theta_{\mathcal{P}} = \mathcal{P}(F)$ where $\mathcal{P} \in \mathbb{R}[\mathfrak{g}]^{\mathrm{Ad}(G)}$ and F is the Lie algebra-valued curvature form so that θ_P determines a real characteristic class. Hence these terms, loosely speaking, give some information about non-trivial characteristic classes (hence topology) of the space constructed from the minimization of either S_{YM}, S_{CS}

[3] Hadrons were particles that were once thought to be fundamental; however, once thirty or so of them were discovered physicists began to believe that they were made up of tinier particles. This guess came to fruition with the discovery of the quark. Thus a hadron can be defined as any particle made up of three quarks (Baryons) or of one quark and one antiquark, the charge conjugate of a quark.

particles and interpreted the harmonic oscillator that connected the incoming and outgoing particles as a one-dimensional object or string. This interpretation gave a small troupe of physicists the idea that the fundamental particle of nature was not point-like and 0-dimensional as quantum field theory implicitly assumes, but instead a one-dimensional object whose various vibrational modes give rise to the particle-like behavior observed in experiments. By instead starting from this assumption, various physicists were able to construct a physical theory based on these one-dimensional objects by minimizing the area swept out over time by these objects. Mathematically, the "area swept out" is a two-dimensional Riemann Surface Σ that is known as the **worldsheet** In particular, one attempts to minimize the **Nambu- Goto functional** $S_{NB} : A \to \mathbb{R}, A = \{f : \Sigma \to \mathbb{R}^D\}$, which is defined as,

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int \sqrt{-\det g}\ dq^0 dq^1 \qquad \text{Extremized via: } \frac{d}{d\epsilon} S_{NG}(x + \epsilon y)|_{\epsilon=0} = 0 \tag{5}$$

where $g = x^*(\eta)$ is the metric on the worldsheet Σ induced by the parametrization $x : \Sigma \to \mathbb{R}^D$ for some $D > 0$ and such that g is Lorentzian, $\det g < 0$. Note that this action is conformally-invariant since a standard result of Lorentzian geometry shows that every two-dimensional Lorentz manifold is conformally flat; in other words if (M, g) is a Lorentzian manifold then,

$$\exists \Omega \in C^\infty(M), \psi : M \to \mathbb{R}^{1,1} \text{ such that } \psi^* g = \Omega^2 \eta$$

where η is the standard metric on $\mathbb{R}^{1,1}$. As such, any solution to (5) determines an entire conformal class of solutions. In order to make meaningful physical computations, one needs to choose a specific conformal representative. In particular if (σ, τ) are local coordinates on Σ, the **conformal gauge** is the following set of constraints:

$$\begin{gathered} \frac{\partial^2 x}{\partial \sigma^2} - \frac{\partial^2 x}{\partial \tau^2} = \eta\left(\frac{\partial x}{\partial \sigma}, \frac{\partial x}{\partial \tau}\right) = 0 \\ \eta\left(\frac{\partial x}{\partial \sigma}, \frac{\partial x}{\partial \sigma}\right) + \eta\left(\frac{\partial x}{\partial \tau}, \frac{\partial x}{\partial \tau}\right) = 0 \quad \eta\left(\frac{\partial x}{\partial \sigma}, \frac{\partial x}{\partial \tau}\right) < 0 \end{gathered} \tag{6}$$

At the time of discovery, the intuitive choice of $D = 4$ was not enforced since it turned out that there are anomalous, physically-unrealistic solutions if $D = 4$. In particular, if $D \neq 26$ then upon enforcing an ersatz canonical quantization, it is possible to construct states with negative mass. The fact that one requires a 26-dimensional spacetime as well as the lack of fermions (e.g. electrons) made this theory quite unsatisfactory and it was treated as a toy model until the advent of supersymmetry.

4 Supersymmetry

The advent of the concept of **Supersymmetry** in the 1980s allowed for the (Bosonic) 26-dimensional theory to become a ten-dimensional theory that admits both **Bosons** and **Fermions**. Physically, Bosons are particles that have spin (defined in the proceeding sections) in $\mathbb{Z}$ while Fermions are particles that have spin angular momentum in $\frac{1}{2}\mathbb{Z} - \mathbb{Z}$. The mathematical definitions will be given in the first subsection with physical intuition for why such objects are needed in the subsequent section. Heuristically, one can define supersymmetry as an operation that takes a boson (resp. fermion) and yields a fermion (resp. boson). Supersymmetry has been exploited in differential geometry as a purely mathematical tool in the form of Seiberg-Witten, Donaldson-Floer-Witten and Gromov-Witten invariants and as such there is a relatively strong mathematical definition.

4.1 Mathematics of Supersymmetry

Before describing Bosons and Fermions, one needs to consider the representation theories of $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$ which represent the fundamental difference between spin angular momentum and orbital angular momentum, respectively.

$\mathsf{SU}(2), \mathsf{SO}(3)$

In physics, representations of Lie group and Lie algebra actions are of importance because they represent the **symmetries** of a system. In quantum mechanics one is given a complex, separable, infinite-dimensional Hilbert

Space $(\mathcal{H}, (\cdot,\cdot)\rangle)$ and is interested in characterizing the transformations $\mathcal{H} \to \mathcal{H}$ that preserve leave the map $\psi \mapsto |(\psi,\psi)|^2$ invariant. Note that if $c \in S^1 \hookrightarrow \mathbb{C}$, then $|(c\psi, c\psi)|^2 = |(\psi,\psi)|^2$ so that we can quotient by the action of $S^1 \cong U(1)$. In particular, we define the **Projective Hilbert Space** $\mathbf{P}(\mathcal{H}) = \mathcal{H}/\sim$ where $\sim$ is the equivalence relation $\psi \sim c\psi$, if $c \in U(1)$. We are now in a position to formally define a symmetry:

Definition 4.1. *A **symmetry** of a quantum system $\mathcal{H}$ is a bijection $\Lambda : \mathbf{P}(\mathcal{H}) \to \mathbf{P}(\mathcal{H})$ such that $|(\Lambda[\psi], \Lambda[\psi])|^2 = |([\psi],[\psi])|^2, \forall \psi \in \mathbf{P}(\mathcal{H})$*

From a mathematical perspective, one the main reasons that Quantum Mechanics is a linear theory is due to the following theorem of Wigner:

Theorem 4.1 (Wigner). *Every symmetry is induced by a unitary or antiunitary operator of $\mathbf{P}(\mathcal{H})$ and the set of symmetries form a group denoted $\mathcal{S}(\mathcal{H})$. In particular, if G is a connected Lie group and $\lambda : G \to \mathcal{S}(\mathcal{H})$ is a group homomorphism, then for each $g \in G$ there exists a unitary operator $L(g) \in \mathcal{S}(\mathcal{H})$ such that $\lambda(g)$ is induced by $L(g)$. As such any representation $G \to \mathcal{S}(\mathcal{H})$ is a unitary representation*

Before leaving the realm of purely abstract definitions, let us recall the notion of a central extension of a Lie group:

Definition 4.2. *Suppose that A is an abelian group and G is an arbitrary Lie group. One says that the Lie group E is an extension of G by A if the following sequence is exact:*

$$\{e\} \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to \{e\}$$

*where $\iota : A \to E, \pi : E \to G$ are the natural inclusion and projection, respectively. E is known as a **central extension** if $\iota(A) \subset Z(E)$, the center of E*

In particular, if G is a connected Lie group and $p : \tilde{G} \to G$ is its universal cover, then $\ker p := p^{-1}(e_G)$ has the discrete topology and is contained in the center of $\tilde{G}$. This allows us to regard $\tilde{G}$ as a central extension of G by $\ker p$.

We can now speak of $SU(2), SO(3)$ directly. Recall that $SU(2)$ is treated as a real Lie group of Hermitian matrices of determinant 1, whereas $SO(3)$ is defined as the connected component of the identity of the set of matrices $O(3) \subset GL(3,\mathbb{R})$ that preserve the quadratic form $x_1^2 + x_2^2 + x_3^2$. Moreover, recall that $\text{Lie}(SU(2)) := \mathfrak{su}(2)$ is the set of trace 0, Hermitian 2×2 matrices, which can always be put in the form,

$$H = \begin{pmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_3 & -x_1 \end{pmatrix} \quad x_i \in \mathbb{R}$$

In particular, this means that $\det H = -x_1^2 - x_2^2 - x_3^2$. Note that under the action of $\text{Ad}(G)$, we have $\det gHg^{-1} = \det H, \forall g \in SU(2)$ so that we can treat any element of $SU(2)$ as an element of $O(3)$. Note that under this (surjective) map $f : SU(2) \to O(3)$, $\mathbb{Z}_2 \cong \{\pm 1\} \in SU(2)$ are sent to the same element so that $\ker f \cong \mathbb{Z}_2$. Moreover as $SU(2)$ is connected, $f(SU(2)) = SO(3)$. From the previous discussion, this means that $SU(2)$ is the central extension of $SO(3)$ by $\mathbb{Z}_2$ and as $\{e\}$ is the only proper subgroup of $\mathbb{Z}_2$, $SU(2)$ is the universal cover of $SO(3)$.

Finally let's discuss the implications that this has on representation theory. A known fact (for example, see [27]) is that for a vector space V, any irreducible projective unitary representation $SO(3) \to PGL(V)$ is finite-dimensional and arises from any ordinary irreducible unitary representation $SU(2) \to GL(V)$. Now note that a general representation of $SU(2)$ is parametrized by $j \in \frac{1}{2}\mathbb{Z}$ and is of dimension $2j+1$. Subsequently, representations of $SO(3)$ are those for which $j \in \mathbb{Z}$. The first fact can be seen from the action of $SU(2)$ on the vector space of homogeneous polynomials of degree $2j$ in $z_1, z_2 \in \mathbb{C}^2$, which is vector space of dimension $2j+1$. On the other hand, if the dimension of the representation is $2j+1$, then the spin representation sends $-1 \in SU(2)$ to $(-1)^{2j} \in SO(3)$, so j must be an integer. The number j is the weight of the representation and physically represents the **spin** of the particle.

Bosons and Fermions

The main physical premise behind separating particles into those of integer and half-integer spin arises when one has a system where the constituents are effectively all **identical particles**. This is a quantum mechanical phenomena and mathematically we require some tenets of tensor algebra to describe the two systems. From a physical perspective, the two main examples of a Boson and a Fermion are the photon γ and electron e^-, respectively. We will use these two examples as building blocks for the general definition. Now suppose that we have N-particles of interest with associated Hilbert Spaces of states $\{\mathcal{H}_i\}_{i=1}^N$. Since we are attempting to look at the *simultaneous* state of the system, we need to consider the tensor product of these spaces, $\mathcal{H} := \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ so that if the ith particle in state ψ_i then the state of the whole N-particle system is $\psi_1 \otimes \cdots \otimes \psi_N$. However in order to take into account the fact that the particles are indistinguishable, we have to consider different permutations equivalent, depending on whether a particle is has half-integer spin (Fermion) or has integer spin (Boson).

For the case of N electrons, the antisymmetric case holds — that is, the ordered state $(\psi_1, \psi_2, \cdots, \psi_n) \in \bigoplus_{i=1}^N \mathcal{H}_i$ should be the negative of $(\psi_2, \psi_1, \cdots, \psi_n) \in \bigoplus_{i=1}^N \mathcal{H}_i$. Hence we define a Fermion to be an element of the exterior algebra $\wedge^N(\mathcal{H}_{e^-})$ where $\mathcal{H}_{e^-}$ is the *single electron Hilbert space*. When the system has N photons, experiments and physical rationales dictate that all permutations are equal and the ordered state $(\psi_1, \psi_2, \cdots, \psi_n) \in \bigoplus_{i=1}^N \mathcal{H}_i$ is invariant under the action of the symmetric group S_N on the particle labels. Hence the space of Bosons is $\mathsf{Sym}(\mathcal{H}_\gamma)$, the space of symmetric tensors in $\mathcal{H}_\gamma^{\otimes N}$ where $\mathcal{H}_\gamma$ is the *single photon Hilbert space*.

Let's mention a few properties of these spaces. If $P^{e^-} : \mathcal{H}_{e^i}^{\otimes N} \to \wedge^N(\mathcal{H}_{e^-})$ is an orthogonal projections and $\psi, \zeta, \alpha \in \mathcal{H}_{e^-}$, then we see that

$$P^{e^-}(\psi \otimes \zeta \cdots \otimes \psi \otimes \alpha) = \begin{cases} 0 & \text{if } N \geq 2 \\ \psi & \text{if } N = 1 \end{cases}$$

This is the overly formal statement of the **Pauli Exclusion Principle**, which states that two Fermions in an N particle ensemble cannot occupy the same state simultaneously.

Next, let's construct a system and its associated Hilbert space $\mathcal{H}$ that has a combination of Bosons and Fermions. If $\mathcal{H}_0$ is the Hilbert Space of states for one Boson and $\mathcal{H}_1$ is the Hilbert space of states for one Fermion, then we simply define $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1$. The N-particle space $\mathcal{H}_N$ is then:

$$\mathcal{H}_N = \bigoplus_{1 \leq d \leq N} \mathsf{Sym}^d(\mathcal{H}_0) \otimes \wedge^{N-d}(\mathcal{H}_1) \tag{7}$$

and if the particle number is not fixed then we simply define the total space to be $\mathcal{A} = \mathsf{Sym}(\mathcal{H}_0) \otimes \wedge(\mathcal{H}_1)$. Given this formulation of a many particle system, one might inquire about the symmetries of $\mathcal{A}$ or $\mathcal{H}_N$. However, the physical motivation for such an inquiry only occurred in the 1970s when physicists Zumino and Wess considered these spaces as **graded vector spaces** and introduced a notation of infinitesimal symmetry (e.g. Action of a something analogous to a Lie Algebra) on such vector spaces. If a generator of such an infinitesimal symmetry were g then out of purely mathematical convenience, Zumino and Wess defined the action of g in such a way that $g \cdot \mathsf{Sym}^\bullet(\mathcal{H}_0) \subset \wedge^\bullet(\mathcal{H}_1)$ and $g \cdot \wedge^\bullet(\mathcal{H}_1) \subset \mathsf{Sym}^\bullet(\mathcal{H}_0)$. Any such action became known as a **Supersymmetry** and due to the works of Pierre Deligne and F. A. Berezin solid definitions of the objects involved in describing this symmetry were formulated. One of the unique features of supersymmetry is that the one-parameter unitary semigroups $t \to \mathcal{S}(\mathcal{H})$ associated to Stone-von Neumann theorem became representations $\wedge^\bullet \mathbb{R}^n \to \mathcal{S}(\mathcal{H})_0 \otimes \mathcal{S}(\mathcal{H})_1, \theta_i \mapsto e^{\theta_i}(\psi_0 \otimes \psi_1)$. In order to deal with this peculiarity, one needs to define the notations of **super Lie algebras** and **supermanifolds**. We will go through the most basic definitions and state some of the theorems — however, this is a rather rich area of mathematics and one can find further information in [52, 44, 16] .

Before jumping into the definitions, let's consider what the mathematical intuition is behind prefixing a large number of mathematical objects with the word "super." One of the great achievements of Algebraic Geometry in the 1950s and 1960s was Alexander Grothendieck's introduction of the **spectrum** of a commutative ring. Note that the intuition behind this construction is that one would like to generalize the concept of the spectrum of an operator algebra to a specific structure or function sheaf over an algebraic variety in such a way that ideals of the spectrum correspond to open sets in the Zariski Topology. This allows for one to take an arbitrary commutative ring and associate to it a geometric object, which upon being topologized with the Zariski Topology can be interpreted as a locally-ringed space. This means that to each set we can associate a general ring to each open set that generalizes the notion of the ring of continuous real-valued functions over an open set in a topological space. One of the important examples of something found in this generalization that is not found in the standard structure sheaves such as the sheaf of continuous functions or the sheaf of holomorphic functions over a space is that the local ring associated to an open set can have a *nilpotent element*. From a functional analytic perspective, nilpotent elements can be thought of as an analogue of local (in the Zariski sense) nilpotent operators $A : V \to V, A^n = 0$ over a vector space V. The only difference is that instead of algebraically dealing with the module of polynomials in $A, \mathbb{R}[A]$, the local ring $R(U)$ associated to an open set U is the set of maps $U \to A$, where A is some associative algebra. This seemingly artificial construction of taking the standard structure sheaf of continuous functions and turning it into a structure sheaf of possibly non-commutative functions $U \to A$ ends up playing an important role in physics. It is in this sense, the objects associated with supersymmetry generalize this construction by instead associating to each open set of a topological space, a ring of functions $U \to A$ that *need not be commutative*. By extending the **Hilbert-Gel'fand Principle** which states that the geometric structure[4] of a space X can be recovered from the commutative algebra of functions on X, Grothendieck setup the framework for the close relationship between geometry and physics without realizing it.

Let's now go through some of the definitions and briefly discuss how linear algebra and the theory of smooth manifolds changes when one switches perspectives along the lines of the previous paragraph. Firstly, any object dealing with supersymmetry will have necessarily involve a **grading** that preserves the empirical fact (due to the requirements of the Dirac Equation) that Fermions need to be represented by elements of a Clifford Algebra so that if e_i, e_j are Fermions then $\{e_i, e_j\}$. As such a **super vector space** is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ where elements of V_0 are said to be **even** and elements of V_1 are known as **odd**. We define a map $p : V \to \mathbb{Z}_2$ called the **parity** map such that $p(a) = 0$ if a is even and $p(a) = 1$ if a is odd. Note that in all other constructions, the parity map takes the same definition even though the grading may not be explicitly given. In the case of (7), the Bosons are represented by even elements while the Fermions are represented by odd elements. Finally, let's go through the established notation of the literature. If $\mathbb{K}$ is a field and $V = \mathbb{K}^{p+q}$ with the standard basis $\{e_i\}_{i=1}^{p+q}$ and define e_i to be even if $i \le p$ and odd if $i > p$, then V is a super vector space with,

$$V_0 = \left\{ \sum_{i=1}^{p} k^i e_i : k^i \in \mathbb{K} \right\} \quad V_1 = \left\{ \sum_{i=p+1}^{p+q} k^i e_i : k^i \in \mathbb{K} \right\}$$

This super vector space is denoted $\mathbb{K}^{p|q}$ and will serve as the local model for supermanifolds.

Since Fermions are considered elements of a Clifford Algebra, we will need to speak of superalgebras:

Definition 4.3. *A* ***superalgebra*** *is an a super vector space* A *which is also an associative algebra with (even) unit* $\mathbf{1}_A$ *such that the parity map* $A \otimes A \to A$ *is a morphism,* $p(ab) = p(a) + p(b), \forall a, b \in A$. *A superalgebra is said to be* ***supercommutative*** *if,*

$$ab = (-1)^{p(a)p(b)} ba \qquad a, b \in A \tag{8}$$

The most common example of a superalgebra A is $A = \mathsf{End}(V)$ for a super vector space V. Note that we define $\mathsf{End}(V)$ as the set of all linear maps and one says that the even maps preserve the grading while the odd maps reverse the grading. For physics considerations, this is the most common example since quantum mechanics says that one looks at expectation values of operators in $\mathsf{End}(V)$. In the beginning of this section, it was mentioned that supersymmetry manifests itself as an "infinitesimal symmetry" similar to the action of a Lie Algebra. In order to speak of such symmetries we need to define a **Super Lie Algebra**:

Definition 4.4. *A* ***Super Lie Algebra*** *is a super vector space* $\mathfrak{g}$ *with a bracket* $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ *such that:*

a) $[a, b] = -(-1)^{p(a)p(b)}[b, a]$

b) $[a, [b, c]] + (-1)^{p(a)p(b)+p(a)p(c)}[b, [c, a]] + (-1)^{p(a)p(c)+p(b)p(c)}[c, [a, b]] = 0$

Note that the decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *allows one to regard* $\mathfrak{g}_1$ *as a left* $\mathfrak{g}_0$*-module and* $\mathfrak{g}_0$ *as a right* $\mathfrak{g}_1$*-module.*

The last piece of the definition represents the mathematical essence of supersymmetry: We want to consider the G-sets of actions of $\mathfrak{g}_i$ on $\mathfrak{g}_j, i \neq j$. Furthermore the bracket preserves the physical intuition that for Bosons (i.e. even degree) we the bracket is a commutator whereas for Fermions the bracket is an anticommutator. Note

[4]Examples of this are:
{Compact Hausdorff Spaces} $\cong$ {Commutative Banach $*$-algebras},
{Affine $\mathbb{C}$-Algebraic varieties} $\cong$ {Finitely-generated algebras over $\mathbb{C}$with no nonzero nilpotents}

that when we speak about generalized brackets in part III of this thesis, we will be brackets that generalize the bracket of a Super Lie Algebra, so all of the results that hold here will *automatically* hold in the case of generalized complex geometry. Note that if we have a super algebra $A = \mathsf{End}(V)$ for some super vector space V, we can define a bracket $[a, b] := ab - (-1)^{p(a)p(b)}ba$ to turn $\mathsf{End}(V)$ into a super Lie Algebra that we will denote $\mathfrak{gl}(V)$. If $V = \mathbb{R}^{p|q}$, one finds (in *both* the physics and mathematics literature) this Lie Algebra denoted $\mathfrak{gl}(p|q)$. Moreover note that the adjoint action $\mathsf{Ad}X\mathfrak{g} \to \mathfrak{g}, \mathsf{Ad}X(Y) = [X, Y]$ satisfies the same naturality property as in the case of a standard Lie Algebra. Recall that in (3) there was an operator Tr which was vaguely defined as a specific G-invariant bilinear form on the Lie algebra of $\mathfrak{g}$. We are now in a position to define Tr:

Definition 4.5. *Let $V = V_0 \oplus V_1$ be a finite-dimensional super vector space and let $X \in \mathfrak{gl}(V)$ be represented as*

$$\begin{pmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{pmatrix} \tag{9}$$

where $X_{ij} : V_j \to V_i$ is a linear map such that for all $v \in V_j$, $X_{ij}v$ is the projection of Xv onto V_i. The ***supertrace*** *of X is defined by:*

$$Tr(X) = tr(X_{00}) - tr(X_{11}) \tag{10}$$

Due to physical considerations, all of the Lie super algebras $\mathfrak{g} = \mathfrak{g}$ considered in this thesis will satisfy the following conditions:

1. $\mathfrak{g}$ is a **real Lie super algebra**.

2. $\mathfrak{g}_1$ is a **spinorial** $\mathfrak{g}_0 -$ module. This means that $\mathfrak{g}_1$ is a quadratic module (i.e. it has a $\mathbb{C}$-quadratic form) such that $\mathsf{End}(\mathfrak{g}_0) \cong C\ell(\mathfrak{g}_1)$, the Clifford Algebra[5] of $\mathfrak{g}_1$. This says that the space of Bosons acts on the set of Fermions in a manner that preserves the spinorial nature of Fermions.

3. The spacetime momenta are described by commutators $[A, B], A, B \in \mathfrak{g}_1$. That is, the translations of object in $\mathfrak{g}_0$ (Bosons) are generated by commutators of objects in $\mathfrak{g}_1$ (Fermions).

Now let's consider the most important example, the Super Poincaré Algebra which generalizes Poincaré Invariance to the case of super vector spaces.

Super Poincaré Algebra

Recall that the **Poincaré group**[6] P is the set of inhomogeneous Lorentz Transformations, $P = L \rtimes \mathbb{R}^{1,3}$ where L is the group of isometries of the Minkowski metric, $L \cong \mathsf{SO}(1,3)^0$. It's Lie algebra is known as the **Poincaré algebra** $\mathfrak{p}$. Physically, the action of $\mathfrak{p}$ on $\mathbb{R}^{1,3}$ represents the set of infinitesimal isometries of $\mathbb{R}^{1,3}$. To make this more precise, we can think of generators of P as generators of rotations (physically, angular momentum) and generators of translations (physically, momentum). In order to construct the algebra, it make sense to consider L and $\mathbb{R}^{1,3}$ separately. Since $\mathsf{SL}(2,\mathbb{C})$ is the universal cover of $\mathsf{SO}(1,3)$, we can consider $\mathsf{SO}(1,3)$ as a subset of the diffeomorphisms of $\mathbb{R}^4$ so that $\mathfrak{so}(1,3)$ can be thought of as set of vector fields. There are six generators of this group, labelled $M_{\mu\nu}, \mu, \nu \in \{0,1,2\}$, which correspond to two null rotations (i.e. rotations of elements with null norm), one boost (i.e. change of inertial frame in one direction) and three rotations around each of the spatial axes. These have non-trivial commutation relations that are preserved in the Poincaré algebra. On the other hand, the Lie algebra of $\mathbb{R}^{1,3}$ is generated by four vector fields $P_\eta, \eta \in \{0,1,2,3\}$ corresponding to a translation in each direction. Since translations commute, we only need to consider the commutation relation of a generator of $\mathsf{SO}(1,3)$ and a P_η. By respecting the semi-direct product, one can define $\mathfrak{p}$ as the algebra generated by the set of $\{M_{\mu\nu}, P_\eta\}$ under the commutation relations,

$$[P_\eta, P_\zeta] = 0 \tag{11}$$
$$-i[M_{\mu\nu}, P_\eta] = \eta_{\mu\eta}P_\nu - \eta_{\nu\eta}P_\mu \tag{12}$$
$$-i[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} \tag{13}$$

[5]Clifford Algebras are defined in part III

[6]For reference, the product operation for this group is $(L, x)(L', x') = (x + Lx', LL')$

Note that the i comes from the fact that in physics one determines the Lie algebra via representations of the universal cover $SL(2, \mathbb{C})$.

To pass to the Super Poincaré Algebra $\overline{\mathfrak{p}} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$, we set $\mathfrak{p}_0 \cong \mathfrak{p}$, so that the aforementioned Poincaré invariance is preserved for Bosons. This decomposition implies that there exists an action of the Poincaré Group P on $\mathfrak{p}_1$. The main constraint used to determine $\overline{\mathfrak{p}}$ is that we require $\overline{\mathfrak{p}}$ to be the *minimal* extension of $\mathfrak{p}_0$ in the sense that there does not exist a sub super Lie Algebra $\mathfrak{p}'$ such that $\mathfrak{p}_0 \subset \mathfrak{p}' \subset \hat{\mathfrak{p}}$. An argument involving Lie's Theorem on solvable actions in [52] forces $\mathfrak{p}_1$ to be irreducible, in the sense that the Clifford action of $\mathfrak{p}_0$ on $\mathfrak{p}_1$ is irreducible.

Now let's introduce physics notation for Group Representations. In the case at hand, we want to find the induced representations of $\mathfrak{sl}(2, \mathbb{C})$ (as a *real* Lie algebra) on $\mathbb{C}^{2n}$ (the space of **Dirac spinors**). Recall that a representation of a Lie Algebra is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ for a n-dimensional vector space V. In the case of a Lie group G, since the Lie algebra is isomorphic to the tangent space at the identity, the pushforward of a group representation $\rho : G \to GL(V)$, ρ_* is known as the induced Lie algebra representation. The representations of this group are tensor products of the holomorphic representations of dimension r with the complex conjugate of a representation of dimension k. Physicists denote one such representation $\mathbf{r} \otimes \mathbf{k}$. Since P acts on $\mathbb{R}^4 \cong \mathbb{C}^2$ and the Dirac Spinors[7] lie in $\mathbb{C}^2$, we want to consider actions on $\mathbb{C}^2 \oplus \mathbb{C}^2$. In particular, the representation of $\mathbf{2} \oplus \overline{\mathbf{2}}$, presented as an action of $g \in SL(2, \mathbb{C})$ on $\mathbb{C}^2 \oplus \mathbb{C}^2$ of the form:

$$(u, v) \mapsto (g \cdot u, \overline{g} \cdot \overline{u})$$

where $\cdot$ is standard matrix multiplication. We can think of $\mathbf{2} \oplus \overline{\mathbf{2}}$ as a free spinorial module since $C\ell(\mathbf{2}) \cong C\ell(1, 3) \cong M_2(\mathbb{R}) \otimes M_2(\mathbb{C})$, the 2×2 complex matrices and as $\mathrm{End}(\mathbf{2}) = M_2(\mathbb{R}) \otimes M_2(\mathbb{C})$, where an endomorphism acts by conjugation to preserve determinant. If we define the map $\sigma : (u, v) \mapsto (\overline{v}, \overline{u})$, then physicists Gol'fand and Likhtman showed that $\mathfrak{p}_1 \cong (2 \oplus \overline{2})^{\sigma}$, e.g. the conjugation of the representation of $2 \oplus \overline{2}$ by σ. They explicitly defined the bracket on $\overline{\mathfrak{p}}$ as an extension of the bracket on $\mathfrak{p}$ by the a map $[a, b] = L(a \otimes b), a, b \in \mathfrak{p}_1$, where $L : \mathfrak{p}_1 \otimes \mathfrak{p}_1 \to \mathbb{R}^{1,3} \otimes \mathbb{C} \subset \mathfrak{p} \otimes \mathbb{C}$ is a linear map induced by the inclusion $\mathfrak{p}_0 \hookrightarrow \mathfrak{p}$ (See [52], Sec. 2, Page 13-15 for details). It turns out that this superalgebra is the **unique** super Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}_1$ is spinorial, $[,]|_{\mathfrak{g}_1 \otimes \mathfrak{g}_1} \neq 0$ and $\mathfrak{g}$ is minimal.

Given this extremely brief survey of the representation theoretic arguments for the existence and uniqueness of the super Poincaré algebra, let's consider it's expression in coordinate form. This will be useful in understanding why the moduli space for supersymmetric gauge theories is a complex manifold. As before, let $P_\mu, M_{\mu\nu}$ be the generators of $\mathfrak{p} = \mathfrak{p}_0$ and let Q_α be a basis for $2 \oplus \overline{2}$ (recall that it is a free, spinorial module). Then the defining relations of the super Poincaré algebra are:

$$[P_\mu, Q_\alpha] = 0 \quad [M^{\mu\nu}, Q_\alpha] = -i\sigma^{\mu\nu}_{\alpha\beta} Q_\beta \quad \{Q_\alpha, Q_\beta\} = -2(\gamma^\mu C^{-1})_{\alpha\beta} P_\mu$$

where:

- γ^μ are the **Dirac Matrices**, which are representations of $C\ell(\mathbf{2}) \cong C\ell(1, 3)$ on $\mathbb{C}^{2n}$
- $\sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$
- $C := i\gamma_2\gamma_0$ is the **Charge Conjugation** matrix. Physically, C it represents an operator that sends the charge q of a particle to $-q$. It negates the flavor, isospin and baryon quantum numbers.[8]
- $\{\cdot, \cdot\}$ is the anticommutator (i.e. the commutator on the odd elements of a superalgebra)

Before getting to the physics of supersymmetry and concrete examples of the aforementioned structures, let's briefly define what a **supermanifold** is since these will serve as models of space-time. As mentioned before, a mathematical description of supermanifolds requires thinking about smooth manifolds from the perspective of algebraic geometry. Suppose that M is a topological manifold; that is, M is a topological space with an open cover $\{U_\alpha\}$ locally homeomorphic to $\mathbb{R}^n$ and M is Hausdorff and Second-Countable. Instead of thinking of a smooth structure on M as a set of C^∞ transition functions, we can think of it as a space such that for each open

[7]See Part III for a definition

[8]Quantum numbers are physically relevant quantities that uniquely characterize a state in quantum system. As in the case of spin, quantum numbers are mathematically the weights of certain Lie algebra representations

set U_α with homeomorphism $h_\alpha : U_\alpha \to U \subset \mathbb{R}^n$, h_α induces an isomorphism between $C^\infty(U)$ and $C^\infty(U_\alpha)$. We need to be a bit more careful and add compatibility conditions so that h_α also induces an isomorphism between $C^\infty(V)$ and $C^\infty(h_\alpha(V)), \forall V \subset U$. While this definition isn't as geometrically appealing as the usual definition of a smooth structure, it provides an easy way to make the notion of a smooth structure algebraic via the notion of a *locally ringed space*, which is defined below.

Definition 4.6. *Let X be a topological space. The pair $(X, \mathcal{O}_X)$ is known as a* ***locally ringed space***

4.2 Physics of Supersymmetry

In the mid-1970s, before QCD was experimentally validated, String Theories discovered that the worldsheet of a string required every bosonic (resp. fermionic) particle to have an associated fermionic (resp. bosonic) particle. These partner particles came to be known as **superparticles**. For example, the fermionic quark q has a superparticle partner (often shortened to "superpartner") known as the *squark*. Supersymmetries are introduced as a way to relate the internal symmetries of a particle, such as spin, isospin and baryon number, to the symmetries of spacetime (i.e. diffeomorphism group of a manifold). While the notion of additional internal symmetries seems physically unmotivated, it turns out that there are stringent restrictions on the symmetry groups that relate and preserve both internal and spacetime symmetries. These restrictions are encapsulated in the **Coleman-Mandula theorem**, which we will discuss in a simplified situation below. Upon discovering these restrictions and realizing the incompatibility of Quantum Field Theory, which focuses with internal symmetries, and General Relativity, which focuses on spacetime symmetries, Theoretical Physicists began began to construct symmetry groups that obeyed the Coleman-Mandula theorem and contained both types of symmetries. The "groups" that were constructed turned out to be actually be super Lie algebras. The physicists who played a major role in early construction of these algebras in the 1970s include:

- Jean-Loup Gervais
- Yuri Gol'fand
- Evgenii P. Likhtman
- Bunji Sakita
- Julius Wess
- Bruno Zumino

From a mathematician's perspective, it is easy to dismiss such symmetry considerations and physical assumptions as mathematically negligible or trivial. However the connection to modern mathematics hidden within the unification of internal and spacetime symmetries is precisely the same global-local interface within the development of manifolds, sheaves and schemes: one wants to find a way to unite local properties (internal symmetries) with global properties (spacetime symmetries) in a manner that preserves a mathematical structure, which in physics can be anything from a Spin Bundle to Symplectic Structure.

Let's explore the Coleman-Mandula theorem by considering a simplistic physical system. We will only consider the most simple scattering reactions where we have two particles a, b which are initially in a state $\overline{a} \oplus \overline{b} \in \mathcal{H}_a \oplus \mathcal{H}_b$, the Hilbert Space of possible simultaneous states of a and b. These particles will then come together, interact and form various products and then separate. The statement that Quantum Mechanics is a linear theory means that $\exists$ (densely-defined) operator $S \in \mathsf{End}(\mathcal{H}_a \oplus \mathcal{H}_b)$ such that the final state after the reaction is $(A \oplus B)(\overline{a} \oplus \overline{b})$. If $\mathcal{H}_a, \mathcal{H}_b$ are finite-dimensional of dimensions n and m, respectively, we define the **Scattering Matrix** or **S-matrix** to be the $n \times m$ real matrix $\mathfrak{S}$ defined as:

$$\mathfrak{S}_{ij} := \left(i \oplus j, S(\overline{a} \otimes \overline{b})\right)_{\mathcal{H}_a \oplus \mathcal{H}_b} \tag{14}$$

where $(\cdot, \cdot)_{\mathcal{H}_a \oplus \mathcal{H}_b}$ is the inner product induced by the inner product on the individual Hilbert Spaces. Each term $\left(i \oplus j, (A \oplus B)(\overline{a} \oplus \overline{b})\right)_{\mathcal{H}_a \oplus \mathcal{H}_b}$ represents a **scattering amplitude** which is interpreted as the probability that one finds state $i \oplus j$ in an experiment, given an initial state $\overline{a} \oplus \overline{b}$. In our case, we will assume that $\mathcal{H}_a = \mathcal{H}_b$ so that S is an operator and $\mathfrak{S}$ is an $n \times n$ matrix (in the finite-dimensional case). If the Hilbert Space is not finite-dimensional, we can construct $\mathfrak{S}_{ij}$ in an analogous manner using spectral measures.

Since the S-matrix contains most of the statistical data that one would need to evaluate a theory at an accelerator, it can serve as a simplistic model of a Quantum Field Theory. Coleman and Mandula ([12]) define a **symmetry transformation** of the S-matrix to be a unitary operator U on the many particle Hilbert Space, $\mathcal{H}_{tot} = \bigoplus_{i \in \mathbb{N}} \mathcal{H}^{(n)}$, where the n particle states $\mathcal{H}^{(n)}$ are defined to be subspaces of $\mathcal{H}^{\otimes n}$, where $\mathcal{H}$ is the Hilbert Space of single particle states. This operator needs to satisfy three conditions:

1. U maps one-particle states to one-particle states, i.e. $U(\mathcal{H}) \subset \mathcal{H}$
2. U acts on many-particle states as if they were tensor products of one-particle states. Mathematically this is a naturalness condition that says that U extends to the $\mathcal{H}_{tot}$ while respecting the morphisms of the category of Hilbert Spaces.
3. $[S, U] = 0$. This implies that $\mathfrak{S}$ doesn't change under the action of U

If a set of operators satisfies these conditions and is a group, this is known as the **symmetry group** G_S of S and S is Lorentz-invariant if G_S is isomorphic to the Poincaré group or its universal cover $SL(2, \mathbb{C}) \rtimes \mathbb{R}^{1,3}$. In the initial paper ([12]), whether G was a Lie Group and/or whether G was finite-dimensional was unclear. However, some of the techniques used in the proof force G to be a Lie group. Since infinite-dimensional Lie groups are difficult to deal with mathematically and as most Lie groups in physics are finite-dimensional, we will assume without the loss of generality that G is finite-dimensional. The condition of Lorentz-invariance is forcing the S-matrix to have the global, spacetime symmetries of Special Relativity. Hence, we need to define what the internal symmetries. One of the easiest compatibility conditions that we can enforce is commutativity of an internal symmetry and a spacetime symmetry so that *both* leave $\mathfrak{S}$ invariant. This is precisely the definition of Coleman and Mandula. Note that this is an important physical fact since it means that the experimental data collected, which can be directly compared to the S-matrix, is *not* affected by internal and spacetime symmetries, as desired. The Coleman-Mandula theorem places restrictions on the types of symmetry groups that admit both internal symmetries and Poincaré invariance. The statement of the theorem is:

Theorem 4.2. ***(Coleman-Mandula)*** *Let G be a connected symmetry group of the S matrix such that the following hold:*

1. ***(Test Functions)****:*
   The set of one-particle states of the system $\mathcal{H}^{(1)}$ is isomorphic to the **Schwartz Space** *$\mathcal{S}$ of test functions, where*
   $$\mathcal{S} = \left\{ f \in L^2(\mathbb{R}^n) \mid \|x^n f^{(m)}\|_{W^{m,2}(\mathbb{R}^n)}, \forall n, m \right\}$$
   where D^β is the standard notation for a multi-index differential operator on $\mathbb{R}^n$
2. ***(Lorentz Invariance)****:*
   G contains a subgroup isomorphic to either the Poincaré group P or it's universal cover $\tilde{P}$.
3. ***(Integral Kernel Assumption)****:*
   The generators of G can be considered integral operators that act on the set of momentum eigenstates.[9] *As integral operators with kernel K, we find that K is a tempered distribution. More precisely, $\exists$ a neighborhood N of the identity e in G such that every element of N lies on a one-parameter subgroup $g : \mathbb{R} \to G$ of G. Given this one-parameter group, the distributional assumption states that the functional,*
   $$\frac{1}{i}\frac{d}{dt}(x, g(t)y) = (x, Ay)$$
   is continuous in the Schwartz topology for all $x, y \in \mathcal{H}^{(1)}$ and linear in y, anti-linear in x
4. ***(Particle Finiteness)***
   Physical Statement*: "All particle types correspond to positive energy representations and for a finite mass M*

[9]The state $p \in \mathcal{H}_{tot}$ is a **momentum eigenstate** if it an eigenstate of a representation of P_μ. That is, if $\rho : P \to GL(\mathcal{H}_{tot})$ is a representation of the Poincaré group on $\mathcal{H}_{tot}$, then $\rho(P_\mu)p = \lambda p$ for some $\lambda \in \mathbb{C}$.

there are only a finite number of particle types with mass less than M*."*
Mathematical Statement*: Any element* $h \in \mathcal{H}_{tot}$ *transforms only under positive weight representations of* G *and that for each weight* w*, there are only finitely many orbits of the* G*-action that correspond to representations with weight* $w' \leq w$.

5. ***(Weak elastic analyticity)***
   Physical Statement*: "Elastic-scattering amplitudes are analytic functions of center-of-mass energy* s *and invariant momentum transfer* t *in some neighborhood of the physical region except at normal thresholds."*
   Mathematical Statement*: The variables* s, t *are known as* ***Mandelstam Variables*** *and correspond to quadratic functions of the incoming momenta* p_1, p_2 *and outgoing momenta* p_3, p_4 *in a scattering reaction. This conditions says that the entries of the S-matrix are real analytic functions of* s, t*. Note one of the early attempts at constructing a String Theory, the theory of Regge's trajectories and pomerons, defined* s, t *to be complex functions. This theorem still holds in that situation, except that we require the S-matrix to have holomorphic entries.*

6. ***(Occurrence of scattering)***
   If $p, p' \in \mathcal{H}_{tot}$ *are momentum eigenstates with* $P_\mu p = \lambda_p p, P_\mu p' = \lambda_{p'} p'$ *and* $p \otimes p'$ *is the two-particle state for this system then* $T(p \otimes p') \neq 0$*, where* T *is defined as* $S = \mathbf{1}_{\mathcal{H}_{tot}} - (2\pi)^4 \delta(\lambda_p - \lambda_{p'}) T$ *where* δ *is the Dirac distribution.*

Then $G = G_I \times P$ *or* $G = G_I \times \mathfrak{P}$ *where* G_I *is the group of internal symmetries.*

The hypotheses of this theorem seem specific and appear to apply to a very specific physical system. However, many particle reactions obey the physical conditions and the mathematical conditions are mild given that we are talking about the symmetries associated to a *single* operator. This theorem effectively says that under reasonable physical and mathematical assumptions, any combination of spacetime symmetries and internal symmetries **must be a trivial product of groups**. In the construction of the Super Poincaré Algebra we see this immediately, since the algebra is isomorphic to $\mathfrak{p} \oplus \mathfrak{m}$, where $\mathfrak{p}$ is the Poincaré algebra. Many of the most prominent, classically anomalous internal symmetries are those related to intrinsic spin. As such the assumption that any Lie super algebra in physics $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is such that $C\ell(\mathfrak{g}_1) \cong \mathsf{End}(\mathfrak{g}_0)$ is natural, since it "builds in" the necessary mathematical machinery[10] of Clifford Algebras (and Spin bundles)

Given these facts, one might wonder if supersymmetry "supercedes" all of the previous use of Lie groups in physics to represent symmetries. There is however an important caveat to this — supersymmetry **has not been found in nature**. It serves as a wonderful mathematical tool and a decent physical tool in its ability to link local and global pieces of information. The mathematical use of supersymmetry is easy to grasp when one considers the large in-roads that have been made in the classification of 4-manifolds, mirror symmetry and generalized complex geometry due to supersymmetric considerations. However, the physical implications are hard to decipher without an explicit example.

[10]An interesting fact to note is that we've now run into a **topological condition** enforced by supersymmetry on spacetime. In order for one to have global supersymmetry, one needs to be able to consider the adjoint $\mathfrak{g}$-bundle for super Lie Algebras $\mathfrak{g}$ that act on supermanifolds. The isomorphism $C\ell(\mathfrak{g}_1) \cong \mathsf{End}(\mathfrak{g}_0)$ implies that such a bundle has a principal subbundle that is Spin. Recall that a manifold M only admits a spin bundle iff the second Stiefel-Whitney class of M, $w_2(M)$ vanishes.

Part II

AdS/CFT and Analysis

5 Introduction

The Anti-de-Sitter Space/Conformal Field Theory correspondence is a derived result of String Theory that posits a geometric relationship between Gauge Theories and the space-times of General Relativity [40, 2]. Physically, the AdS/CFT correspondence states that if one has the n-dimensional Anti de Sitter Space, $M = AdS_n$ as a model of spacetime, then the partition function associated to a given conformal structure on M is determined by the vacuum expectation value of a linear functional J on $\mathrm{Sym}(2, \partial M)$, the set of symmetric 2-tensors on ∂M. On the other hand, the correspondence conjectures a mathematical relationship between an asymptotically hyperbolic, Einstein metric g on a $n+1$-dimensional manifold with boundary M and the restriction to ∂M of the n^{th} term in the Fefferman-Graham expansion of geodesically equivalent Einstein metric. In physics, such a relationship arose from the discovery that certain 5-dimensional Riemannian Manifolds X_5 give rise to a string background $AdS_5 \times X_5$ such that the choice of metric g on X_5 could determine the central charge of a Conformal Field Theory on $\partial(AdS_5)$.

The goal of this part is to describe the AdS/CFT correspondence from a mathematical perspective as a problem in analysis. Unlike many other areas of String Theory and String Theory-related research, the AdS/CFT correspondence provides a direct connection to geometric analysis since the conjecture relates the partition function of a field theory with a Boltzmann-like sum of Einstein-Hilbert actions. In particular, the AdS/CFT correspondence can be thought of as a specific example of the **Yamabe problem** on Asymptotically Hyperbolic Einstein Manifolds for which String Theoretic considerations as well as the Einstein Field Equations provide enough data to guarantee existence and uniqueness up to local isometry. After briefly describing this relation to analysis, we will look at the recently discovered $Y^{p,q}$ manifolds that disprove a conjecture of Tian and Cheeger that states that no irregular Sasaki-Einstein manifolds exist. This solution was motivated by M-theory solutions related to AdS/CFT and serves as an excellent example of the recent interplay between mathematics and physics.

This paper is structured so that both the physical and mathematical problems at hand are explicitly described. § will provide a brief introduction to the breadth of mathematics required to elucidate the AdS/CFT correspondence while providing the analytical definition of what exactly the correspondence is claiming. § will provide a review of the physics involved and the motivations for studying the manifolds $Y^{p,q}$. § will provide a brief description of Sasaki-Einstein manifolds as well as the explicit metrics involved. § will conclude this paper with a description of the problem at hand.

6 Formal statement of the AdS/CFT correspondence

6.1 Formulating a Conformal Field Theory

While many mathematicians tend to look at String Theory as most similar to Algebraic Geometry, the AdS/CFT correspondence provides both an analytic and differential geometric point-of-view of String Theory and Conformal Field Theories. Before arriving at the more analytic description of the correspondence, let's consider the bare necessities for a mathematical formulation of a Conformal Field Theory. Let M be an n-dimensional Manifold that will serve as our *Configuration Space* of possible physical states. In order to describe a *Conformal Field Theory*, one needs a linear functional[11] $J : \mathcal{T}M \otimes \mathcal{T}^*M \to R$, called an *action*, that is (locally) invariant under the group of conformal transformations of $\mathbb{R}^n, \mathrm{Conf}(\mathbb{R}^n)$. In order to determine quantities of interest, one would ideally like to compute how the action (which contains all the physics) describes the time-evolution of a particle over a specific path $\gamma : [0,1] \to M$. However, this implies that we are integrating over the set of all C^∞ paths γ. This set is necessarily an infinite-dimensional space, since for each chart $\varphi_\alpha : U_\alpha \subset M \to \mathbb{R}^n$, the set of all paths contained in U_α is $C^\infty([0,1], \mathbb{R}^n)$. It is a standard result of Real Analysis that there exists no translational-invariant, infinite-dimensional measure on $C^\infty([0,1], \mathbb{R}^n)$ and as such a "sum over all paths" is not mathematically well-defined. A crucial symmetry in physics in Poincaré Symmetry, which is invariance of J under the action of the n-dimensional Poincaré Group $SO(n) \rtimes \mathbb{R}^n$. Intuitively, this invariance says that the energetics of a physical system, as predicted by the action J, should not change under a change of origin (translation) or a rotation. As such it is physically necessary that the action J be translational-invariant, so a precise formulation of the "expectation of physical event A with respect to the constraints contained in J" is currently unknown. However, physicists have come up with clever ways to approximate such "non-existent" expectations.

[11]Notation: $\mathcal{T}M$ is the tensor algebra of TM amd $\mathcal{T}^*M$ is the tensor algebra of T^*M

In the physics literature, the expectation value of the functional J is typically presented in the form of a Feynman Path Integral that sums over the restriction to ∂M of "all Einstein metrics" on M. While this initially seems to be an ill-defined object, G. Segal and M. Kontsevich have proposed a mathematical definition[12] of the expectation with respect to a Gaussian measure on Teich(M), the Teichmüller Space of M [46]. From an analytic perspective this should be well-defined, since the Teich(M) is in general a Banach Space. This paper will not be concerned with the ambiguity in the expectation value since the main goal of this thesis is analytic in nature.

6.2 Analytic Definition of the correspondence

We will assume that any manifold M defined is of class C^∞. Moreover, we will restrict ourselves to the category of *Einstein Manifolds* (M, g_E), which have $\mathrm{Ric}(u,v) = \kappa g_E, u, v \in TM, \kappa \in \mathbb{R}$. Let us first define what exactly a Conformally Compact metric is [7]:

Definition 6.1. *Suppose that we have an $n+1$ dimensional Riemannian Manifold with non-empty boundary, $(\overline{M}, \overline{g})$. Let M be the interior of $\overline{M}$ and let ∂M be the boundary of $\overline{M}$. A complete Riemannian metric g on M is* ***conformally compact*** *if there exists a function $\rho \in \Omega^0(\overline{M})$ such that $\overline{g} = \rho^2 g$, $\rho^{-1}(0) = \partial M$ and $d\rho \neq 0$ on ∂M. Such a function ρ is called a* ***defining function***[13] *for the pair $(M, \overline{M})$*

Define $\gamma := \overline{g}|_{\partial M}$ which is the boundary metric on $\overline{M}$ and let $\mathcal{M}, \mathcal{M}_\partial$ be the moduli spaces of metrics and boundary metrics on conformally compact, $n+1$ dimensional Einstein manifolds with non-empty boundary, respectively. The choice of γ is in general not unique because there can be many defining metrics for a given $\overline{M}$. Instead, we will work with boundary metrics γ defined up to conformal transformation; that is, we consider the equivalence class $[\gamma]$ under the relation,

$$\gamma', \gamma \in \mathcal{M}_\partial, \gamma' \sim \gamma \Longleftrightarrow \exists f : \overline{M} \to R, \mathrm{Im}(f) \subset (0, \infty), \gamma' = f\gamma$$

Note that since we are considering boundary metrics up to conformal transformation we are indeed only interested in $\mathcal{M}_\partial := \mathrm{Teich}(M)/\mathrm{MCG}(M)$ as opposed to Teich(M), since diffeomorphic manifolds will have the same conformal structure. Moreover, note that the physical interpretation of the boundary metric $[\gamma]$ is embodied in Penrose's notion of *Conformal Infinity*. The idea is that the causal structure of spacetime (i.e. whether a geodesic is timelike, null or spacelike) is preserved under conformal transformations so that singularities (such as Black Holes) can be more easily analyzed.

In order to simply the mathematical content of the AdS/CFT correspondence, assume that we are in a regime that satisfies the vacuum Einstein equations, $G_{\mu\nu} = 0$, where $G_{\mu\nu}$ is the Einstein Tensor. We define the *Einstein-Hilbert action*, $S_{EH} : \mathcal{M} \to \mathbb{R}$ by

$$S_{EH}(g) := \int_M K dV_g$$

where K is a Gaussian Curvature of (M, g) and dV_g is the volume form associated to g. In coordinates, this reduces to the more recognizable action, $S_{EH} = \int_M d^{n+1}x\sqrt{g}R$, where R is the Ricci Scalar Curvature. Heuristically, one can say that the AdS/CFT correspondence takes boundary data $(\partial M, [\gamma])$ and derives a partition function for $(\overline{M}, \overline{g})$. Finally let $\mathcal{M}_{(\partial M, [\gamma])} := \left\{ g \in \mathcal{M} | \exists \rho \in \Omega^0(\overline{M}) \ni \overline{g} = \rho^2 g, \gamma = \overline{g}|_{\partial M} \right\}$ so that given these definitions, the AdS/CFT correspondence can be summarized by the following equation:

$$Z_{CFT}(\partial M, [\gamma]) = \sum_{g \in \mathcal{M}_{(\partial M, [\gamma])}} e^{-S_{EH}(g)} \tag{15}$$

[12]Segal and Kontsevich formulate two-dimensional Conformal Field Theories in terms of Cobordism Classes of compact, connected 2-manifolds that resemble the worldsheets of String Theory. This functorial definition constructs Conformal Field Theory in terms of a Generalized Cohomology Theory, with two compact, connected Riemann Surfaces M, N deemed equivalent iff M is cobordant to N. I have a bit of trouble connecting this description with the analytic aspects of Field Theories, so I've chosen to ignore it for the most part.

[13]Note that we assuming that ρ is of class C^∞. This definition has been expanded in the following way: A metric g is said to be $L^{k,p}$ or $C^{m,\alpha}$ conformally compact if there exists a defining function such that $\overline{g}$ has an $L^{k,p}$ or $C^{m,\alpha}$ extension to $\overline{M}$, where $L^{k,p}$ is the $(k,p)-$Sobolev Space and $C^{m,\alpha}$ is the $(m,\alpha)-$Hölder space. See [3] for details

where Z_{CFT} is the partition function associated to a conformal field theory associated with the conformal structure $[\gamma]$ on ∂M. Note that the principal concept behind the sum on the right hand side is that we are summing over *all* Conformally Compact Einstein Manifolds (M, g) *given* the boundary data $(\partial M, [\gamma])$. However, it is clear that (15) doesn't provide any intuition as to the analytic aspects of the AdS/CFT conjecture. This is where the Fefferman-Graham expansion and the asymptotic hyperbolicity of the metric come into play. Recall that a metric g is hyperbolic if it has constant, negative sectional curvature. We can now define what an asymptotically hyperbolic metric is:

Definition 6.2. *Again suppose that we have an $n+1$ dimensional Riemannian Manifold with non-empty boundary, $(\overline{M}, \overline{g})$. Let M be the interior of $\overline{M}$ and let ∂M be the boundary of $\overline{M}$. We say that g is* ***asymptotically hyperbolic*** *if $\overline{g}|_{\partial M}$ is hyperbolic.*

Now there are some interesting properties about asymptotically hyperbolic Einstein metrics, namely that they have a canonical form in which they can be expressed as a cone over a a family of hypersurface metrics. This splitting comes via the Gauss Lemma and represents the first analytic formulation of the AdS/CFT conjecture. Let detail this process a bit. Suppose that we choose a defining function $\rho(x) = d_{\bar{g}}(x, \partial M)$ in a collar neighborhood[14] of ∂M. A defining function of this type is called a *geodesic* defining function, since the minimization of $\rho(x)$ will solve a geodesic-like problem on the conformal infinity $(\bar{M}, \bar{g})$.

It is possible to prove that given a boundary metric $[\gamma]$ in the conformal infinity of (M, g) there exists a unique geodesic defining function ρ_γ that has γ as the boundary metric on $(\bar{M}, \bar{g})$ [4]. The proof boils down to reformulating the uniqueness problem in terms of a Cauchy problem on the collar neighborhood U. Recall the Gauss Lemma of Riemannian Geometry [49]:

Lemma 6.1. *(Gauss) The radial geodesic through the point $p = \exp(\xi), \xi \in T_pM$ is orthogonal to the Riemann Hypersurface Σ_p that passes through p.*

If we choose $p \in \partial M$, then this says that in some normal (or inertial) neighborhood U of p, we can write the metric $\bar{g}|_U = dt^2 + g_{\Sigma_p}|_U$. Since we can choose a boundary metric up to conformal transformation, the choice of a geodesic defining function gives a conformal class of metrics such that *globally* we have $\bar{g} = dt^2 + g_t$, where g_t is a family of metrics on hypersurfaces where $t = t'$. Now we can define the *Fefferman-Graham expansion* of the metric $\bar{g}$ as the truncated Taylor Series expansion of g_t. Explicitly for an n-dimensional Riemannian Manifold (M, g) with conformal infinity, this expansion is [23]:

$$g_t = g_0 + tg_1 + t^2 g_2 + t^3 g_3 + \ldots + t^n g_{(n)} + O(t^{n+\alpha}) \tag{16}$$

Given the above background, the conjecture is effectively the dimension $n = 5$ version of the following dimension $n = 4$ theorem [3, 6]:

Theorem 6.1. *Suppose that $\dim M = 4$ and the boundary metric γ is of class[15] $C^{7,\alpha}$. Then the pair $(\gamma, g_{(3)})$ on ∂M uniquely determine an Asymptotically Hyperbolic Einstein metric up to local isometry. This means that if g^1, g^2 are two AH Einstein metrics on manifolds M_1, M_2 with $\partial M = \partial M_1 = \partial M_2$ such that with respect to the aforementioned compactifications $(\bar{M}_1, \bar{g}_1), (\bar{M}_2, \bar{g}_2)$, we have:*

$$\gamma^1 = \gamma^2 \quad \textit{and} \quad g^1_{(3)} = g^2_{(3)}$$

then g^1, g^2 are locally isometric and M_1, M_2 have diffeomorphic universal covers.

This effectively says that given a boundary metric γ and an n-th order approximation of the interior metric g, we can compute g up to local isometry. For physical purposes, one desires knowledge of the null and timelike geodesics, so that this uniqueness up to local isometry is "good enough." Moreover, the choice of boundary

[14] Recall that a collar neighborhood U of an n-manifold M with boundary is an open set $U \subset \mathbb{R}^{2n}$ such that U is diffeomorphic to $\partial M \times [0, \epsilon)$. The Whitney Embedding Theorem guarantees the existence of such a neighborhood in $\mathbb{R}^{2n}$.

[15] Recall that the Hölder Spaces $C^{k,\alpha}(\Omega), \Omega \in \mathbb{R}^n$ are the topological vector spaces of functions $f : \Omega \to \mathbb{R}$ that are k times continuously differentiable and such that f is Hölder continuous with exponent α. This means that $\forall x, y \in \Omega, |f(x) - f(y)| \leq C|x - y|^\alpha$.

metric γ effectively defines the boundary Energy-Momentum Tensor, so the above theorem will yield the Energy-momentum tensor for the entire spacetime M. The case of $\dim M = 5$ appears to not have been proved completely yet. In the next section, we will show how a choice of String Theory background (a geometric constaint) fixes the equivalence class $[\gamma]$ so that the above theorem can be used.

6.3 Geometric Description of the correspondence

As mentioned in the introduction, the AdS/CFT correspondence is closely tied together with Complex Geometry and String Theory. Let's first give a short description of the geometric structures associated with String Theory. String Theory purports that if strings that obey known symmetries[16] as well as supersymmetry exist, then the total spacetime manifold M must be 10-dimensional. As such, most early formulations of string theory assumed that $M = \mathbb{R}^{1,3} \times X_6$, where X_6 is either a 6-dimensional real manifold or a 3-dimensional complex manifold. The idea is that if $X_6 \hookrightarrow \mathbb{R}^{12} \cong \mathbb{C}^6$ is contained in a ball of radius r in $\mathbb{R}^{12}$ or $\mathbb{C}^6$, then as $r \to 0$, M would begin to look like Minkowski Space, $\mathbb{R}^{1,3}$. This means that all known physics, which requires a background of Minkowski space or a locally-Minkowski space (i.e. a Lorentzian Manifold), could be preserved if the embedding radius r of X_6 is quite small. Note that because Complex Manifolds inherently have an integrable structure with an easy way to check symplectic and differentiable structure integrability (i.e. via the Newlander-Nirenberg Theorem), almost all models for X_6 were complex.

However in the early-1990s, there was some compelling evidence that suggested that one should consider spacetime manifolds M that decompose as $M = X \times Y$ with $\dim X = \dim Y = 5$. The idea stemmed from the fact that the 5-dimensional Anti de Sitter space, AdS_5 can be viewed as the Lorentzian analogue of the hyperbolic space $\mathbb{H}^n$ [2]. More precisely, we can define AdS_5 via the locus of the quadratic polynomial f on $\mathbb{R}^{4,2}$ with coordinates $(x_1, x_2, x_3, x_4, S, T)$,

$$f(x_1, x_2, x_3, x_4, S, T) := x_1^2 + x_2^2 + x_3^2 + x_4^2 - S^2 - T^2 + 1 \tag{17}$$

It is clear that f is a submersion, since $dF(\vec{x}) = (2x_1, 2x_2, 2x_3, 2x_4, -2S, -2T)$ so that the submersion level set theorem says that this zero locus is an embedded, 5-dimensional submanifold of $\mathbb{R}^6$. Note that the induced metric on AdS_5 is $g_{AdS_5}(\partial_1, \dots, \partial_4, \partial_S, \partial_T) = dx^1 + \dots dx^4 - dS^2 - dT^2$, so that AdS_5 is the Lorentzian analogue of the Hyperbolic Space $\mathbb{H}^5$. Moreover, note that the surfaces with constant x_1, T are copies of $\mathbb{R}^{1,3}$ As such we can view AdS_5 as a set of hypersurfaces $\Sigma_{x_1,T}$ that are isometric to $\mathbb{R}^{1,3}$ with a scaling factor of $R^2 := x_1^2 - T^2$. The idea is that we are using one of the six remaining (and remember, required) dimensions to construct a space that heuristically "warps" Minkowski Space.

As it turns out, our current knowledge of physics and our desire to have supersymmetry heavily constrains the choice of manifold Y we choose. Let us briefly review some of the main assumptions in the decomposition of spacetime as $M = \mathbb{R}^{1,3} \times X_6$. The main condition enforced is that X_6 has a Kähler metric and is Ricci-Flat[17]. The Kähler metric can be heuristically justified on the grounds that classical mechanics formally requires a symplectic form while quantum mechanics requires symmetries to be preserved under unitary transformations. The Ricci-Flat condition implies that our X_6 satisfies the vacuum Einstein equations $G(\Theta_\mu, \Theta_\nu) = T(\Theta_\mu, \Theta_\nu)$ where G is the Einstein 2-tensor, $G(\Theta_\mu, \Theta_\nu) := \mathrm{Ric}(\Theta_\mu, \Theta_\nu) - \frac{1}{2} g_{X_6}(\Theta_\mu, \Theta_\nu) R$ and $T \in TM \otimes TM$ is a symmetric 2-tensor[18]. A complex manifold that is Kähler and Ricci-Flat is known as a *Calabi-Yau Manifold.* Physicists have been interested in these manifolds precisely because they are compatible with the symmetries of nature and serve as a good starting point. However, while Yau proved that such manifolds exist, no explicit metric has ever been found so this has made analysis on these spaces difficult. For completeness and for use in §, let us give some equivalent definitions of a Calabi-Yau Manifold:

Theorem 6.2. *For a compact complex* n*-manifold* (M, g)*, the following are equivalent:*

[16] Recall that for a symplectic manifold (M, ω), the Lagrangian formulation of physics on M allows for all physical quantities to be written in terms of a linear functional that can be extremized via an Euler-Lagrange equation. For purposes of this paper, we will consider the Lagrangian to be a linear functional $\Lambda : L^2(\bigwedge^\bullet M, dm) \to \mathbb{R}$. In this situation, a *symmetry* of a Lie Group G with an action on $\Gamma(TM)$ such that $\forall f \in L^2(\bigwedge^\bullet M, dm)$, f, ω are invariant under the flows of $g \cdot \vec{v}, \vec{v}$ where $\vec{v} \in \Gamma(TM)$

[17] This means that the Ricci Scalar $R := \mathrm{tr}(\mathrm{Ric}(\Theta_i, \Theta_j))$, for any local frame $\{\Theta_i\}$ vanishes

[18] Physically, this is the *Energy-Momentum Tensor*

- M *is a Calabi-Yau*
- $Hol_g(M) \subset SU(n)$
- *The first Chern Class of* M, $c_1(M)$ *vanishes*
- *The canonical bundle of* M *is trivial*
- M *admits a global, non-vanishing holomorphic* n*-form*

The AdS/CFT correspondence effectively conjectures that for a certain class of 5-manifolds Y, we can still preserve the symmetries required to have a well-defined string theory and that the choice of metric on Y *uniquely* determines the conformal boundary $[\gamma]$ of AdS_5 [2, 40, 4]. The conjecture was initially formulated with $AdS_5 \times S^5$, where S^5 is given the round metric and with $AdS_5 \times S^2 \times S^3$, where $S^2 \times S^3$ is given the homogeneous metric.[19] However, these examples are considered trivial in that they admit free, proper $U(1)$ actions so that given a $U(1)$ action, any quotient of S^5 or $S^2 \times S^3$ by this action will be a Kähler-Einstein Manifold with positive curvature [41]. However, this condition is too restrictive both mathematically and in physical applications. We can loose the type of $U(1)$ action by requiring that we have a periodic $U(1)$ orbit as opposed to a free $U(1)$ orbit. Recently, a new infinite class of 5-manifolds that are compatible with AdS/CFT correspondence and admit both free and non-free $U(1)$ orbits have been constructed. This class of manifolds subsumes $S^5, S^2 \times S^3$ and represent *Sasaki-Einstein Manifolds*. These manifolds are defined and analyzed in §. The physical importance of these manifolds stems from their closed relationship with Calabi-Yau Manifolds, which have been studied extensively as models for the internal space of String Theory. This allows for a much simpler description of a $\mathcal{N} = 1$ superconformal field theory than one could hope for on an arbitrary Kähler manifold. See [1] for more details.

7 The geometry of the the $Y^{p,q}$ Manifolds

7.1 Brief Overview of Sasaki-Einstein Manifolds

In this section, we will only consider manifolds of dimension 5 and higher. Let us start with the most cogent definition of a Sasaki-Einstein Manifold:

Definition 7.1. *An odd-dimensional, compact, real Riemannian manifold* (M, g) *is* ***Sasaki-Einstein*** *iff it is Einstein and its metric cone* $(C(M), \overline{g}), C(M) \cong \mathbb{R}_+ \times M, \overline{g} = dr^2 + r^2 g_M$ *is Kähler and Ricci-flat, or in other words Calabi-Yau. We will naturally identity* M *via as* $\{1\} \times M \subset C(M)$

This brief introduction will will follow § of [48] and portions of Chapters 3,6 and 11 of [8], which is relatively recent and complete monograph on Sasakian Geometry. Since a Sasaki-Einstein Manifold has a Kähler cone, it inherits many of the nice features of Kähler and Symplectic manifolds. In particular, the odd dimensional cousins of Kähler and Symplectic geometries are ***CR*** and *Contact Geometries*, respectively. One can think of contact geometry as an odd-dimensional analogue of symplectic geometry, inspired by classical mechanics with a configuration space that also depends on time. More formally, we define a *Contact Structure* (M, η) on an $2n + 1$ manifold M with one-form η if $\eta \wedge (d\eta)^n$ is a volume form. In that case of a Sasaki-Einstein Manifold M, if ω_g is the Kähler form of $C(M)$, then one can prove ([8], §.4-6.5) that the Kähler potential can always be put in the form $\omega_g = d(r^2\eta)$ for a one-form η on $C(M)$. This means that $r^2\eta$ is a Kähler potential and moreover that for $r = 1$, we get a non-vanishing one-form on M. Since $d\eta$ is also globally non-vanishing, by Kählerity, we get an induced contact structure on M. The following proposition is quite important (from [8], §.1):

Proposition 7.1. *On a contact manifold* (M, η) *of dimension* $2n + 1$*, there exists a unique vector field* ξ *called the* ***Reeb Vector Field*** *satisfying the conditions,* $\xi \lrcorner \eta = 1, \xi \lrcorner d\eta = 0$

The proof is relatively straightforward and gives some intuition about where ξ comes from, so let's go through it:

[19] Recall that $S^2 \approx SO(3)/SO(2)$ and $S^3 \approx SU(2)$ so that the product can be considered a homogeneous space

Proof. Since (M, η) is a contact manifold[20] we have a volume form $\eta \wedge (d\eta)^n$. By the musical isomorphism $TM \cong T^*M$, there exists a unique $\xi \in \Gamma(TM)$ such that $\xi \lrcorner \eta = 1$ so that $\xi \lrcorner \eta \wedge (d\eta)^n = (d\eta)^n$. since $(d\eta)^n = \underbrace{d\eta \wedge d\eta \cdots \wedge d\eta}_{n}$, is alternating this means that $\xi \lrcorner (d\eta)^n = 0$. □

For a Sasaki-Einstein manifold, the Reeb Vector Field takes a rather simple form, via a slightly opaque construction. Consider the *homothetic vector field*, ζ on $C(M)$ is simply $\zeta := r\partial_r$. Let $\nabla, \overline{\nabla}$ be the Levi-Civita connections associated to $g, \overline{g}$, respectively. In order to construct a vector field on M that extends naturally to a vector field on $C(M)$, we need to ensure that we have a real analytic vector field on M. Let's first look at the transport behavior of ζ via the following formulas [48] hold for $X, Y \in \Gamma(\{1\} \times M) \hookrightarrow \Gamma(C(M))$:

$$\overline{\nabla}_\zeta \zeta = \zeta \tag{18}$$

$$\overline{\nabla}_\zeta X = \overline{\nabla}_X \zeta = X \tag{19}$$

$$\overline{\nabla}_X Y = \nabla_X Y - g(X, Y)\zeta \tag{20}$$

Heuristically equations (18),(19) imply that a vector field X on the base M doesn't change as we move it up the cone via ζ. Now since $C(M)$ is Kähler, we know that the Almost Complex Structure $J : TC(M) \to TC(M)$ associated to $C(M)$ is parallel, i.e. $\overline{\nabla} J = 0$. Using these facts we can prove the following claim:

Claim 7.1. *The homothetic vector field ζ is real analytic, i.e. $\mathcal{L}_\zeta J = 0$.*

Proof. Since $\overline{\nabla}, \nabla$ are Levi-Civita connections, the torsion tensor for both connections vanishes. This means that $\mathcal{L}_\zeta J = \overline{\nabla}_\zeta J - \overline{\nabla}_J \zeta$. Using (19), it is clear that this vanishes. Similarly if we restrict to the hypersurface $\{1\} \times M$, then using (19), (20), we have,

$$\mathcal{L}_{\zeta|_M} J = \nabla_{\zeta|_M} J - \nabla_{J|_M} \zeta = \overline{\nabla}_{\zeta|_M} J + g(\zeta|_M, J)\zeta - \overline{\nabla}_J \zeta|_M - g(J, \zeta|_M)\zeta = 0$$

where the last equality holds since the metric is symmetric. □

Now define $\xi = J(r\partial_r)$. We now will show that the $\eta = \frac{1}{r^2}\xi^\flat$ is a contact form. Recall that given an almost Hermitian Manifold (M, g, J), we define the Kähler form $\omega_g : TM \otimes TM \to \mathbb{R}$ by $\omega_g(X, Y) := g(X, JY)$. The musical isomorphism gives the formula $\eta(X) = \frac{1}{r^2} g(X, \xi) = \frac{1}{r^2}\omega_g(X, r\partial_r)$. Since $r^2\eta$ is the Kähler potential, it is clear that η is non-vanishing and more over, since the Kähler form is also symplectic, $(d\eta)^n$ is also non-vanishing. As such, $\eta \wedge (d\eta)^n$ is a non-vanishing $2n + 1$-form, or in other words it is a volume form. Finally, note that there is a closed relationship between the Kähler form ω on the metric cone and the contact 1-form η:

$$\omega = \frac{1}{2} d(r^2\eta) \tag{21}$$

7.2 The $Y^{p,q}$ metrics

Background

Until 2004, it was widely believed that irregular Sasaki-Einstein Manifolds did not exist as per a conjecture by Tian and Cheeger [10]. However, in 2004 a landmark paper of Sparks, Martelli, Gauntlett and Waldram [22] constructed an infinite sequence of Sasaki-Einstein metrics on $S^2 \times S^3$ which included irregular and quasiregular Sasaki-Einstein Manifolds. These manifolds, denoted $Y^{p,q}$ are indexed by $p, q \in \mathbb{Z}, (p, q) = 1$. Initially, the metrics were described in coordinates; however, soon after an argument using the Gysin Spectral Sequence to the natural $U(1)$ fibration, $U(1) \hookrightarrow Y^{p,q} \twoheadrightarrow Y^{p,q}/U(1)$ was presented in order to show that $Y^{p,q}$ was homotopically $S^2 \times S^3$. Finally, the Smale-Poincaré Theorem implies that homotopy spheres are homeomorphic, so that we have a homeomorphism $Y^{p,q} \cong S^2 \times S^3$. Later, an explicit diffeomorphism between $S^2 \times S^3$ and $Y^{p,q}$ was found [17]. Soon thereafter, a paper that generalized the construction of $Y^{p,q}$ to a larger family of metrics that could be defined in any odd dimension $2n + 1$ on $S^n \times S^{n+1}$ was completed [14, 15]. One of these larger family of metrics is denoted $L^{p,q,r}, p, q, r \in \mathbb{Z}, 0 \le p \le q, 0 < r < p + q, (p, q) = (p, r) = (q, r) = 1$ and its construction is far less complicated than the original construction in [22]. Note that if $p + q = 2r$, then $L^{p,q,r} = Y^{p,q}$. This section will construct the $L^{p,q,r}$ metric due to simplicity and subsequently, we will restrict ourselves to the Laplacian for the $Y^{p,q}$ case.

[20] When the phrase 'contact manifold' is used, we will mean an *strict contact manifold* in the sense of [8], page 181

Construction

We will follow the methodology of [14], which provides a direct route to the $L^{p,q,r}$ metric from a well-known AdS_n solution of the vacuum Einstein equations with negative cosmological constant. The method of construction can be summarized as follows:

1. Start with the five-dimensional Kerr-de Sitter Black Hole Metrics (in local coordinates) found in [30]. These metrics represent solutions to Einstein's Field Equations that admit charged, rotating black holes
2. Consider the "Euclideanization" of these metrics, which amounts to a formal analytic continuous of a real n-manifold to an almost complex n-manifold (i.e. with real dimension $2n$)
3. Implement a supersymmetry constraint known as the *BPS Scaling Limit*
4. Compute the Killing Vectors and Killing Spinors[21] associated to the given metric

The five-dimensional Kerr-de Sitter metric for a rotating, charged black hole (as per Hawking, et. Al, [30]) in local coordinates $(t, \phi, \psi, r, \theta)$ on an open set $U_\alpha \subset AdS_5$ is:

$$g(\partial_t, \partial_\phi, \partial_\psi, \partial_r, \partial_\theta) = -\frac{\Delta}{\rho^2}\left(dt - \frac{a\sin\theta}{\Xi_a}d\phi - \frac{b\cos^2\theta}{\Xi_b}\right)^2 + \frac{\Delta_\theta\sin^2\theta}{\rho^2}\left(a\,dt - \frac{(r^2+a^2)}{\Xi_a}d\phi\right)^2 + \frac{\Delta_\theta\cos^2\theta}{\rho^2}\left(b\,dt - \frac{r^2+b^2}{\Xi_b}d\psi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{(1+r^2\ell^{-2})}{r^2\rho^2}\left(ab\,dt - \frac{b(r^2+a^2)\sin^2\theta}{\Xi_a}d\phi - \frac{a(r^2+b^2)\cos^2\theta}{\Xi_b}d\psi\right)^2 \tag{22}$$

where we have:

- a,b are the conserved quantities of the SO(4) $\cong$ SU(2) $\times$ SU(2) symmetry in this metric. Effectively, they scale the Killing Vectors that are the infinitesmal generators of these symmetries
- ℓ is another conserved quantity from the Killing Vector that corresponds to the conserved Energy of the system
- $\Delta = \frac{1}{r^2}(r^2+a^2)(r^2+b^2)(1+r^2l^{-2}) - 2M$
- $\Delta_\theta = (1 - a^2l^{-2}\cos^2\theta - b^2l^{-2}\sin^2\theta$
- $\rho^2 = (r^2 + a^2\cos^2\theta + b^2\sin^2\theta)$
- $\Xi_a = (1 - a^2l^{-2})$
- $\Xi_b = (1 - b^2l^{-2})$

Now we can (somewhat) informally extend this metric (at least locally) to the complexification $U_\alpha \otimes \mathbb{C} \approx \mathbb{R}^n \otimes \mathbb{C}$. To see why this is feasible, consider the metric localized in local coordinates over a coframe $\{dx^i(y)\}_{i=1}^n$ for $y \in U_\alpha$ to be defined as as $g(y) = g_{ij}(y)dx^i dx^j$. The idea is that since we are complexifying the chart (U_α, ψ_α), we can also complexify the local trivialization of Sym$(2, M)$ over U_α. Explicitly, the authors of [14] do this as follows. Consider the following coordinate transformations:

$$\tau := \frac{\sqrt{\lambda}t}{i}, \lambda := -l^2, a' := -ia, b := -ib \tag{23}$$

Now the next simplification step requires taking a limit that relates a, b, r to λ. Effectively, this ties all of the three open parameters a, b, l as well as one coordinate to a single length scale λ. This will immensely simply the metric allowing us to compute the new simplified Killing Vectors. These transformations come from supersymmetry considerations as well as conservation of mass and energy. These transformations represent the **Bogomol'nyi − Prasad − Sommerfield Limit** (BPS Limit) which is a way to bound the conserved quantity derived from the t or τ coordinates (Energy). Effectively, we are scaling our free parameters as well as our r

[21]The introduction of spinors and particular spin bundles will be explained as spinors are encountered in this derivations. Keep reading!

coordinate in such a way that the energy E goes to the BPS Limit as one takes a limit $\epsilon \downarrow 0$, for a perturbation ϵ. Explicitly, these transformations are:

$$a = \lambda^{-1/2}\left(1 - \frac{1}{2}\alpha\epsilon\right), b = \lambda^{-1/2}\left(1 - \frac{1}{2}\beta\epsilon\right), r^2 = \lambda^{-1}(1 - x\epsilon), M = \frac{1}{2}\lambda^{-1}\mu\epsilon^2 \qquad \alpha, \beta, \mu \in \mathbb{R} \tag{24}$$

Under a combination of the coordinate transformations (23) and (24), where the limit $\epsilon \downarrow 0$ is taken, our new metric $\tilde{g}$ is defined as:

$$\lambda\tilde{g}(\partial_\tau, \partial_x, \partial_\theta, \partial_\phi, \partial_\psi) = (d\tau + \sigma)^2 + h(\partial_x, \partial_\theta, \partial_\phi, \partial_\psi) \tag{25}$$

where h is defined as:

$$h(\partial_x, \partial_\theta, \partial_\phi, \partial_\psi) = \frac{\rho^2 dx^2}{4\Delta_x} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{\rho^2 d\theta^2}{\Delta_\theta}\left(\frac{\sin^2\theta}{\alpha}d\phi + \frac{\cos^2\theta}{\beta}d\psi\right)^2 + \frac{\Delta_\theta \sin^2\theta\cos^2\theta}{\rho^2}\left(\frac{\alpha - x}{\alpha}d\phi - \frac{\beta - x}{\beta}d\psi\right)^2$$

where we have:

$$\begin{aligned}
\sigma &= \frac{(\alpha - x)\sin^2\theta}{\alpha}d\phi + \frac{(\beta - x)\cos^2\theta}{\beta}d\psi \\
\Delta_x &= x(\alpha - x)(\beta - x) - \mu \\
\rho^2 &= \Delta_\theta - x \\
\Delta_\theta &= \alpha\cos^2\theta + \beta\sin^2\theta
\end{aligned} \tag{26}$$

One quick note about why such a transformation requires using an analytic continuation. In physics, one tends to ignore when one is dealing with a spin bundle and when one is dealing with the tangent or cotangent bundles in order to simplify computation. However, in General Relativity, one tends to deal with the standard, real tangent and cotangent bundles of a spacetime manifold M to solve the classical equations of motion. However, when one deals with relativistic quantum mechanics, a spin bundle S is introduced (either implicitly or explicitly) so that solutions to the Dirac equation, which involve spinors $s \in \Gamma(S)$, can be developed. As such, this ad-hoc complexification illustrated above serves as a way to explicitly construct a local trivialization of $S \twoheadrightarrow M$ over some trivializable chart domain $U_\alpha \subset M$.

Using Mathematica, one can quickly establish that the Ricci Tensor for the above metric is related to the $\tilde{g}$ by $\mathrm{Ric} = 4\lambda\tilde{g}$ so this metric is Einstein. A further computation shows that $R = 0$, so that this metric is Ricci-flat. Moreover, one can use an $\mathbb{R}^n$ diffeomorphism to set the free parameter $\mu = 1$ so that the only free parameters are α, β. Now there are a few angle forms in the above metric (the exact ranges aren't established in [14]) and in particular the ψ, ϕ, θ coordinates are periodic (i.e. their chart domains are $[0, k\pi]$, where the choice of k isn't explicit). This generates a $U(1) \times U(1) \times U(1)$ isometry that will help us elucidate the Killing Vectors associated to $\tilde{g}$. Now note that if we regard $\tilde{g}$ as representing a local fibration $U(1) \hookrightarrow U_\alpha \subset M \twoheadrightarrow U_\alpha/U(1)$, where the last quotient is over any of the angle forms ψ, ϕ, θ, we can show that the induced metric on the quotient space represents a Kähler-Einstein manifold with Kähler 2-form $\omega_{\tilde{g}}|_{U_\alpha} = \frac{1}{2}d\sigma$. Let us sketch out the argument in [22]. Let the metric on $N := M/U(1)_\theta$ be denoted $\hat{g}$. Firstly, one considers the quotient over the $U(1)$ fiber corresponding to θ and then uses a computation of the first Chern number to show that there exists a coordinate transformation which reduces to the round metric on $S^2 \times S^2$ with trivial clutching function.[22] Moreover, using the other circle coordinates, we can construct a complete open cover of $(M, \tilde{g})$ (i.e. so that the North Pole, South Pole in the θ coordinate have a non-singular coordinate representation). One can then construct a basis for $H_2(N;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ by looking at the intersection number of chains based in the three different $M/U(1)$ quotients. Finally, we can dualize these chains and get an explicit basis for $H^2(S^2 \times S^2;\mathbb{Z})$ in the coordinates we are looking at. This basis is [22, 41]:

$$\begin{aligned}
\omega_1 &= \frac{1}{4\pi}\cos\zeta d\zeta \wedge (d\psi - \cos\theta d\phi) + \frac{1}{4\pi}\sin\theta d\theta \wedge d\phi \\
\omega_2 &= \frac{1}{4\pi}\sin\theta d\theta \wedge d\phi
\end{aligned} \tag{27}$$

[22]Recall that a clutching function for S^n is a map $S^{n-1} \to S^n$ that serves at the attaching map for the two n-cells D_1, D_2 that one glues along $\partial D_i \cong S^{n-1}$ to construct S^n. Given a clutching function, one can represent S^n as a CW complex

Finally, by taking a linear combination of the above basis vectors for $H^2(N;\mathbb{Z})$ and enforcing hermiticity, one arrives at the Kähler form on the quotient: $\omega_{\hat{g}} = \frac{1}{2}d\sigma$.

We have sketched an argument that shows that $(M, \tilde{g})$ is the total space of three $U(1)$ fibrations over a Kähler-Einstein space that is homeomorphic to $S^2 \times S^2$. As it turns out, (25) is in a "standard form" for the local expression of a Sasaki-Einstein metrics so that it is almost automatic that $\tilde{g}$ is Sasaki-Einstein [22]. In the future, this section will include the full derivation of the Killing Vectors associated to $\tilde{g}$ and how the relationship between the Killing Vectors and the constraints on $p, q, r \in \mathbb{Z}$ is established. However, for now, the Killing Vectors and the relationship between the moduli of the Killing Vectors and p, q, r will simply be stated. Firstly note that we have four Killing Vectors for this space:

- Killing Vectors that are compatible with the unsuitability of these coordinates at $\theta = 0, \frac{\pi}{2}$:
$$\partial_\phi, \partial_\psi \tag{28}$$
This intuitively makes sense, since these vectors vanish when $\theta = 0, \frac{\pi}{2}$ as all the $d\phi, d\psi$ terms in h have $\sin\theta$ in front of them.

- Killing Vectors that are compatible with the unsuitability of these coordinates at the roots of the cubic Δ_x, denoted x_1, x_2, x_3:
$$\ell_i = c_i\partial_\tau + a_i\partial_\phi + b_i\partial_\psi \tag{29}$$
where $i \in \{1, 2\}$ and,
$$\begin{aligned} a_i &= \frac{\alpha c_i}{x_i - \alpha} \\ b_i &= \frac{\beta c_i}{x_i - \beta} \end{aligned} \tag{30}$$
$$c_i = \frac{(\alpha - x_i)(\beta - x_i)}{2(\alpha+\beta)x_i - \alpha\beta - 3x_i^2} \tag{31}$$

suppose that $u, v \in \mathbb{Q}$ are such that $0 < v < 1, -v < u < v$. Then we define $p, q, r, s \in \mathbb{Z}$ by [35]:
$$\frac{q-p}{p+q} := \frac{2(v-u)(1+uv)}{4-(1+u^2)(1+v^2)}, \frac{r-s}{p+q} := \frac{2(v+u)(1-uv)}{4-(1+u^2)(1+v^2)}, p+q-r = s \tag{32}$$
These conditions ensure that $\alpha, \beta, \mu, x_1, x_2$ and x_3 are all rational. This is an important consequence of the Chern number restrictions on M. As it turns out, the authors of [22] show that $\frac{l^{-1}}{2\pi}\sigma$ (as per the definition in (25)) is a connection on M if we treat $M \hookrightarrow M/U(1)$ as a $U(1)$ bundle. Since this is independent of the choice of $U(1)$ quotient (the quotient is completely described by the metric h of (25)) and since line bundles over $S^2 \times S^2$ are classified by Chern Classes in $H^2(S^2 \times S^2; \mathbb{Z})$ ([29], §.1), we have implicit restrictions on the two parameters α, β. Moreover, if we assume that we don't know the diffeomorphism $\mathbb{R}^5 \to \mathbb{R}^5$ that sends μ to unity.[23] Given this disclaimer, here is the relationship between u, v and $\alpha, \beta, \mu, x_1, x_2, x_3, x_3$:
$$\begin{aligned} &\alpha = 1 - \frac{1}{4}(1+u)(1+v), \beta = 1 - \frac{1}{4}(1-u)(1-v), \mu = \frac{1}{16}(1-u^2)(1-v^2) \\ &x_1 = \frac{1}{4}(1+u)(1-v), x_2 = \frac{1}{4}(1-u)(1+v), x_3 = 1 \end{aligned} \tag{33}$$
Finally, we can relate the moduli a_i, b_i, c_i of the Killing Vectors to u, v [35]:
$$\begin{aligned} a_1 &= \frac{(1+v)(3-u-v-uv)}{(v-u)[4-(1+u)(1-v)]} & a_2 &= -\frac{(1+u)(3-u-v-uv)}{(v-u)[4-(1-u)(1+v)]} \\ b_1 &= \frac{(1-u)(3+u+v-uv)}{(v-u)[4-(1+u)(1-v)]} & b_2 &= -\frac{(1-v)(3+u+v-uv)}{(v-u)[4-(1-u)(1+v)]} \\ c_1 &= -\frac{2(1-u)(1+v)}{(v-u)[4-(1+u)(1-v)]} & c_2 &= \frac{2(1+u)(1-v)}{(v-u)[4-(1-u)(1+v)]} \end{aligned} \tag{34}$$

[23]This is a good assumption, at least based on the scant efforts at numerical computation related to these manifolds. For instance in the Master's Thesis [35], the author notes that he could not find a closed form diffeomorphism (or even an approximation) using Mathematica.

Recall that the goal of this thesis is to study the one-form spectrum of a slightly simpler object, the Sasaki-Einstein metrics $Y^{p,q}$. Given these quite complicated expressions, it will be wise to restrict our initial scope to a specific choice of p, q. The first choice of p, q, $p = q = 1$ is simply the homogeneous metric on $S^2 \times S^3$; this space is known as $T^{1,1}$ and it's spectrum is well-established [47, 26]. The first non-trivial manifold is $Y^{2,1}$ which is an irregular Sasaki-Einstein manifold — that is, the orbits of one of the allowable $U(1)$ actions is dense in $Y^{2,1}$. In general, note that the metric in the form (25) is invariant has $U(1) \times U(1) \times U(1)$ isometries. However, if placed in the original form from [22], one can show that there is actually an *graded* isometry group isomorphic to $SU(2) \times_{\mathbb{Z}_2} \times U(1) \times U(1)$. For completeness, we present this metric:

$$
\begin{aligned}
g_{Sparks}(\partial_\theta, \partial_\phi, \partial_y, \partial_\psi, \partial_\alpha) = \frac{(1-cy)}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1}{w(y)q(y)}dy^2 + \frac{q(y)}{9}[d\psi - \cos\theta d\theta]^2 \\
+ w(y)\left[d\alpha + \frac{ac - 2y + y^2 c}{6(a - y^2)}[d\psi - \cos\theta d\phi]^2\right]
\end{aligned}
\tag{35}
$$

where we have,

$$
\begin{aligned}
w(y) &= \frac{2(a - y^2)}{1 - cy} \\
q(y) &= \frac{a - 3y^2 + 2cy^3}{a - y^2} \\
0 \le \theta \le \pi, \quad 0 \le \phi, \psi \le 2\pi, &\quad y_1 \le y \le y_2, \quad y < 1, \quad 0 \le \alpha \le 2\pi\ell
\end{aligned}
$$

where y_1, y_2 are roots of the cubic $q(y)$ and due to symmetry considerations $\ell = \frac{q}{3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}}$. For reasons explained on page 3 of [22], we are forced to restrict $a \in (0, 1)$. From this definition, it is apparent that ψ, α have $U(1)$ isometries. However, note that one can rewrite the expressions for θ, ϕ in terms of the round metric on S^3, so that θ has an $SU(2)$ isometry. When one rewrites (35) in terms of the Maurer-Cartan forms of $SU(2)$, it is apparent that there is a $U(1)$ right action (from the Killing Vector ∂_ϕ) and an $SU(2)$ left action (derived from the Killing Vector ∂_θ).

7.3 The Scalar Laplacian

The Scalar Laplacian (i.e. the Laplacian-Beltrami Operator on Functions $C^\infty(M)$) of the $Y^{p,q}$ manifolds has been studied in [35, 36, 45]. Unfortunately, we have two metrics to deal with, (25), (35). The solutions of (35) are a bit harder to elucidate, but they will be discussed in this section. From §4.3.1 onwards, we will solely use (25) simply because it is easier to work with. Using separation of variables, the authors were able to deduce that harmonic functions associated to this operator (up to a scalar multiple) for (35) are of the form [36],

$$
\Psi_{Sparks}(y, \theta, \phi, \psi, \alpha) = \exp\left(i\left[N_\phi \phi + N_\psi \psi + \frac{N_\alpha}{\ell}\alpha\right]\right) R(y)\Theta(\theta) \tag{36}
$$

The expressions for $R(y), \Theta(\theta)$ are a bit complicated and require a more delicate analysis than the angular solutions. The authors make many analogies to the solutions of the non-relativistic, time-independent, Schrödinger Equation and consider $R(y)$ to be the "radial" function for $Y^{p,q}$ and $\Theta(\theta)$ to be the "angular" function for $Y^{p,q}$. These analogies are quite valid since we are dealing with a manifold that is diffeomorphic to $S^2 \times S^3$ and we can rewrite our metric in terms of the Maurer-Cartan forms on S^3. In fact, the Θ equation, while not explicitly known, is an eigenfunction of the Casimir operator $\hat{K}$ of $SU(2)$, which has the coordinate expression:

$$
\hat{K} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\left(\frac{\partial}{\partial\phi} + \cos\theta\frac{\partial}{\partial\psi}\right)^2 + \left(\frac{\partial}{\partial\psi}\right)^2 \tag{37}
$$

and as per the physics convention we have $\hat{K}\Theta(\theta) = -L(L+1)\Theta(\theta), L \in Z$. It turns out that the ordinary differential equation for $R(y)$ obtained via separation of variables is in fact *Heun's equation*, an equation of Fuchsian-type with four regular singularities at $y = y_1, y_2, y_3, \infty$. The explicit solution will be discussed in the next section after we write the Ordinary Differential Equations for each variable.

The Scalar Laplacian in Coordinates

Without further ado, the Scalar Laplacian $\Delta_{(5)}$ in coordinates (with regard to the metric (25)) is [35]:

$$\begin{aligned}\Delta_{(5)} \to & \frac{4}{\rho^2}\frac{\partial}{\partial x}\left(\Delta_x \frac{\partial}{\partial x}\right) + \frac{4}{\rho^2}\frac{\partial}{\partial y}\left(\Delta_y \frac{\partial}{\partial y}\right) + \frac{\partial^2}{\partial \tau^2} \\ & + \frac{\alpha^2\beta^2}{\rho^2\Delta_x}\left(\frac{(\beta - x)}{\beta}\frac{\partial}{\partial \phi} + \frac{(\alpha - x)}{\alpha}\frac{\partial}{\partial \psi} - \frac{(\alpha - x)(\beta - x)}{\alpha\beta}\frac{\partial}{\partial \tau}\right)^2 \\ & + \frac{\alpha^2\beta^2}{\rho^2\Delta_y}\left(\frac{(1+y)}{\beta}\frac{\partial}{\partial \phi} - \frac{(1-y)}{\alpha}\frac{\partial}{\partial \psi} - \frac{(\alpha - \beta)(1-y^2)}{2\alpha\beta}\frac{\partial}{\partial \tau}\right)^2 \end{aligned} \tag{38}$$

where $\Delta_y := (1-y^2)\Delta_\theta$.

While these coordinates are slightly different that those in , the only change in the resulting eigenfunction will arise in different normalization constants (i.e. the process of defining Ψ so that $\|\Psi\|_{L^2(M)} = 1$) with the eigenfunctions of (38) having normalization constants that depend on α, β, ρ as opposed to a and c. Note that we can simplify (38) significantly if we express Δ_y, Δ_x in terms of their roots. That is, if x_1, x_2, x_3 are the roots of Δ_x and y_1, y_2, y_3 are the roots of Δ_y (see (26)), the metric becomes [35]:

$$\begin{aligned}\Delta_{(5)} \to & \frac{\partial^2}{\partial \tau^2} + \frac{4}{\rho^2}\frac{\partial}{\partial x}\left(\Delta_x \frac{\partial}{\partial x}\right) + \frac{\Delta_x}{\rho^2}\left(\frac{1}{(x-x_1)}\nu_1 + \frac{1}{(x-x_2)}\nu_3 + \frac{1}{(x-x_3)}\nu_5\right)^2 \\ & + \frac{4}{\rho^2}\frac{\partial}{\partial y}\left(\Delta_y \frac{\partial}{\partial y}\right) + \frac{\Delta_y}{\rho^2}\left(\frac{1}{(y-y_1)}\nu_2 + \frac{1}{(y-y_2)}\nu_4 + \frac{1}{(y-y_3)}\nu_6\right)^2 \end{aligned} \tag{39}$$

The separable solutions in these coordinates are quite similar to those in (8.3) except that we now have lost the angular function Θ and we have replaced it with another solution to a Heun's Differential Equation. Explicitly, we have the eigenfunction,

$$\Psi(\tau, \phi, \psi, x, y) = \exp\left(i\left[N_\tau \tau + N_\phi \phi + N_\psi \psi\right]\right) F(x) G(y) \tag{40}$$

where F, G are defined by the *Heun's Differential Equations*,

$$\frac{d^2F}{dx^2} + \left(\frac{1}{(x-x_1)} + \frac{1}{(x-x_2)} + \frac{1}{(x-x_3)}\right)\frac{dF}{dx} + Q_x F = 0 \tag{41}$$

$$\frac{d^2G}{dy^2} + \left(\frac{1}{(y-y_1)} + \frac{1}{(y-y_2)} + \frac{1}{(y-y_3)}\right)\frac{dF}{dy} + Q_y F = 0 \tag{42}$$

where we have,

$$Q_x = \frac{1}{\Delta_x}\left(\mu_x - \frac{1}{4}Ex - \sum_{i=1}^{3}\frac{\alpha_i^2}{x - x_i}\frac{d\Delta_x}{dx}(x_i)\right), \quad Q_y = \frac{1}{H_y}\left(\mu_y - \frac{1}{4}Ey - \sum_{i=1}^{3}\frac{\beta_i^2}{y - y_i}\frac{dG}{dy}(y_i)\right) \tag{43}$$

$$\alpha_i = -\frac{1}{2}(a_i N_\phi + b_i N_\psi + c_i N_\tau), \quad \beta_1 = \frac{1}{2}N_\phi, \ \beta_2 = \frac{1}{2}N_\psi, \ \beta_3 = \frac{1}{2}(N_\tau - N_\phi - N_\psi) \tag{44}$$

$$H_y = (y-y_1)(y-y_2)(y-y_3), \quad \mu_x = \frac{1}{4}C - \frac{1}{2}N_\tau(\alpha N_\phi + \beta N_\psi) + \frac{1}{4}(\alpha+\beta)N_\tau^2 \tag{45}$$

$$\mu_y = \frac{1}{2(\beta - \alpha)}\left(-C + \left(\frac{\alpha+\beta}{2}\right)E + 2(\alpha N_\phi + \beta N_\psi)N_\tau - (\alpha+\beta)N_\tau^2\right) \tag{46}$$

Part III

Generalized Complex Geometry

8 History and Motivation

8.1 Motivation

- We want to unite Complex and Symplectic Geometry by finding a *natural* integrability condition for both structures.
- The main idea is that instead of treating these structures as group actions on TM, consider their action on the sum $TM \oplus T^*M$ for a manifold M
- There are quite a few nice features of this bundle. It has a natural Lie Algebroid structure under the *Courant Bracket*. There is a canonical metric on $TM \oplus T^*M$ and as such we can define isotropic subbundles *without* a specified symplectic form
- There is a natural homological classification of whether certain structures can be put on a manifold when one considers $TM \oplus T^*M$. This comes from the natural *gerbe* on $TM \oplus T^*M$
- This generalized structure on a manifold which subsumes complex and symplectic geometry will make some of the subtleties of mirror symmetries a lot clear, conceptually

8.2 History

9 Linear Algebra on $V \oplus V^*$

One of the things that Koerber tends to brush through quickly are some of the subtleties and changes in definitions that occur when one considers $TM \oplus TM^*$. To rectify this, I will summarize Chapter 2 of M. Gualtieri's Thesis in this section. One quick notational note: If V is an m-dimensional vector space with dual V^*, then Roman letters will indicate elements of V, Greek letters will indicated elements of V^* and an element of $V \oplus V^*$ will be denoted $X + \xi$ (Roman + Greek).

Before moving to the vector bundle $TM \oplus TM^*$, let's consider some analytic facts about the vector spaces in each fiber. Let V be an n-dimensional vector space with dual V^*. We can define a natural symmetric and skew-symmetric bilinear forms $\langle \cdot, \cdot \rangle_+, \langle \cdot, \cdot \rangle_-$ on $V \oplus V^*$ by,

$$\langle X + \xi, Y + \eta \rangle_+ := \frac{1}{2}\left(\xi(Y) + \eta(X)\right)$$
$$\langle X + \xi, Y + \eta \rangle_- := \frac{1}{2}\left(\xi(Y) - \eta(X)\right)$$

We will only need the first form. Now for completeness, let's show that it is a non-degenerate form. Suppose the form is degenerate: $\exists X + \xi$ such that $\langle X + \xi, Y + \eta \rangle = 0, \forall Y + \eta$. Without the loss of generality, assume that $X = \hat{e}_i, \xi = \tilde{f}_j, 1 \leq i, j \leq n$, where $\{\hat{e}_i\}$ is a basis for V, $\{\tilde{f}_i\}$ is the basis of V^* dual to $\{\hat{e}_i\}$. But note that $\langle \hat{e}_i + \tilde{f}_j, \hat{e}_j + \tilde{f}_i \rangle_+ = 1$, which contradicts the degeneracy hypothesis. As such, we will now refer to $\langle \cdot, \cdot \rangle_+$ as $\langle \cdot, \cdot \rangle$, an inner product on V. It is clear that $O(V \oplus V^*) \cong O(n, n)$, so that this inner product defined the orthogonal group for this space. Moreover we have $\wedge^{2n}(V \oplus V^*) \cong \wedge^m V \otimes \wedge^n V^*$ and by applying the **det** functor, one can show that there is a natural pairing between $\wedge^k V^*, \wedge^k V$. Hence there is a canonical orientation on $V \oplus V^*$ and we can define the special orthogonal group $SO(V \oplus V^*) \cong SO(m, m)$.

9.1 Symmetries of $V \oplus V^*$

Since the symmetries of each fiber affect the set of allowable G-structures on a fiber bundle over a manifold, it is pretty important to know the symmetries of $V \oplus V^*$. Let us first define the $\mathfrak{so}(V \oplus V^*) \cong \text{Lie}\,(SO(V \oplus V^*))$:

$$\mathfrak{so}(V \oplus V^*) := \{T \in \text{End}(V \oplus V^*) : \langle Tx, y \rangle + \langle x, Ty \rangle = 0, \forall x, y \in V \oplus V^*\} \tag{47}$$

Now let's trying to figure out how a linear operator T looks. Let $p_1 : V \oplus V^* \to V, p_2 : V \oplus V^* \to V^*$ be projections onto the first and second coordinates, respectively. Now note that we need a pair of operators

$A \in \mathrm{End}(V \oplus V^*)$, $C : V^* \to V$ so that $p_1 T(X+\xi) = AX + C\xi \in V$. Such a decomposition is always possible since T is linear. Similarly, we need a pair of operators $B : V \to V^*, D \in \mathrm{End}(V^*)$ such that $p_2 T(X+\xi) = BX + D\xi \in V^*$. What restrictions does the $\mathfrak{so}(V \oplus V^*)$ symmetry place on A, B, C, D? As it turns out, we require C, B to be antisymmetric, i.e. $C = -C^\dagger, B = -B^\dagger$ and that $D = -A^*$, the transpose of A. According to convention C is denoted by β and we (formally) represent T as,

$$T := \begin{bmatrix} A & \beta \\ B & -A^* \end{bmatrix} \tag{48}$$

Note that the antisymmetry conditions let us regard $\beta \in \wedge^2 V^*, B \in \wedge^2 V$ with $B(X) = \iota_X(B)$. Using the formal matrix representation of (48), we can break up the group actions:

- B-transform: Note that $\exp(B) = \begin{bmatrix} 1 & \\ B & 1 \end{bmatrix}$, so that $\exp(B)(X+\xi) = X + \xi + \iota_X B$
- β-transform: On the other hand, $\exp(\beta) = \begin{bmatrix} 1 & \beta \\ & 1 \end{bmatrix}$ so that $\exp(\beta)(X+\xi) = X + \xi + \iota_\xi \beta$
- $GL(V)$Action Let $A \in \mathfrak{so}(V \oplus V^*)$. Then $\exp(A) = \begin{bmatrix} \exp A & \\ & (\exp A*)^{-1} \end{bmatrix}$. This is an embedding of $GL^+(V)$ into the component containing $e \in SO(V \oplus V^*)$. This extends to a orientation preserving map $GL(V) \to SO(V \oplus V^*)$

9.2 Maximal Isotropic Subspaces

The notion of isotropic subspaces is important in symplectic geometry; however, the set of *Maximal Isotropic Subspaces* of $V \oplus V^*$ plays an important role in the theory of generalized complex structures. As we will show in §.3, the Maximal Isotropic Subspaces are in bijective correspondence with an important class of spinors on $V \oplus V^*$. Let us recall the main definition from Symplectic Geometry, adapted to the case of $V \oplus V^*$:

Definition 9.1. *A subspace $L < V \oplus V^*$ is* ***isotropic*** *when $\langle X, Y\rangle = 0, \forall X, Y \in L$. Since we have a metric signature (m, m), the maximal dimension of such a subspace is m, so if $\dim L = m$, L is called* ***maximal isotropic*** *or* ***a linear Dirac structure***

It turns out that there is a cogent way to construct all of the maximal isotropic subspaces. Let $E \le V$ be any subspace and let $\epsilon \in \wedge^2 E^*$ so that we can treat $\epsilon : E \to E^*, \epsilon(X) = \iota_X \epsilon$. Define the subspace $L(E, \epsilon)$ as,

$$L(E, \epsilon) := \{X + \xi \in E \oplus V^* : \xi|_E = \epsilon(X)\} \tag{49}$$

Using this $L(E, \epsilon)$ we can get any maximal isotropic subspace:

Proposition 9.1. *Every maximal isotropic subspace in $V \oplus V^*$ is of the form $L(E, \epsilon)$*

Proof. Let L be a maximal isotropic subspace. Now let $E := p_1 L \le V$. Since L is maximal isotropic, $\dim L = m$, so that $W := L \cap V^* \le V^*$ is a subspace of dimension $m - \dim E$. Now if $X, Y \in E, \xi, \eta \in W$, then $\langle X + \xi, Y + \eta\rangle = \frac{1}{2}(\eta(X) + \xi(Y))$ which vanishes for all $X, Y \in E$ iff $\eta, \xi \in \mathrm{Ann}(E)$ and $L = E \oplus \mathrm{Ann}(E)$. Hence, $E^* \cong V^*/\mathrm{Ann}(E)$ so that we can define $\epsilon : E \to E^*$ via $e \mapsto p_2(p_1^{-1}(e) \cap L) \in V^*/\mathrm{Ann}(E)$. □

Given this proposition, we are now in a position to categorize the Maximal Isotropic subspaces. From this construction, one of the easiest characterizations of a maximal isotropic subspace is that of *type*:

Definition 9.2. *Let (L, ϵ) be a maximal isotropic subspace. The* ***type*** *of a maximal isotropic subspace is defined to be the codimension k of the projection $L \twoheadrightarrow V$. In other words,*

$$k := \dim \mathit{Ann}(E) = m - \dim E \tag{50}$$

We can now use the various actions of $\mathsf{End}(V \oplus V^*)$, namely the B, β-transforms, to further classify the maximal isotropic subspaces. From the action of a B-transform, $X + \xi \to X + (\xi + \iota_X B)$, we see that a B-transform doesn't affect the dimension of the subspace E since $p_1(e^{-B}(X+\xi)) = X$. More concretely, suppose that $X + \xi \in L(E, \epsilon)$ so that $e^B(X+\xi) = X + (\iota_X B + \xi)$. We want to find $\tilde{\epsilon} : E \to E^*$ such that $X + (\iota_X B + \xi) \in L(E, \tilde{\epsilon})$. This means that $\tilde{\epsilon}(X) = (\iota_X B + \xi)|_E = \iota_X B|_E + \epsilon(X)$, so $\tilde{\epsilon} = \epsilon + i^* B$ where $i : E \hookrightarrow V$ is the natural inclusion. As such, B transforms do not affect type and one can obtain any maximal isotropic subspace using a B-transform of $V = L(E, \{0\})$.

However the action of the β-transform, $X + \xi \to (X + \iota_\xi \beta) + \xi$ *can* affect $\dim E$ and hence the type k. It turns out that β transforms increase the dimension of E (if possible) by an even number (i.e. $\dim E \xrightarrow{\beta} \dim E + 2n$) and as such, they are also type preserving. Let us prove this statement:

Proposition 9.2. *Let* $L(E, \epsilon)$ *be a maximal isotropic subspace of* $V \oplus V^*$. β*-transforms change* $\dim E$ *and the type of* $L(E, \epsilon)$ *by an even number*

Proof. Recall that we can consider β as a map $V^* \to V$. Since β-transforms can change the X in a vector $X + \xi$, our goal to try to construct something analogous to $L(E, \epsilon)$ for V^* so that we can make dimensionality arguments as in Proposition 6.1. As such, define $F \subset V^*$ by $F := p_2(L(E, \epsilon))$ and $\gamma \in \wedge^2 F^*$ by $\gamma(f) = p_1\left(p_2^{-1}(f) \cap L\right)$. Note that we are treating γ as a map $V^* \to V$, just as ϵ is a map $E \to E^*$. Treating (F, γ) like the pair (E, ϵ), we define,

$$L(F, \gamma) := \{X + \xi \in V \oplus F : X|_F = \gamma(\xi)\} \tag{51}$$

Now let $X + \xi \in L(F, \gamma)$. Then $e^\beta(X|_F + \xi) = (X|_F + \iota_\xi|_F \beta) + \xi = (\gamma(\xi) + \iota_\xi|_F \beta) + \xi$ so that $e^\beta(X + \xi) \in L(F, \gamma + j^*\beta)$, where $j : F \hookrightarrow V^*$ is the natural inclusion. Hence, a β transform of $L(F, \gamma)$ leaves $\dim F$ invariant. This will be useful, since we can quotient out the annihilator of F.

Note that from (51) we have $E = p_1(L(F, \gamma))$ which contains $L \cap V = \mathrm{Ann}(F) \subset E$. Hence we have F^* defined by:

$$F^* := \frac{E}{L \cap V} = \frac{E}{\mathrm{Ann}(F)} = \mathrm{Im}(\gamma) \tag{52}$$

so that $\dim E = \dim L \cap V + \mathrm{rank}(\gamma)$. Since γ is a skew 2-form it has even rank and if $\beta \in SO(V \oplus V^*)^+$, then we can change the dimension of E and subsequently the type of $L(E, \epsilon)$ by an even number.

□

Hence, one is generally interested in whether the type of a maximal isotropic subspace $L(E, \epsilon)$ is odd or even. With a little more effort, the above proposition gives a classification of maximal isotropics and the relationship between $E, E' = L(E, \epsilon)/E$:

Proposition 9.3. *[11] III.1.0* *Let* E *be a maximally isotropic subspace of* V *and let* $E' = L(E, \epsilon)/E$. *If* $\dim E = r$ *and if* $\dim L \cap V = h$, *then* E, E' *are of the same type (both odd or both even) if and only if,*

$$h \equiv r \quad (\textit{mod } 2)$$

Lastly, note that if L, L' are maximal isotropic subspaces of $V \oplus V^*$ with $L \cap L' = \{0\}$ then $V \oplus V^* = L \oplus L'$ and the inner product defines an isomorphism $L' \cong L^*$. This follows from the Witt Decomposition associated to $\langle \cdot, \cdot \rangle$ [11], I.3.2.

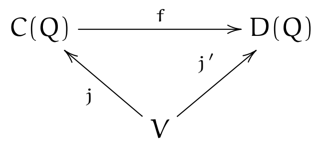
9.3 Spinors

Before moving onto the natural spin representation afforded to $V \oplus V^*$, let us review some of the basis mathematical principles about spinors. The proofs will be omitted since they are standard in most textbooks about spin geometry, e.g. [11, 19].

Clifford Algebras and Spin Representations

Recall that for a vector space V with quadratic form Q, we have the fundamental theorem of Clifford Algebras over (V, Q):

Theorem 9.1. *For every quadratic form* (V, Q)*, there exists a Clifford Algebra*[24] $(C(Q), j)$ *and if* $(C(Q), j), (D(Q), k)$ *are both Clifford Algebras for the same quadratic form* (V, Q)*, then there exists an isomorphism* $f : C(Q) \to D(Q)$ *such that the following diagram commutes:*



Note that the easiest definition of $C(Q)$ is $C(Q) := \tau(V)/I$, where $\tau(V)$ is the tensor algebra of V, $\tau(V) = \oplus_{i=0}^{\infty} T^i V$, $T^i V = \underbrace{V \otimes V \otimes \cdots V}_{i}$ and I is the ideal generated by $\nu \otimes \nu - q(\nu) \cdot \mathbf{1}$. Now in all cases dealt with in these notes, V (or $V \oplus V^*$) will be a complex vector space with such that $C(V, Q) \cong C(\mathbb{C}^n, z_1^2 + \cdots z_n^2)$. A standard fact about the Clifford Algebra $\mathcal{C}_n := C(\mathbb{C}^n, z_1^2 + \cdots z_n^2)$ that will be useful is the following:

Fact 9.1.

$$\mathcal{C}_n = \begin{cases} \overbrace{M_2(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C})}^{k} = \mathsf{End}(\overbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}^{k}) = \mathsf{End}(\mathbb{C}^{2^k}) & \textit{if } n = 2k \in 2\mathbb{Z} \\ \{\overbrace{M_2(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C})}^{k}\} \oplus \{\overbrace{M_2(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C})}^{k}\} = \mathsf{End}(\mathbb{C}^{2^k}) \oplus \mathsf{End}(\mathbb{C}^{2^k}) & \textit{if } n = 2k+1 \in \mathbb{Z} \setminus 2\mathbb{Z} \end{cases}$$

Now we are in a position to define what a spinor is:

Definition 9.4. *A* ***(Dirac) Spinor*** *is simply an element of* $\Delta_n := \mathbb{C}^{2^k}$ *that* $\mathcal{C}_n, n \in \{2k, 2k+1\}$*. The* ***spin representation of the Clifford Algebra*** $\mathcal{C}_n$ *is a map* $\kappa_n : \mathcal{C}_n^c \xrightarrow{\cong} \mathsf{End}(\Delta_n)$ *for* n *even and* $\kappa_n : \mathcal{C}_n^c \cong \mathsf{End}(\Delta_n) \oplus \mathsf{End}(\Delta_n) \to \mathsf{End}(\Delta_n)$*, which is simply the composition of the isomorphism and a projection.*

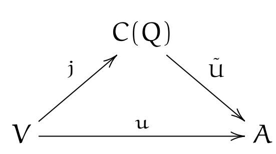
Recall that $\mathsf{Pin}(n)$ is defined to be group of vectors multiplicatively (and formally) generated by all $x \in S^{n-1}$. The spin group $\mathsf{Spin}(n)$ is $\mathsf{Pin}(n) \cap \mathcal{C}_n^0$, where $\mathcal{C}_n^0$ is the connected component containing the identity. We know recall the following facts about $\mathsf{Spin}(n)$ being the double cover of $\mathsf{SO}(n)$:

Fact 9.2. *Let* $\mathsf{Pin}(n), \mathsf{O}(n), \mathsf{SO}(n), \mathsf{Spin}(n)$ *be as discussed. Then:*
a) $\lambda : \mathsf{Pin}(n) \to \mathsf{O}(n)$ *is a continuous surjective homomorphism*
b) $\lambda^{-1}(\mathsf{SO}(n)) = \mathsf{Spin}(n)$
c) $\ker(\lambda) = \{1, -1\} \cong \mathbb{Z}_2$
d) For $n \geq 2, \mathsf{Spin}(n)$ *is connected*
e) For $n \geq 3, \mathsf{Spin}(n)$ *is simply-connected and* $\lambda : \mathsf{Spin}(n) \to \mathsf{SO}(n)$ *is the universal (2-sheeted) covering of the group* $\mathsf{SO}(n)$ *f) The linear subspace* $\mathfrak{m}_2 \subset \mathcal{C}_n, \mathfrak{m}_2 := \textit{Span}\{e_i e_j : 1 \leq i \leq j \leq n\}$ *is a Lie algebra that coincides with the Lie algebra of* $\mathsf{Spin}(n)$ *under the standard matrix commutator. Hence* $\mathsf{Spin}(n) \subset \mathcal{C}_n$*.*

---

[24] Recall:

Definition 9.3. *A pair* $(C(Q), j)$ *is called a* Clifford Algebra *for* (V, Q) *if:*
1) [Associative Algebra] $C(Q)$ *is an associative algebra over a characteristic zero field with identity* $\mathbf{1}$*.*
2) [Clifford Multiplication] $j : V \to C(Q)$ *is a linear map and* $j(\nu)^2 = Q(\nu) \cdot \mathbf{1}, \forall \nu \in V$*.*
3) [Uniqueness] If A *is another associative algebra over a characteristic zero field with identity and* $u : V \to A$ *satisfies* 2)*, then* $\exists$ *a unique algebra homomorphism* $\tilde{U} : C(Q) \to A$ *such that the following diagram commutes:*



Finally we have arrive at the main result about Clifford Algebras that we need. Recall that a representation of a group G on a vector space V is said to be *faithful* if the map $\rho : G \to \mathsf{GL}(V)$ is faithful. In the case of the spin representation, this simply means that the map κ_n is injective. Our main result is the following:

Theorem 9.2. *The Restricted Spin Representation $\kappa_n|_{\mathsf{Spin}(n)}$ of a Clifford Algebra $\mathcal{C}_n$ is a faithful representation of* $\mathsf{Spin}(n)$

Spinors on $V \oplus V^*$

We will use the inner product from **§** to define the quadratic form Q:

$$v^2 := Q(v) = \langle v, v\rangle, \forall v \in V \oplus V^* \tag{53}$$

Let $\mathcal{C}_V$ be the Clifford Algebra $C(V \oplus V^*, Q)$. Note that $\mathcal{C}_V$ has a natural representation on $S = \wedge^\bullet V^*$, $(X+\xi)\cdot\varphi = \iota_X\varphi + \xi \wedge \varphi$. Let's verify that this is a representation, i.e. $(X+\xi)^2 \cdot \varphi = \langle X+\xi, X+\xi\rangle\varphi$:

$$\begin{aligned}
(X+\xi)^2 \cdot \varphi &= (X+\xi)\cdot(\iota_X\varphi + \xi\wedge\varphi)\\
&= \iota_X(\iota_X\varphi) + \xi\wedge\iota_X\varphi + \iota_X(\xi\wedge\varphi)\\
&= \iota_X(\iota_X\varphi) + \xi\wedge\iota_X\varphi + \iota_X\xi\wedge\varphi - \xi\wedge\iota_X\varphi\\
&= \iota_X\xi\wedge\varphi = \xi(X)\varphi\\
&= \langle X+\xi, X+\xi\rangle\varphi
\end{aligned}$$

This is in fact the spin representation for $\mathcal{C}_V$. If $\{\hat{e}_i \oplus \hat{f_j}\}$ is an adapted basis for $V \oplus V^*$, then the n-vector $\omega := \hat{e}_1\hat{e}_2\cdots\hat{e}_n$ has $\omega^2 = 1$ so that the spin representation splits into the ±1 eigenspaces of ω, i.e. $S = S^+ \oplus S^-$ or $\wedge^\bullet V^* = \wedge^{\text{even}}V^* \oplus \wedge^{\text{odd}}V^*$. In analogy to helicity in quantum mechanics, this decomposition is called the helical decomposition. Now it is claimed that even though $\mathcal{C}_V \cdot (S^+ \oplus S^-) \neq \mathcal{C}_V \cdot S^+ \oplus \mathcal{C}_V \cdot S^-$, the spin group $\mathsf{Spin}(V \oplus V^*) \subset \mathcal{C}_V$ preserves this splitting. Analogously to the case of §2.3.1, we have,

$$\mathsf{Spin}(V \oplus V^*) = \{v_1 \cdots v_r : v_i \in V \oplus V^*, \langle v_i, v_i\rangle = \pm1, r \text{ even} \subset \mathcal{C}_V\}$$

Note that we have r even so that the map $v_1\cdots v_r \mapsto \prod_i \langle v_i, v_i\rangle$ is always 1. This is equivalent to the intersection with the connected component containing the identity, since $v_1\cdots v_r \mapsto \prod_i\langle v_i, v_i\rangle$ is continuous. As $V \oplus V^*$ is a vector space of dimension 2k, the results for even n hold and in particular, $\rho : \mathsf{Spin}(V \oplus V^*) \to \mathsf{SO}(V \oplus V^*), \rho(x)(v) = xvx^{-1}, x \in \mathsf{Spin}(V \oplus V^*)$ is again a two-sheeted universal cover of $\mathsf{SO}(V \oplus V^*)$.

Now that we know that there is a natural Clifford Algebra structure on $V \oplus V^*$, how does this relate to the symmetries of the past few sections? The symmetries, such as the $\{B, \beta\}$-transformations, were defined as action by the exponentiating a bivector or 2-form. Moreover, B, β were skew-symmetric so that $B, \beta \in \mathfrak{so}(V \oplus V^*) \cong \wedge^2(V \oplus V^*) \subset \mathcal{C}_V$. As such, we can relate the nature spinors to the symmetries by considering the action of $\mathfrak{so}(V \oplus V^*)$ on $V \oplus V^*$ via the pushforward of ρ, $d\rho_x : T_x\mathsf{Spin}(V \oplus V^*) \cong \mathfrak{so}(V \oplus V^*) \to T_x\mathsf{SO}(V \oplus V^*) \cong \mathfrak{so}(V \oplus V^*), d\rho_x(v) = [x, v], x \in \mathfrak{so}(V \oplus V^*), v \in V \oplus V^*$. Let's consider three cases: the B-transform, the β-transform and the $\mathsf{GL}(V \oplus V^*)$ action:

B-Transformation

Without the loss of generality, we can write $B = \frac{1}{2}B_{ij}e^i \wedge e^j, B_{ij} = -B_{ij}$. Under the quotient $\mathcal{C}_V = \tau(V)/I$, the image of $B, \tilde{B}$ is[25] $\tilde{B} = \frac{1}{2}B_{ij}e^je^i$ and we have:

$$d\rho_{e^je^i}(e_i) = [e^je^i, e_j] = (e^je^i)e_i - e_i(e^je^i) = e^j(e^ie_i) + (e_ie^i)e^j = \{e^i, e_i\}e^j = e^j$$

Finally, let's consider the action of B on a form $\varphi \in \wedge^\bullet V^*$. Note that $B\cdot\varphi = \frac{1}{2}B_{ij}e^j \wedge (e^i) \wedge \varphi) = -B \wedge \varphi$ so that $e^{-B}\varphi = (1 - B + \frac{1}{2}B \wedge B + \cdots) \wedge \varphi$.

[25]Remember, $\hat{e}^i \wedge \hat{e}^j$ can be considered a map $V \to V^*$ and note that $\hat{e}^i \wedge \hat{e}^j(\hat{e}i) = \iota_{\hat{e}_i}\hat{e}^i \wedge \hat{e}^j = \hat{e}^j$ so that $e_i \mapsto e^j, e_j \mapsto e^i$.

β-transform

Now if $\beta = \frac{1}{2}\beta^{ij}e_i \wedge e_j, \beta^{ij} = -\beta^{ji}$, then its quotient $\tilde{\beta} \in \mathcal{C}_V$ is, $\tilde{\beta} = \frac{1}{2}\beta^{ij}e^i e^j$. As before, this leads to $e^{-\beta}\varphi = (1 - \iota_\beta + \frac{1}{2}\iota_\beta^2 + \cdots)\varphi$

$\mathsf{GL}(V \oplus V^*)$ Action

If $A \in \mathsf{End}(V) < \mathfrak{so}(V \oplus V^*)$ we can write it as $A = A_i^j e^i \otimes e_j$. Recall that the lower diagonal term of a transformation T ((48)) has a minus sign, so that we have a transformation of,

$$T = \begin{bmatrix} A & \\ & -A^* \end{bmatrix}$$

so that the action of A is $X + \xi \mapsto A(X) - A^*(\xi)$. Hence the image of A in the Clifford Algebra is $\frac{1}{2}A_i^j(e_j e^i - e^i e_j)$ and it's spinorial action on $\varphi \in \wedge^\bullet V^*$ is,

$$\begin{aligned} A \cdot \varphi &= \frac{1}{2}A_i^j(e_j e^i - e^i e_j) \cdot \varphi = \frac{1}{2}A_i^j\left(e_j(e^i \wedge \varphi) - e^i(\iota_{e^j}\varphi)\right) \\ &= \frac{1}{2}A_i^j\left(\iota_{e_j}(e^i \wedge \varphi) - e^i \wedge \iota_{e^j}\varphi)\right) = \frac{1}{2}\left(A_i^j\delta_j^i\varphi - A_i^j e^i \wedge \iota_{e^j}\varphi\right) \\ &= \frac{1}{2}\left((\mathrm{Tr}A)\varphi - A^*\varphi\right) \end{aligned}$$

Hence the exponential action of $\mathsf{GL}(V)$ on $V \oplus V^*$ is $e^A = \sqrt{\det(A)}(A^*)^{-1}\varphi$ which induces a splitting of $S = \wedge^\bullet V^* \otimes (\det V)^{1/2}$. This can be extended to a full acton of $\mathsf{GL}(V) \oplus \mathsf{GL}(V) < \mathsf{SO}(V \oplus V^*)$ in a relatively natural way.

These actions will serve an important role in determining the diffeomorphism pseudogroup of a Generalized Complex Manifold. Moreover, it turns out that the deformation classes of Generalized Complex Manifolds will be related to the allowable B, β transforms. Finally, note that this natural spin structure coupled with the invariances of B, β has a direct meaning in String Theory where B plays the role of the Neveu-Schwarz 2-form potential.

9.4 The Mukai Pairing and Pure Spinors

We will see later that generalized complex geometry involves the specification of maximal isotropic subbundle L and that these bundles correspond to certain spinors called pure spinors. In order to establish this correspondence, we will need a bilinear form on the set of spinors so that the 2-form ϵ needed to define $L(E, \epsilon)$ can be constructed. The bilinear form Gualtieri and Hitchin used for this purpose is called the Mukai Pairing:

Definition 9.5. *The **Mukai Pairing** is defined to be* $(\cdot,\cdot)_{\mathcal{M}} : S \otimes S \to \det V^*, (s,t)_{\mathcal{M}} \mapsto (\alpha(s) \wedge t)_{\mathtt{top}}$, *where* α *is the antiautomorphism* $\mathcal{C}_V \to \mathcal{C}_V$ *defined by the quotient of* $v_1 \otimes \cdots \otimes v_k \mapsto v_k \otimes \cdots \otimes v_1$ *and the* top *indicates that we take the top degree component of the polyform.*

The Mukai Pairing is a different, but still natural (relative to the Clifford Algebra structure) bilinear form that is compatible with the natural spin action. It was first described by Chevalley [11], Page 143, where he termed it "the bilinear form on spinors." Later, Mukai [43] discovered that it was useful in classifying which symplectic manifolds admit spin structures. The main theorem about the Mukai Pairing is the following:

Proposition 9.4. *([11] Page 143, III2.1, III 2.2) The Mukai Pairing is invariant under the identity component of* $\mathsf{Spin}(V \oplus V^*)$*:*

$$(x \cdot s, x \cdot t)_{\mathcal{M}} = (s,t)_{\mathcal{M}}, \forall x \in \mathsf{Spin}_0(V \oplus V^*)$$

In order to prove this, we first need to establish a bit of algebra. Since V, V^* are maximal isotropic subspaces, $\wedge^\bullet V, \wedge^\bullet V^*$ are subalgebras of $\mathcal{C}(V \oplus V^*)$ since any element of $\wedge^\bullet V$ or $\wedge^\bullet V^*$ will still have norm 0 under the Clifford Norm, $\langle \cdot,\cdot \rangle$. In particular, this means that $\det V$ is a 1-dimensional subspace of $\mathcal{C}(V \oplus V^*)$. It is claimed that $\det V$ generates a special ideal:

Claim 9.1. $\det V$ *generates a left ideal* I *such that if* $f \in \det V$ *is a generator, then every element of the ideal has a unique representation as* $sf, s \in \wedge^\bullet V^*$

The proof of this claim is almost immediate. Since $v \wedge f = 0, \forall v \in \wedge^\bullet V$ and as $\mathcal{C}(V \oplus V^*) \cong \wedge^\bullet(V \oplus V^*)$, we immediately get the desired result. As a result, this induces an action of $\mathcal{C}(V \oplus V)$ on $\wedge^\bullet V^*$,

$$(\rho(x)s)\, f = xsf, \quad \forall x \in \mathcal{C}(V \oplus V^*), \rho : \mathsf{Spin}(V \oplus V^*) \to \mathsf{SO}(V \oplus V^*) \tag{54}$$

Now we can express the Mukai pairing in terms the generator f of I,

10 The Vinogradov and Courant Brackets

Notation:

- $\Gamma(\cdot)$ means "sections of"
- $\Omega^\bullet(M) = \oplus_{i \geq 0} \Omega^i(M) = \Gamma\left(\oplus_{i\geq 0} \wedge^i (T^*M)\right)$ is the exterior algebra of T^*M
- $V^\bullet(M) = \oplus_{j\geq 0} V^p(M) = \Gamma\left(\oplus_{j\geq 0} \wedge^j TM\right)$ is the exterior algebra of TM
- Note that $V^\bullet(M) \otimes \Omega^\bullet(M) \subset \mathsf{End}(\Omega^\bullet(M))$

The goal of this section is to introduce two types of Brackets on $V \oplus V^*$ that when carried over to $TM \oplus TM^*$ will allow us to define a Generalized Complex Structure $\mathcal{J}$. In order to do this we need to define the concept of a *derived bracket*. We will have many brackets in this section and in order to avoid the confusion of some of the references, we will explicitly label all brackets as follows:

$[\cdot,\cdot]$ — Loday Bracket
$[\cdot,\cdot]_{\mathcal{D}}$ — Derived Bracket
$[\cdot,\cdot]_{\mathcal{L}}$ — Lie Bracket on $C^\infty(TM)$
$[\cdot,\cdot]_{\mathcal{G}}$ — Graded Commutator on $\mathsf{End}(\Omega^\bullet(M))$
$[\cdot,\cdot]_{\mathcal{V}}$ — Vinogardov Bracket
$[\cdot,\cdot]_{\mathcal{S}}$ — Schouten Bracket
$[\cdot,\cdot]_{\mathcal{C}}$ — Courant Bracket

Before starting, let's give a few definitions to establish notation and to help with readability. Firstly, note that any graded vector space or graded algebra in this thesis will be $\mathbb{Z}$ or $\mathbb{Z}_2$-graded and the set I will be used to denote the index set of the grading.[26] Furthermore if $V = \oplus_{i\in I} V_i$ is a graded vector space then we define the *degree map* $|\cdot| : V \to I$ to be the index of the homogeneous space of the input. That is, if $v \in V_i \subset V$, then $|v| = i$. In order to speak of brackets, one (generally) needs to speak of Lie Algebras in some shape or form. Since differential complexes are in general sequences of graded vector spaces (or injective resolutions), we will require the notion of a *Graded Lie Algebra*:

Definition 10.1. *([24], Page 415)* A **Graded Lie Algebra** L is a graded vector space $L = \oplus_{i\in I} L_i$ together with a linear map of degree zero,[27] $L \otimes L \to L, x \otimes y \mapsto [x, y]$ satisfying the following:
i) $[x, y] = -(-1)^{|x|\cdot|y|}[x, y]$
ii) $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}\, [y[x, z]]$

Finally, we want to introduce the concept of a graded, differential Lie Algebra for a graded vector space V that has the familiar differential structure of the Exterior Algebra as well as a bracket:

Definition 10.2. *([24], Page 415)* A **Graded Differential Lie Algebra** is a Graded Lie Algebra L equipped with an operator $D : L \to L$ of degree 1 that is a derivation, i.e. D satisfies:

$$D[x, y] = [Dx, y] + (-1)^{|x|}[x, dy]$$

for all $x, y \in L$

[26] While one can generalize the index set to be an arbitrary monoid, we will only consider index sets I that are groups

[27] Recall that a linear map f from a graded vector space $V = \oplus_{i\in I} V_i$ to a graded vector space $W = \oplus_{j\in I} W_j$ is of degree k if $f(V_i) \subset f(W_{i+k}), \forall i \in I$. This degree is related to but different from the degree map $|\cdot|$

A few last considerations to state: $\mathsf{End}(\Omega^\bullet(M))$ is a graded algebra, where the grading is given by endomorphisms of degree k for $k \in \mathbb{Z}, k \leq \dim M$. Also note that we will be moving between the vector space formulation and the vector bundle formulation of the brackets developed in this section interchangeably; if one has qualms with this, consult [38, 13] for full justification.

10.1 Derived Brackets

This section will summarize the results in [38]. First, let's consider a familiar example from the category of smooth manifolds. Let M be n-manifold and let $C^\infty(TM) = \{X \in \mathsf{End}(C^\infty(M)) | X(fg) = X(f)g + fX(g)\}$. Recall that TM naturally has the structure of an infinite-dimensional Lie Algebra under the Lie Bracket $[\cdot,\cdot]_{\mathcal{L}} : C^\infty(TM) \to C^\infty(TM)$ and that there is a natural extension of the differential on functions, *exterior derivative* d which is defined on forms as,

$$\begin{aligned} d : \Omega^k(T^*M) &\to \Omega^{k+1}(T^*M) \\ \omega(X_1,\dots,X_K) \mapsto \Omega(X_1,\dots,X_{K+1}) &= \sum_i (-1)^i X_i \omega(X_1,\dots,\hat{X}_i,\dots,X_{K+1}) \\ &+ \sum_{i<j} (-1)^{i+j} p([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{K+1}) \end{aligned}$$

where $\Omega^k(T^*M)$ is the sheaf of sections of $\wedge^k T^*M$. Since the differential complex $\Omega^\bullet(M)$ is a graded algebra, we can define the degree of a map $\Omega^\bullet \to \Omega^{\bullet+l}$ to simply be l. In the case of the exterior derivative, d has degree 1. Finally recall that if $\iota_X : \Omega^k(M) \to \Omega^{k+1}(M)$ is the inner product, a derivation of degree -1, we have Cartan's Formulas:

$$[d,d]_{\mathcal{G}} = 0 \quad [\iota_X, d]_{\mathcal{G}} = \mathcal{L}_X \quad [\iota_X, \iota_Y]_{\mathcal{G}} = 0 \quad [\mathcal{L}_X, \iota_Y]_{\mathcal{G}} = \iota_{[X,Y]_{\mathcal{L}}} \tag{55}$$

Now note that that if we combine the second and fourth of the Cartan Formulae, we have:

$$\iota_{[X,Y]_{\mathcal{L}}} = [\mathcal{L}_X, \iota_Y]_{\mathcal{G}} = \left[[\iota_X, d]_{\mathcal{G}}, \iota_Y\right]_{\mathcal{G}} \tag{56}$$

This is one of the inspirations for the *derived bracket* — the idea being that one can compute the Lie Bracket of some exotic differential algebra in terms of a "standard" Lie Bracket and a differential operator. In this case, we see that one can use the standard Lie Bracket to define the (graded) bracket for graded endomorphisms of $\Omega^\bullet(M)$. In order to define a derived bracket, we first need to define a *Loday Algebra of degree* n:

Definition 10.3. *[38],* (1.1) *A* ***Loday Algebra*** *of degree* n *is a graded vector space* V *over a field* $\mathbb{K}$, *char*$(K) \neq 2$ *and a* K*-bilinear map* $[\cdot,\cdot] : V \otimes V \to V$ *satisfying the following identity:*

$$[a,[b,c]] = [[a,b],c] + (-1)^{(n+|a|)(n+|b|)}[b,[a,c]] \tag{57}$$

where $|a|$ *is the degree of* $a \in V$.

Note that this looks very similar to the Jacobi Identity and if $[\cdot,\cdot]$ is antisymmetric, then a Loday algebra is simply a graded Lie Algebra. In this paper, one can assume that all fields $\mathbb{K}$ are either $\mathbb{R}$ or $\mathbb{C}$. Now we are in a position to define the *Derived Bracket* associated to a Loday Algebra of degree n.

Definition 10.4. *[38],* (1.2) *If* $(V,[\cdot,\cdot],D)$ *is a graded, differential Lie Algebra over a field* $\mathbb{K}$ *with Loday bracket of degree* n, *we define the* ***Derived Bracket***, $[\cdot,\cdot]_{\mathcal{D}} : V \otimes V \to V$, *by*

$$[a,b]_{\mathcal{D}} = (-1)^{n+|a|+1}[Da,b] \tag{58}$$

In the case with ι_X, d, we have $V = \mathsf{End}(\Omega^\bullet(M)), [,] = [\cdot,\cdot]_{\mathcal{G}}, D = \mathcal{L}_X$ and $[\cdot,\cdot]_{\mathcal{D}} = [\cdot,\cdot]_{\mathcal{L}}$. All of the brackets dealt with in this paper are Loday Brackets. Once we describe the Vinogradov Bracket and the Courant Bracket, we will see our main examples of derived brackets. Notice that since we don't force the Jacobi Identity to hold, replacing it with (28), we will find structures that are not necessarily Lie Algebras. For reference, the main algebraic statements about the derived bracket relative to the initial Loday Bracket can be summarized in the following theorem [38], Theorem 1.1:

Theorem 10.1. *The following hold:*
i) The derived bracket of a Lie bracket[28] *of degree* n *is a Loday bracket of degree* $n+1$
ii) The map D *is a morphism of Loday algebras from* $(V, [\cdot,\cdot]_{\mathcal{D}})$ *to* $(V, [\cdot,\cdot])$
iii) The map D *is a derivation of the Loday bracket* $[\cdot,\cdot]_{\mathcal{D}}$
iv) The bracket $[\cdot,\cdot]_{\mathcal{D}}$ *induces a Lie Bracket of degree* $n+1$ *on* $V/D(V)$

The first three statements are a matter of definition chasing, while the last statement is a bit more subtle to prove. We can generalize the case of (58) by instead considering the nested commutator definition. Recall that for a graded differential Lie Algebra $(V, [,], D)$, the derivative operator D is said to be act via *interior derivation* if $\forall a \in V, Da = [d, a]$ where d is a square-zero element of V, i.e. $[d, d] = 0$. Using this definition we have,

Theorem 10.2. *Let* $(V, [,], D)$ *be a graded differential Lie Algebra with bracket of degree* n*. If* D *acts via interior derivation by an element* $d \in V$ *such that* $|d| + n$ *is odd and* $[d, d] = 0$*, then for all* $a, b \in V$ *the derived bracket is:*

$$[a, b]_{\mathcal{D}} = [[a, d], b] \tag{59}$$

Proof. This is a straight-forward computation:

$$\begin{aligned}
[a, b]_{\mathcal{D}} &= (-1)^{n+|a|+1}[Da, b] = (-1)^{n+|a|+1}\,[[d, a], b] \\
&= -(-1)^{n+|a|+1}(-1)^{|d||a|}\,[[a, d], b] = (-1)^{n+|a|+2+|a||d|}\,[[a, d], b] \\
&= (-1)^{n+|d|+(|a|-1)|d|+(|a|+|2|)}\,[[a, d], b] = -(-1)^{(|a|-1)|d|+(|a|+|2|)}\,[[a, d], b] \\
&= [[a, d], b]
\end{aligned}$$

where the second to last equality comes from the fact that $n + |d|$ is odd and the last equality is due to the fact that $|a| - 1$ will be odd (resp. even) iff $|a| + 2$ is even (resp. odd). □

This paper will only focus on derivative operators that act via interior derivation, since many such operators in Complex Geometry behave this way. Note that the generalization in theorem 3.2 will be useful, for it allows us to cast the Vinogradov bracket and eventually the Courant bracket in terms of derived brackets. Before getting to these brackets, there is one more definition to state.

If one traces through the definitions, it is clear that the graded, differential Lie Algebra $(V, [,], D)$ of degree n need not have a skew-symmetric derived bracket, for if $X, Y \in V, |X| \cdot |Y|$ is odd, then $[X, Y] = [Y, X]$. In order to rectify this we define the *skew-symmetrization* of the derived bracket as,

Definition 10.5. *Let* $(V, [,], D)$ *be a graded, differential Lie Algebra and* $[\cdot,\cdot]_{\mathcal{D}}$ *its derived bracket. The* ***skew-symmetrization*** $[\cdot,\cdot]^{-}_{\mathcal{D}}$ *of* $[\cdot,\cdot]_{\mathcal{D}}$ *is,*

$$[\cdot,\cdot]^{-}_{\mathcal{D}} = \frac{1}{2}\Big([a, Db] - (-1)^{n+|a|}[Da, b]\Big)$$

10.2 The Vinogradov and Courant Brackets

One of the motivations for introducting the Vinogradov bracket is that rapidly leads to the Courant bracket. Most of the treatments of Generalized Complex Geometry tend to ignore the motivation for the Courant Bracket, as Hitchin [32] started with the form of the Courant Bracket in [13] and expanded it to arbitrarily p-forms. The Vinogradov Bracket $[\cdot,\cdot]_{\mathcal{V}} : \mathrm{End}(\Omega^{\bullet}(M)) \otimes \mathrm{End}(\Omega^{\bullet}(M)) \to \mathrm{End}(\Omega^{\bullet}(M))$ is defined, for $a, b \in \mathrm{End}(\Omega^{\bullet}(M))$ to be

$$[a, b]_{\mathcal{V}} := \frac{1}{2}\Big([[a, d], b] - (-1)^{|b|}\,[a, [b, d]]\Big) \tag{60}$$

Note that this bracket does not satisfy the Jacobi identity and that it is almost the skew-symmetrization of $[\cdot,\cdot]_{\mathcal{G}}$ except for the factor of $(-1)^{|b|}$. This factor plays an important role into the subspaces of $\mathrm{End}(\Omega^{\bullet}(M))$ for which the Jacobi identity holds and it turns out that specification of such subspaces is related to Maximally Isotropic

[28]A Lie Bracket of Degree n is an antisymmetric bracket that satisfies . Note that by taking the derived bracket of this Lie Bracket, we are increasing changing the premultiplying factor as in (60)

Bundles. It turns out that the Vinogradov Bracket is actually the skew-symmetrization of the derived *graded* bracket, $[a,b]_{\mathcal{D}} = \left[[a,d,,]_{\mathcal{G}}\, b\right]_{\mathcal{G}}$ where d is the de Rham Differential. Since $V^{\bullet}(M) \otimes \Omega^{\bullet}(M) \subset \mathrm{End}(\Omega^{\bullet}(M))$, we can restrict to be the Vinogradov Bracket to a subbundle of interest, namely $TM \oplus T^{*}M$. In particular the *Courant Bracket* is defined to be the restriction of the Vinogradov Bracket to $TM \oplus T^{*}M$. This is summarized in the following theorem, [38], Theorem 3.3:

Theorem 10.3. *We have the following results:*
i) The Vinogradov Bracket is a Loday Bracket of odd degree ii) The skew-symmetrization of the derived bracket is the Vinogradov bracket iii) The skew-symmetrized derived bracket of a vector field X and a 1-form ξ is

$$[X,\xi]^{-}_{\mathcal{D}} = \mathcal{L}_X \xi - \frac{1}{2} d\iota_X \xi \tag{61}$$

iv) The restriction of the skew-symmetrized derived bracket to $TM \oplus T^{*}M$ is given by,

$$[X+\xi, Y+\eta]^{-}_{\mathcal{D}} = [X,Y]_{\mathcal{L}} + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(\iota_X \eta - \iota_Y \xi) \tag{62}$$

This is the ***Courant Bracket***

Now that we have something similar to the Lie Bracket defined on the space $TM \oplus T^{*}M$, the Courant Bracket, we have two main questions to answer:

- Will we get a Lie Algebra from the Courant Bracket?
- Will we get an analogue of the Frobenius or Newlander-Nirenberg Theorems? Can integrability and/or involutivity of a Tangent distribution be determined in terms of this bracket

It turns out that the answer to the first question is no, but if we expand the definition of a Lie Algebra to that of a *Lie Algebroid* we can answer the second question positively.

10.3 Lie Algebroids

Let's try to motivate the Lie Algebroid construction a bit before giving the algebraic definition. The most "geometrically-natural" Lie Algebra is the infinite-dimensional Lie Algebra of vector fields on an n-manifold with the standard Lie Bracket $[\cdot,\cdot]_{\mathcal{L}}$. Let $f, g \in C^{\infty}(M)$ and recall that since vector fields $v, w \in \Gamma(TM)$ are linear derivations of $C^{\infty}(M)$, we have,

$$[v,w]_{\mathcal{L}}(fg) = [v,w]_{\mathcal{L}}(f)g + f\,[v,w]_{\mathcal{L}}\, g \tag{63}$$

so that the Lie Bracket is also a derivation of $C^{\infty}(M)$. On the other hand, standard Differentiable Manifolds texts will show that following the Leibniz rule holds,

$$f \in C^{\infty}(M), [v, fw]_{\mathcal{L}} = f\,[v,w]_{\mathcal{L}} + v(f)w \tag{64}$$

However, what happens if we have a vector bundle $p : E \to M$ such that $\Gamma(E)$ admits a Lie Algebra but with a bracket that *is not* a derivation? We can reduce this situation to the case of the tangent bundle and preserve (66) by associating the bracket on E to a map $a : E \to TM$ that takes the role of $v(f)$ in (66). This is precisely what the *anchor* of a Lie Algebroid does; more formally we have,

Definition 10.6. *Let $p : E \to M$ be a vector bundle over a manifold M such that there exists a Lie Algebra structure $(E, [\cdot,\cdot])$ on $\Gamma(E)$. A* ***Lie Algebroid*** *structure on E is the specification of a map $\rho : E \to TM$ called the anchor such that for all $u, v \in \Gamma(E), f \in C^{\infty}(M)$ we have,*

$$[u, fv] = f[u,v] + (\rho(u)f)v \tag{65}$$

Recall that one extends the Lie Bracket of TM to a Lie Bracket on the exterior algebra $\wedge^{n} TM$ by using the *Schouten Bracket*, $[\cdot,\cdot]_{\mathcal{S}}$. One can do the same for a Lie Algebroid:

Proposition 10.1. *Let* $L \to M$ *be a Lie Algebroid with bracket* $[,]$*. The Schouten bracket acting on* $X_1 \wedge \cdots \wedge X_p \in C^\infty(\wedge^p L)$, $Y_1 \wedge \cdots \wedge Y_q \in C^\infty(\wedge^q L)$ *is defined as,*

$$[X_1 \wedge \cdots \wedge X_p, Y_1 \wedge \cdots \wedge Y_q]_{\mathcal{S}} = \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \hat{X}_i \cdots \wedge X_p \wedge Y_1 \wedge \cdots \hat{Y}_j \cdots \wedge Y_q \tag{66}$$

$$[X, f]_{\mathcal{S}} = -[f, X]_{\mathcal{S}} \quad \forall X \in C^\infty(L), f \in C^\infty(M) \tag{67}$$

This bracket turns $C^\infty(M)$ *into a graded Lie Algebra with*

$$[A, B]_{\mathcal{S}} = -(-1)^{|a|-1}(-1)^{|b|-1} [B, A]_{\mathcal{S}}$$
$$[A, [B, C]_{\mathcal{S}}]_{\mathcal{S}} = [[A, B]_{\mathcal{S}}, C]_{\mathcal{S}} + (-1)^{|a|-1}(-1)^{|b|-1} [B, [A, C]_{\mathcal{S}}]_{\mathcal{S}}$$

Given this and the results of the last section, if we define a differential operator and an inner product, one may guess that it will be possible to speak of derived brackets. The main goal of this section from this point on is to define these structures in order to get a better grasp of the underlying geometry of a manifold that admits a Lie Algebroid structure. As expected, we define the derivative and inner product operators as:

Definition 10.7. *The* ***Lie Algebroid derivative*** $d_L : C^\infty(\wedge^k L^*) \to C^\infty(\wedge^{k+1} L^*)$ *is a degree* 1 *linear operator defined by*

$$d_L \sigma(X_0, \ldots, X_k) = \sum_i (-1)^i a(X_i) \sigma(X_0, \ldots, \hat{X}_i, \ldots, X_k)$$
$$+ \sum_{i<j} (-1)^{i+j} \sigma\left([X_i, X_j]_{\mathcal{S}}, X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k\right) \tag{68}$$

Definition 10.8. *Let* $X \in C^\infty(L)$*. We define the* ***interior product*** ι_X *to be the degree* -1 *derivation on* $C^\infty(\wedge^\bullet L^*)$ *defined by*

$$\iota_X \sigma = \sigma(X, \cdots) \tag{69}$$

Using Cartan's formula we define the Lie Derivative by,

$$\mathcal{L}_X^L = d_L \iota_X + \iota_X d_L \tag{70}$$

Now that we have used the Lie Algebroid bracket and anchor to extend the notion of the exterior derivative to other bundles, we can answer the second question of the last section with an affirmative answer by providing a theorem analogous to the Frobenius theorem. This is quite an involved theorem that requires much of the derived bracket formalism presented above. Sussmann [50] actually proves a very broad theorem that extends the Frobenius Theorem using a notion of tangent distributions of *finite type*.

Definition 10.9. *A tangent distribution* $D \subset TM$ *is said to be* ***locally of finite type*** *if* $\forall m \in M$ *there exist vector fields* $\{X^1, \ldots, X^k\} \in D_m$ *such that:*
1) $X^1(m), \ldots, X^k(m)$ *span*

Finally before continuing onto the Courant Algebroids, let's look at some examples.

Examples

Let's start with the tautological examples:

- The Tangent Bundle — The anchor is simply the identity map, $a \in \mathsf{End}(TM), a = \mathbf{1}_{TM}$. Any integrable sub-bundle $S \subset M$ also qualifies, with $a = \iota : S \to M$
- Complex Structures — Since $TM_+ < TM \otimes \mathbb{C}$ is bundle isomorphic to that uncomplexified tangent bundle on a Complex Manifold, it has a natural Lie Algebra structure and the inclusion map is anchor.
- CR Structures — Recall from § that a CR structure on a real, $2n-1$ dimensional manifold is a complex $n-1$ dimensional sub-bundle $L < TM \otimes \mathbb{C}$ such that $L \cap \overline{L} = \{0\}$ and is closed under the Lie Bracket. The inclusion map again serves as an anchor

As a more complicated example, we consider the *Atiyah Sequence*. This example will give a Lie Algebroid structure on a rather large class of fiber bundles and is quite useful, because it will allow for the introduction of a few more elementary structures in Differential Geometry. The main point to grasp here is that the set of G-invariant vector fields on a Principal G-bundle is a Lie Algebroid, analogous to the fact that the vector fields on a manifold form a Lie Algebra. This will serve as an example where the anchor will allow the definition of connections on Lie Algebroids, which allows for geometry to be deciphered.

Let $\pi : P \to M$ be a fiber bundle over a manifold M and G a Lie Group with Lie Algebra $\mathfrak{g}$. An important type of bundle $P \to M$ is one that admits a free group action of G:

Definition 10.10. *A* G-***bundle*** *is a fiber bundle* $\pi : P \to M$ *that admits a left action by* G *such that* M *is homeomorphic to the orbit space* P/G *and* $\pi : P \to M$ *is bundle isomorphic to the bundle* $\pi_{P/G} : P \to P/G$*. If* G *acts freely on* P *then* $\pi : P \to M$ *is known as a* ***principal*** G-***bundle***.

Note that from this definition, we see that a fiber of P is isomorphic to G. Conversely, if we have a manifold P' such that P' admits a free left action $\rho : G \times P' \to P'$ such that $\pi : P' \to P'/G$ is a submersion, then the Ehresmann Fibration Theorem guarantees that $P' \to P'/G$ is a fiber bundle. Any such manifold P' is also known as a principal G-bundle.

Given a principal G-bundle, one can define an **Ehresmann Connection** that generalizes the affine connection and covariant derivative. The Ehresmann Connection separates each fiber into a **horizontal space** $H_pP \subset T_pP, p \in P$, which is related to the Ehresmann connection analogously to the relationship between the tangent space of a manifold and the Levi-Civita connection. More precisely, the horizontal space is defined as the image of the map π_* on tangent spaces induced by the projection $\pi : P \to M$, i.e. $\pi_* : T_pP \to T_{\pi(p)}M$. As such, the Horizontal Subspace gives a **splitting** of T_pP into two subspaces. In particular, one receives the split exact sequence,

$$0 \longrightarrow V_p \hookrightarrow T_pP \xrightarrow{\pi_*} T_{\pi(p)}M \longrightarrow 0$$

where V_p is known as the **vertical space** at p. Finally recall that the **adjoint bundle** $\mathrm{Ad}(P)$ is an extension of P as a fiber bundle, $\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} \mathfrak{g} \cong (P \times \mathfrak{g})/G$ such that we have a fiber-wise bracket $[p \cdot g, x] = [p, g^{-1}xg][p, \mathrm{Ad}_{g^{-1}}(x)], \forall p \in P, g \in G, x \in \mathfrak{g}$. Under this construction, each fiber is naturally a Lie algebra, isomorphic to $\mathfrak{g}$.

Now we can define the Atiyah Sequence:

Definition 10.11. *Suppose that* $\pi : P \to M$ *is a principal* G*-bundle. The* ***Atiyah Sequence*** *is the exact sequence of fiber bundles,*

$$0 \longrightarrow Ad(P) \to TP/G \xrightarrow{\pi_*} TM \longrightarrow 0 \tag{71}$$

where the first map is sends $(P \times \mathfrak{g})/G$ *to* $P \times e/G \hookrightarrow TP/G$*.*

Note that TP/G is the set of G-invariant vector fields on P. The bundle is TP/G is a Lie Algebroid with anchor π_*.

By analogy with the construction connections on TM, any splitting of TP by the anchor π_* will allow one to define a $\mathfrak{g}$-valued connection on the image of π_*, i.e. the Horizontal Space. As such, we see that the *anchor allows for the construction of connections on Lie Algebroids*.

10.4 The Courant Algebroid

Given that the construction of the Courant Bracket as a derived bracket is complete from a categorical point-of-view, we will now present results that will aid in computation. The majority of the proofs are technical and tedious in that they require many nested commutator computations. As such, for clarity we will only present the results; more information can be found in [25, 13, 11].

Recall that the Courant Bracket was defined in (64) and in this section the bracket will be denoted $[A, B]_{\mathcal{C}}$. It turns out that this bracket *fails to satisfy the Jacobi Identity*. However, there is a very close relationship between the the **Jacobiator** of the Courant Bracket and the **Nijenhuis Operator** associated to this bracket. As one might expect, we define the Jacobiator and Nijenhuis Operator as,

$$\mathrm{Jac}(A, B, C) = [[A, B]_{\mathcal{C}}, C]_{\mathcal{C}} + [[B, C]_{\mathcal{C}}, A]_{\mathcal{C}} [[C, A]_{\mathcal{C}}, B]_{\mathcal{C}} \tag{72}$$

$$\mathrm{Nij}(A, B, C) = \frac{1}{3} \left(\langle [A, B]_{\mathcal{C}}, C \rangle + \langle [B, C]_{\mathcal{C}}, A \rangle + \langle [C, A]_{\mathcal{C}}, B \rangle \right) \tag{73}$$

The relationship between these two is stated in the following proposition, [25], Proposition 3.16:

Proposition 10.2. $\mathsf{Jac}(A, B, C) = d(\mathsf{Nij}(A, B, C))$

Given that the Newlander-Nirenberg theorem states that an almost complex structure J is integrable iff $\mathsf{Nij}(J, A, B) = 0, \forall A, B \in TM^{\mathbb{C}}$, one might expect that the Courant Bracket will be useful for defining integrability. We will explore this more in the next section. We can now define more generally *Courant Algebroid*, which satisfies the important properties associated to the Courant Bracket ([13], Definition 2.1) :

Definition 10.12. *A* ***Courant Algebroid*** *is a vector bundle* $p : E \to M$ *equipped with a nondegenerate symmetric bilinear form* $\langle \cdot, \cdot \rangle$ *as well as a skew-symmetric bracket* $[\cdot, \cdot]$ *on* $C^\infty(E)$ *and with a smooth bundle map* $\pi E \to TM$ *(the anchor). This induces a natural differential operator* $\mathcal{D} : C^\infty(M) \to C^\infty(E)$ *via the definition* $\langle \mathcal{D}f, A \rangle = \frac{1}{2}\pi(A)f, \forall f \in C^\infty(M), A \in C^\infty(E)$. *The following compatibility conditions are required*[29]*:*
1) $\pi([A, B]) = [\pi(A), \pi(B)], \forall A, B \in C^\infty(E)$
2) $\mathsf{Jac}(A, B, C) = \mathcal{D}(\mathsf{Nij}(A, B, C)), \forall A, B, C \in C^\infty(E)$
3) $[A, fB] = f[A, B] + (\pi(A)f)B - \langle A, B \rangle \mathcal{D}f, \forall A, B \in C^\infty(E), f \in C^\infty(M)$
4) $\pi \circ \mathcal{D} = 0$
5) $\pi(A)\langle B, C \rangle = \langle [A, B] + \mathcal{D}\langle A, B \rangle, C \rangle + \langle B, [A, C] + \mathcal{D}\langle A, C \rangle \rangle, \forall A, B, C \in C^\infty(E)$

Note that from the preceding sections, $TM \oplus T^*M$ is naturally a Courant Algebroid, provided that an anchor is chosen and the Courant Bracket is used.

One of the properties of the standard Lie Bracket is that it commutes with diffeomorphisms, i.e. $f_*([A, B]) = [f_*A, f_*B]$. One might ask whether this diffeomorphism invariance holds in the case of the Courant Bracket for the *generalized diffeomorphisms* (i.e. standard diffeomorphisms as well as B, β-transforms). It turns out that this is true:

Theorem 10.4. *That map* e^B *is an automorphism of the Courant bracket iff the* 2*-form* B *is closed. Subsequently, the set of automorphisms of the Courant bracket* Ω^{diff} *is,*

$$\Omega^{\mathrm{diff}} \cong \mathit{Diff}(M) \rtimes \Omega^2_{\mathit{closed}}(M) \tag{74}$$

where $\Omega^2_{\mathit{closed}}(M)$ *is the set of closed* 2*-forms on* M

Now that we have a bracket, a cogent question to ask is "What does involutivity with regard to the bracket mean, geometrically?" In the case of real manifolds, involutivity with regard to the standard Lie Bracket is equivalent to integrability via the Frobenius theorem. It is natural to thus ask, "Is it possible to relate Courant involutive subspaces of $TM \oplus T^*M$ to some sort of integrable structure?" This is answer in terms of *Dirac structures* on a manifold:

Definition 10.13. *A real, maximal isotropic sub-bundle* $L < TM \oplus T^*M$ *is called an* ***Almost Dirac Structure****. If* L *is involutive, then the almost Dirac structure is said to be integrable and termed a* ***Dirac Structure****. If the same conditions are met for the complexified generalized tangent bundle,* $(TM \oplus T^*M) \otimes \mathbb{C}$ *then the bundle* L *is called a* ***complex Dirac structure***

The main reason for the real, maximal and isotropic hypotheses is because of the following proposition ([25], Propositions 3.26-3.27):

Proposition 10.3. *If* $L < TM \oplus T^*M$ *is involutive, then* L *must either be an isotropic subbundle or a bundle of type* $\Delta \oplus T^*$ *for* Δ *a nontrivial involutive sub-bundle of* T*. Subsequently, if* L *is a maximal isotropic sub-bundle of* $TM \oplus T^*M$ *then the following are equivalent:*

- L *is involutive*
- $\mathsf{Nij}|_L = 0$
- $\mathsf{Jac}|_L = 0$

Let's show that a Dirac Structure subsumes Symplectic and Complex structures as we desire.

[29]For the Courant Bracket as previously defined, these properties are naturally satisfied

Example: Symplectic Geometry

The tangent bundle TM is maximal isotropic and involutive so we have a Dirac structure. Now suppose that we have a closed 2-form $\omega \in \Omega^2_{\text{Closed}}(M)$. Then the subspace $\{X + \iota_X \omega : X \in TM$ is involutive iff $d\omega = 0$ due to the proposition about automorphisms of the Courant bracket.

Example: Complex Geometry

Suppose that $J \in \mathsf{End}(TM)$ is almost complex structure and let $T_{0,1} < TM \otimes \mathbb{C}$ be the $-i$-eigenbundle and $T_{1,0} < TM \otimes \mathbb{C}$ be the i-eigenbundle. Then $L = T_{0,1} \oplus \text{Ann}(T_{0,1})$ is a maximal isotropic from Proposition 11.1. Now if L is Courant integrable, by projecting to the component in TM, we get the standard Lie Bracket which means that $T_{0,1}$ is involutive and J is integrable. If J is integrable, then if $X + \xi, Y + \eta \in C^\infty(L)$ then we have

$$[X + \xi, Y + \eta]_{\mathcal{C}} = [X, Y]_{\mathcal{C}} + \iota_X \overline{\partial} \eta - \iota_Y \overline{\partial} \xi \in L$$

so that L is Courant involutive. Hence:

$$L \text{ is Courant involutive} \iff J \text{ is integrable}$$

Finally, we will briefly discuss the **twisted Courant bracket**. Consider the bracket

$$[X + \xi, Y + \eta]_H = [X + \xi, Y + \eta]_{\mathcal{C}} + \iota_Y \iota_X H$$

for a closed 3-form H. This bracket will be important in classifying line bundles and hence curvature forms associated to Generalized Complex Geometries; the complete description will be given in the section on Gerbes. We end this section with the major result about $[\cdot, \cdot]_H$:

Proposition 10.4. *Let* $A, B, C \in TM \oplus T^*M$ *with* $A = X + \xi, B = Y + \eta, C = Z + \zeta$. *We then have the following properties for the twisted Courant Bracket:*

- $\mathsf{Nij}_H(A, B, C) = \mathsf{Nij}(A, B, C) + H(X, Y, Z), \mathsf{Jac}_H(A, B, C) = d(\mathsf{Nij}_H(A, B, C)) + \iota_Z \iota_Y \iota_X dH$. *Hence we only have a Courant Algebroid if* $dH = 0$.
- *If* b *is a 2-form then,*
$$[e^b X, e^b Y]_H = e^b [X, Y]_{H + db}$$

11 Explicit Generalized Sasakian Metric

This section depicts a novel result from both a mathematical and physical perspective. Using a solution from [31], we nearly find a metric for an explicit example of a Generalized Sasakian Metric. The majority of this section will be written in coordinates and theorems from [31, 21] will be cited as needed. Using the coordinate definitions of a Generalized Complex Structure as in [37] one can show that a SU(2)-structure is specified by three forms, a 1-form Θ, a 2-form J_2 and a 2-form Ω_2. From these components, one can construct a pair of **bispinors**[30] Ψ_+, Ψ_- that serve a similar purpose as the pure spinors in determining a pair of almost Generalized Kähler structures $\mathcal{J}$. These forms are **polyforms** in the ring $\wedge^\bullet TM \oplus T^*M$. The authors of [31] found an example of a **dynamic** SU(2)-structure, which means that the rank of the lowest order forms of $\Omega_\pm$ change with some parameter. These solutions are constructed as fiber bundles over $\mathbb{CP}^2$ and just as in the Sasaki-Einstein case [22], any solution that admits both contact and CR structures will be subject to the classification theorems of Tian and Yau (for Kähler-Einstein 4-manifolds) [51].

We will start with the ansatz for $\Theta, J_2, \Omega_2, \Psi_\pm$ described in [31] and arrive at rather different warp factors. Before describing the metric ansatz, recall that in Generalized Sasakian Geometry one need to define cone coordinates that satisfy the proposition from the previous section. The dynamic SU(2)-structure solution provides

[30] These are very closely related to the pure spinor line bundles that determine the generalized almost complex structure and in fact only differ from a spinor line representative by a B-transformation. Bispinors are constructed by first applying the Fierz identity to the coordinate representation of a pure spinor and subsequently applying the Clifford map, which is an explicit linear isomorphism $C\ell(V) \cong \wedge^\bullet V$

cone coordinates for a cone over the Kähler-Einstein base $\mathbb{CP}^2$ that are almost exactly the same as the coordinates used in [21]. In particular, one complex coordinate $z = re^{i\psi}$ in [31] whereas in [21] the same coordinate is denoted $z = re^{3h}e^{3i\psi}$. Since [31] applies a $\mathbb{Z}_3$-orbifold action to simplify the form of the metric, this slight different makes sense. In order to avoid any potential difficulties regarding compatibility with the orbifold action, we will assume that $h \equiv 0$ and that the domain of ψ specificed in [31] holds so that we can simply rescale the ψ from [21] via a diffeomorphism. If we let $u_1, u_2, \overline{u}_1, \overline{u}_2$ denote our holomorphic and antiholomorphic, respectively, homogenous coordinates in some chart $U_\alpha \subset \mathbb{CP}^4$, the metric ansatz is of the form:

$$g_{\alpha\beta}(x)dx^\alpha dx^\beta = e^{-4B(r)}\left(dr^2 + r^2 d\psi^2\right) + e^{2C(r)}\sum_a du^a d\overline{u}^a \tag{75}$$

The main ansatz made for the bispinors is provided in [37] and is written in a convenient form to apply supersymmetry conditions (which are usually presented in the form of a Killing Spinor equation). We have:

$$\Phi_+ = e^{\frac{1}{2}\Theta\wedge\overline{\Theta}}\left[\cos\varphi\left(1 - \frac{1}{2}J_2^2\right) + \sin\varphi \mathrm{Im}\Omega_2 - iJ_2\right] \tag{76}$$

$$\Phi_- = \Theta \wedge \left[\sin\varphi\left(1 - \frac{1}{2}J_2^2\right) - \cos\varphi \mathrm{Im}\Omega_2 + i\mathrm{re}\Omega_2\right] \tag{77}$$

Before stating what Θ, J_2, Ω_2 are, let's briefly describe how they were found. The physical goal behind Generalized Complex Geometries is to find geometries that serve as extrema of the low-energy effective field theory action for type-IIB string theory, which is of the form (written in the Einstein Frame):

$$S_{LE} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g^{(10)}}\left[\mathcal{R} - \frac{1}{2}\left((\nabla\phi)^2 + e^{-\phi}|H_3|^2 + e^{2\phi}|F_1|^2 + e^{\phi}|\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right)\right] - \frac{1}{4\kappa_{10}^2}\int C_4\wedge H_3\wedge F_3$$

where:

- $\sqrt{-g^{(10)}}$ is the determinant of a local representation of the 10-dimensional metric on a potential spacetime manifold M
- $\mathcal{R}$ is the Ricci Scalar of the manifold M
- $\phi : M \to \mathcal{C}$ is a scalar field
- F_1 is the 1-form field strength, derived from a scalar potential. Physically, is analogous to the electric field.
- F_3 is the 3-form Neveu-Schwarz field strength. This is an exact 3-form analogous to the Faraday 2-form field strength of electromagnetism
- F_5 is the exact 5-form Ramond-Ramond field strength. Note that F_5 is self-dual, $F_5 = \star_{10}F_5$, where $\star_{10}$ is the 10-dimensional Hodge star operator.
- H_3 is the 3-form associated to a choice of gerbe patching in $H^3(M;\mathbb{Z})$ B_2, $H_3 = dB_2$
- C_4 is the potential of F_5, $dC_4 = F_5$

Note that minimizing this functional is similar in spirit to minimizing the Einstein-Hilbert Functional $\int\sqrt{g}\mathcal{R}$, in that an extrema represents manifold (up to diffeomorphism). This action is invariant under an $SL(2,\mathbb{R})$ action and by enforcing invariance under the superconformal algebra, one can determine a series of differential constraints on the terms in the action. These constraints are in [31], equations $(4.34) - (4.40)$. By making an ansatz that allows for the field strengths to be expressed in terms of J_2, Ω_2, Θ, one can convert these into differential constraints on an underlying Generalized Complex Geometry.

The authors of [31] found that the following forms of J_2, Θ, Ω_2 satisfy these differential constraints:

$$J_2 = \frac{i}{2}e^{2C(r)}\left(du^1\wedge d\overline{u}^1 + du^2\wedge d\overline{u}^2\right) \tag{78}$$

$$\Theta = 1 - ire^{-4B(r)}dt\wedge d\psi \tag{79}$$

$$\Omega_2 = e^{2C(r)}\frac{z}{r}du^1\wedge du^2 \tag{80}$$

Note that in order to simplify the Generalized Sasakian Solution, some of the phase factors in the above solutions of [31] were truncated. Give these forms, we can compute the bispinors (defined up to B-transform):

$$\Phi_+ = (\mathbf{1} - ire^{-4B(r)}dr \wedge d\psi)\left[\cos\varphi\left(\mathbf{1} + \frac{1}{4}e^{4C}(du^1 \wedge d\overline{u}^1 + du^2 \wedge d\overline{u}^2)\right) - \sin\varphi\sin\psi e^{2C(r)}du^1 \wedge du^2\right] \tag{81}$$

$$\begin{aligned}\Phi_- = e^{-2B(r)}(dr + ird\psi) \wedge &\left[\sin\varphi\left(\mathbf{1} + \frac{1}{4}e^{4C(r)}(du^1 \wedge d\overline{u}^1 + du^2 \wedge d\overline{u}^2)\right) - \cos\varphi\sin\psi e^{2C(r)}du^1 \wedge du^2\right.\\ &\left. + i\cos\psi e^{2C(r)}du^1 \wedge du^2\right]\end{aligned} \tag{82}$$

Given this ansatz, let's see what the bispinors associated to a Generalized Sasakian Geometry. In [21], the authors construct a Generalized Sasakian Geometry by considering bispinors $\tilde{\Phi}_\pm$ of the form:

$$\tilde{\Phi}_+ = \left(r^3 e^{-3i\psi} d(\log r) - i\eta\right) \cdot e^{-b_2} e^{b_2 + i\omega_0} \tag{83}$$

$$\tilde{\Phi}_- = r^3\left(\mathbf{1} + ie^{2\Delta} d(\log r) \wedge \eta\right) \cdot \left(-i\frac{f_5}{32}e^{-\Delta}e^{ie^{2\Delta}\omega_T}\right) \tag{84}$$

where

- $\eta = \mathcal{J}_-(d\log r)$ is the (generalized) contact 1-form
- ω_0 is the first form of the symplectic triple
- ω_T is the transverse of the symplectic triple
- f_5 is the **charge** of the 5-form Ramond-Ramond flux. That is, $F_5 = f_5 \mathrm{vol}_{g|_{\text{base}}}$
- Δ is the **warp factor**
- $\cdot$ is the Clifford Action

Our goal is to relate $\tilde{\Phi}_\pm$ to $\Phi_\pm$ so that the forms ω_0, ω_T, b_2 associated to $\Phi_\pm$ can be found and checked again the symplectic triple theorem of the previous section. Now to find the contact form η, we need to use that fact that in a Sasakian manifold, one has,

$$\omega = \frac{1}{2}d(r^2\eta)$$

where ω is the Kähler form on the metric cone of a Sasakian manifold. In the case of a Generalized Sasakian Manifold, this is only true up to B-transform. That is, for some $\sigma = e^b\eta$, we have

$$\omega = \frac{1}{2}d(r^2\sigma) \tag{85}$$

The author decided to make the ansatz,

$$\sigma = \eta - re^{-2B(r)}\sin\psi d\psi \tag{86}$$

as it yields a solution and is possible to describe via a B-transform. In [31], a natural Kähler form $\omega = J_1 + J_2$ is constructed, where J_1 satisfies the supersymmetry conditions and is defined by

$$J_1 = \frac{i}{2}e^{-4B(r)}dz \wedge d\overline{z} \tag{87}$$

From this and (87) one can deduce a possible form of σ up to a choice of function depending on $\overline{u}^1, \overline{u}^2$. The form of σ chosen was:

$$\sigma = \frac{i}{r^2}\left(e^{2C(r)}(\overline{u}^1du^1 + \overline{u}^2du^2) - e^{-4B(r)}re^{i\psi}dr\right) \tag{88}$$

Moreover, recall that $\omega_T = \frac{1}{2}d\sigma$ so that we have:

$$\omega_T = \frac{1}{2}\left[-\frac{2e^{2C(r)}}{r^3}(\overline{u}^1dr \wedge du^1 + \overline{u}^2dr \wedge du^2) + \frac{e^{2C(r)}}{r^2}(d\overline{u}^1 \wedge du^1 + d\overline{u}^2 \wedge du^2)\right] \tag{89}$$

which is trivially closed. If we substitute (88) into the expressions for $\tilde{\Phi}_+, \tilde{\Phi}_-$ we have,

$$\tilde{\Phi}_+ = \left(\left(r^2 e^{-3i\psi} - \frac{e^{-4B(r)}e^{i\psi}}{r}\right)dr + \frac{1}{r^2}\left(e^{2C(r)}(\overline{u}^1 du^1 + \overline{u}^2 du^2) - ire^{-2B(r)}\sin\psi d\psi\right)\right)\cdot e^{-b_2}e^{b_2 + i\omega_0} \tag{90}$$

$$\tilde{\Phi}_- = r^3\left(\mathbf{1} - \frac{e^{2\Delta}}{r}dr \wedge \frac{1}{r^2}\left(e^{2C(r)}(\overline{u}^1 du^1 + \overline{u}^2 du^2) - e^{-4B(r)}re^{i\psi}dr - ire^{-2B(r)}\sin\psi d\psi\right)\right)\cdot\left(-i\frac{f_5}{32}e^{-\Delta}e^{ie^{2\Delta}\omega_T}\right) \tag{91}$$

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