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Mathematics Department Senior Thesis

Numerical Simulations of N Point Vortices on Bounded Domains

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Abstract

Behavior of vortices is core to the understanding of phenomenons in nature, such as Jupiter's red spot. The simplification to point vortices provides us with an easier way to model behaviors of large numbers of vortices. Although we know from statistical mechanics that there exists a time invariant measure, it is unclear whether the measure derived from the Mean Field Equation (MFE) reflects the true behavior in nature. We explore the point vortex model on the 2 dimensional disk and test for the convergence to the Mean Field distribution using Kolmogorov-Smirinov test (K-S test). Due to the pairwise interaction of the vortices, we use the Fast Multipole Method (FMM) to decrease the calculations required. We hope that the observed behavior on the disk imply a general result on all simply connected domains via conformal mappings.

1 Motivation

We wish to study the limiting behavior of N-vortices in a bounded, simply connected domain by considerng the behavior of N point vortices. The behavior of regular vortices converges to that of point vortices weakly [4]. Intuitively, this is reasonable, as this is similar to considering a small rigid body as a single point at the center of mass. It is unclear what the limiting behavior of N point vortices is as N approaches infinity. In this paper, we will numerically explore the long run behavior of N point vortices in the unit disk with Dirichlet boundary conditions.

2 Background

2.1 Definitions and Basic Fluid Mechanics

The motion of incompressible fluids in terms of the velocity field $u = u(x,t) \in \mathbb{R}^n$ is given by the Euler Equations:

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla \rho, \nabla \cdot u = 0$$

For incompressible fluids, the density, ρ is constant, so the continuity equation for fluid dynamics reduces to [3],

$$\nabla \cdot u = 0$$

We shall only consider the case when the point vortices are in the second dimension, that is $n = 2$. Intuitively, we define a point vortex as a central rotational potential. If we denote the strength of a point vortex i as λ_i, then the point vortex can be defined as $\lambda_i \delta_{x_i}$. We define $\omega = \nabla \times u$ as the vorticity of velocity field. In a system of N point vortices, this is simply $\omega = \sum_{i=1}^{N} \lambda_i \delta_{x_i}$. Recalling a classical result, in a simply connected domain, there exists a stream function ψ such that $u = \nabla^{\perp} \psi$, where $\nabla^{\perp} = \begin{pmatrix} \partial_y \\ -\partial_x \end{pmatrix}$. Then it follows that $-\triangle \psi = \omega$, where $\omega = \nabla \times u$. The fundamental solution of the negative Laplacian, $-\triangle$, on $\mathbb{R}^2$ is given by the Green's function [6]

$$V(x,y) = \frac{1}{2\pi} \log \|x - y\| + \tilde{\gamma}$$

Intuitively, the Green's function corresponds to the potential of point y acting on point x. In the unit disk, we solve for the boundary condition term, $\tilde{\gamma}$, via the method of images. Overall, we have that the Green's function on unit disk is given by

$$V(x,y) = -\frac{1}{2\pi} \log \frac{\| \|y\|^2 x - y\|}{\|y\| \|x - y\|} = -\frac{1}{2\pi} \log \frac{\|y\| \|x - \frac{y}{\|y\|^2}\|}{\|x - y\|}$$

Therefore, ψ is given by the convolution of the Green's function with the vorticity,

$$\psi(x) = \int V(x,y)\omega(y)dy$$

Because the vorticity ω is the sum of weighted δ functions, the integral reduces to,

$$\psi(x) = \sum_{i=1}^{n} \lambda_i V(x, y_i)$$

Thus the perpendicular gradient of $\psi(x)$ gives us the velocity of the vortices at point x. Because the unit disk is conformally equivalent to any simply connected domain via the Riemann Mapping Theorem, we can study the dynamics on the unit disk and apply it to any simply connected domain via conformal mappings.

2.2 Hamiltonian and Equations of Motion

The Hamiltonian is commonly interpreted as the total energy of the system. In our case, the Hamiltonian is given by

$$H = \sum_{i \neq j} \lambda_i \lambda_j V(x_i, x_j)$$

Intuitively, this is just the sum of the pairwise potentials between vortices, similar to gravitational energy of N point particles or electromagnetic potential energy from N point charges. Notice that the Hamiltonian is given in terms of position only. Let $x_i = (x_i^1, x_i^2)$. Deriving the Hamiltonian equations for our H,

$$\frac{\partial H}{\partial x_i^2} = \lambda_i \frac{\partial}{\partial x_i^2} \sum_{j \neq i} \lambda_j V(x_i, x_j)$$

But the right hand side is exactly the stream function ψ at x_i, so the equations become.

$$\frac{\partial H}{\partial x_i^2} = \lambda_i \frac{\partial}{\partial x_i^2} \psi(x_i)$$

Similarly,

$$-\frac{\partial H}{\partial x_i^1} = -\lambda_i \frac{\partial}{\partial x_i^1} \psi(x_i)$$

But $u = \nabla^{\perp}\psi$, and u is the velocity field. So we get

$$\lambda_i \frac{dx_i^1}{dt} = \frac{\partial H}{\partial x_i^2}$$

$$\lambda_i \frac{dx_i^2}{dt} = -\frac{\partial H}{\partial x_i^1}$$

So our phase space is just our regular x, y plane [10].

2.3 Basic Statistical Mechanics and the Mean Field Equation

In statistical mechanics, a microstate is a microscopic thermodynamical configuration that the system can be in. The configuration is determined by properties such as entropy and internal energy. Let us denote the total energy $E^{(0)} = E' + E_r$, where E' is the energy of the heat reservoir and E_r is the energy of the system at state r. If we let $\Omega(E)$ be the number of accessible states at energy E, then we define entropy as $S = -\kappa_B \ln \Omega$ and the thermodynamical $\beta = \frac{1}{\kappa_B}\frac{\partial S}{\partial E} = \frac{1}{\kappa_B T}$, where T is the temperature and κ_B is the Boltzmann constant. The probability of being in state r is given by $P_r = C'\Omega(E^{(0)} - E_r)$. If $E_r \ll E^{(0)}$, we expand entropy, S, and get that $\ln \Omega(E^{(0)} - E_r) = \ln \Omega(E^{(0)}) - \beta E_r$, or $P_r = \Omega(E^{(0)})e^{-\beta E_r}$. Normalizing the probability, we get $P_r = \frac{e^{-\beta E_r}}{\int e^{-\beta E_r}}$ Letting $P(E) = \sum_r P_r$, we get $P(E) = C\Omega(E)e^{-\beta E}$ [8].

As N approaches infinity, there is a limiting result given by the Mean Field Equation (sometimes known as the Boltzmann distribution to physicists or Gibb's measure or distribution to mathematicians). The MFE is given by

$$\rho = e^{\beta\psi}\left(\int_\Omega e^{\beta\psi}\right)^{-1}, \psi = \int_\Omega V(x,y)\rho(y)dy$$

where Ω here is the entire domain and ψ is the continuous stream function and ρ is the continuous vorticity distribution (previously denoted ω in the discrete case). For positive temperature state, the system is convex, thus the existence of an unique solution to the MFE is guaranteed [1]. We do not need to restrict ourselves to positive temperature states as we are not only concerned with the thermodynamical behavior [10].

For negative temperature states, the solution of the MFE on the unit disk in $\mathbb{R}^2$ exists if and only if $-8\pi < -\beta \leq 0$, in which case the distribution of vortices is given by

$$\rho(r) = \frac{A+1}{\pi} \frac{1}{(1+Ar^2)^2}$$

where $A = \frac{-\beta}{8\pi+\beta}$[1].

2.4 Interpretation of the Negative Temperature

If we think of the number of accessible microstates in a system at a given energy level, denoted $\Omega(E)$, as proportional to the area the system occupies in the phase space, then positive temperature means that increasing the energy increases the area the system occupies in the phase space because the probability, $P(E) = C\Omega(E)e^{-\beta E}$ is bounded and nonzero. Conversely, if the temperature is negative, then increasing the energy of the system decreases the area that the system is likely to occupy in the phase space. This may explain why vortices tend to cluster (stronger vortices would have higher energy and cluster more).

2.5 Basic Ergodic Theory

Because we are exploring convergence to the Mean Field distribution, we may discuss the convergence in terms of ergodicity. Formally, we work with the measure space $(\mathbb{D}, F, \mu)$, where $\mathbb{D}$ is the unit disk, μ is the Lebesgue measure, and F is the associated σ-field. Consider a bijective μ-preserving map $f : \mathbb{D} \to \mathbb{D}$ with $f_{t+s} = f_t f_s$ for $t, s \in \mathbb{R}$. By μ-preserving, we mean $\forall A \in F$, $\mu(f_t(A)) = \mu(A)$ for all t. Hence we can consider the orbit of f given by f^k, $k \in \mathbb{N}$. Then given any μ-measurable function $\rho(x)$, $\rho : \mathbb{D} \to \mathbb{R}$, the time average of the system is given by

$$\bar{\rho}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \rho(f^k(x))$$

The spatial average of ρ is given by

$$\tilde{\rho} = \int_{\mathbb{D}} \rho(x) d\rho$$

A system is ergodic if and only if

$$\bar{\rho}(x) = \tilde{\rho}(x)$$

for a.e. $x \in \mathbb{D}$ with respect to μ.

In our case, our f is given by equations of motion of the vortices in section 2. ρ is given by the Mean Field distribution. It is unknown whether vortex motion is ergodic or not, which is what we are trying to explore numerically [2]. It is also unclear which invariant measure is the true, physical one.

3 Assumptions and Initial Conditions

We only consider the most simple case where each vortex has equal strength, $\lambda_i = \lambda$. To keep the total energy of the system finite as we increase N, we let $\lambda \propto \frac{1}{N^2}$. From $u = \nabla^{\perp}\psi$, we deduce that the equation of motion of each point vortex is then given by $\dot{x}_i = \frac{1}{2}\lambda\nabla^{\perp}\tilde{\gamma}(x,x) + \sum_{j\neq i}\lambda\nabla^{\perp}V(x_i,x_j)$, which can be interpreted by the sum of the effect of the boundary on the vortex at x_i and the other vortices acting on the vortex at x_i with regards to the boundary.

From our assumptions, our Hamiltonian of the discrete case becomes $H = \lambda^2\sum_{i>j}V(x_i,x_j)$ and that of the continuous case becomes

$$H = \lambda^2 \iint\limits_{\Omega} V(x,y)\rho(x)\rho(y)dxdy$$

.

4 Questions

4.1 How do the dynamics of discrete N-vortices converge to the continuous Mean Field distribution or some other distribution, if at all?

As shown in Cassio Neri (2003), the solutions of the MFE are minimizers of the free energy functional, which are associated with the true, physical behavior of the vortices. However, not every solution of the MFE is physical. We hope to make empirical observations of when and how the discrete dynamics converge to solutions of the MFE for large N [5].

4.2 If the distribution given by the MFE is stationary, is it stable or physical?

If the distribution given by the solution of the MFE is stationary, that is, the spatial distribution of vortices does not change over time, what would happen if we perturb it by a small amount? It is unclear whether vortices in the perturbed system would converge to the stationary distribution of the unperturbed system or diverge. It is also unclear whether these distributions can happen in real life (as modeled by the discrete dynamics).

4.3 Which distributions are stable?

As N approaches infinity, due to the symmetrical qualities of the unit disk, given any rotationally symmetric distribution, we see that any motion in a direction that is purely tangential to the disk, therefore not affect the distribution. However, for finite N, even a small perturbation from the purely symmetric case will lead to the velocity field to not be purely tangential to the boundary of the unit disk.

4.4 How does the strength of the vortices affect the stationary distribution?

Since we are assuming equal strengths for all vortices, the strengths of the vortices can be factored out of the discrete dynamics as a constant. It is unclear what the relationship is between the total energy, the thermodynamic β, and the strength of vortices, λ. Assuming the MFE is correct, we can solve for the value of β given λ. However, the system is non-linear and difficult to solve.

5 Numerical Model

Due to the chaotic behavior of N-vortices and the lack of an analytical solution to the equation of motion, we use numerical approximations at discrete time steps to model the dynamics of the system. Specifically, we use the Runge-Kutta method of order 4, which has a local error of $O(h^5)$ and a global error of $O(h^4)$. To account for the Dirichlet boundary conditions on the unit disk, we use the method of images to introduce image points that we update in each time step. Since we each vortex does not interact with itself and only with other vortices, by the principle of superpositioning, we only need to consider pairwise interactions of the particles.

Looking at our Green's function again, we see that

$$V(x, y) = -\frac{1}{2\pi}\left(\log\|x - y\| - \log\|x - \frac{y}{\|y\|^2}\| - \log\|y\|\right)$$

In other words, the potential on x is caused by y and the image point of y at $\frac{y}{\|y\|^2}$ plus the logarithm of the norm of y.

5.1 Rejection Sampling

We wish to use rejection sampling to sample from a given distribution. We sample the radial distance r from an uniform distribution and the mean field distribution on $[0, 1]$. Then we sample an angular coordinate, θ, from an uniform distribution on $[0, 2\pi)$. Combining the two, we have the distribution we desired on the unit disk.

5.2 Fast Multipole Method

Due to the pairwise interaction, naive algorithm would be $O(N^2)$, which is unfeasible as N approaches infinity. Therefore we use a Fast Multipole Method (FMM), which reduces the problem to $O(N \log N)$ or $O(N)$, depending on the particular method used. The basic idea of the Fast Multipole Method is to lump "far" points together into a multipole node in a way similar to calculating the center of mass. The mulitpole node is then treated as a single particle in the calculation. A quick simulation shows that the results of the FMM is arbitrarily close to that of the naive computation. The FMM scheme we use is the package by Greengard [**?**].

5.3 Kolmogorov-Smirnov Test

Let us denote the empirical distribution by ρ_{exp}. Then Kolmogorov-Smirinov test statistic is defined as

$$D(\rho, \rho_{exp}) = \sup \|\rho(x) - \rho_{exp}(x)\|$$

. The test is constructed using the critical values of the Kolmogorov distribution. We reject the null hypothesis at level α if

$$\sqrt{n}D > K_\alpha$$

, where K_α is derived from the Kolmogorov distribution.

5.4 Steps in Our method

1. Use naive rejection sampling to sample the initial locations of the N point vortices based on desired probability distribution.

2. At every time step, we use Runge-Kutta 4 method and perform the following calculations

   a. Find location of image points, which is given by $\frac{y}{\|y\|^2}$.

   b. Use FMM to compute the perpendicular gradient at each vortex, which is given by sum of $\nabla^{\perp} V$ over the N vortices and their corresponding image points.

   c. Analytically calculate $\sum \nabla^{\perp} \log \|y\|$ and add it to the velocity.

   d. Record radial positions of vortices at every k time step, where k is our sample rate.

3. Take average of spatial distribution based on the recorded data about position.

4. Compare average spatial distribution to mean field distribution using K-S test

6 Simulations

6.1 Tests Used to Verify Numerical Stability

6.1.1 2 Vortices in Symmetrical Positions

We know that 2 vortices symmetric on a line passing through the origin will be purely rotational. That is, their orbit is just the circle with radius equal to their distance to the origin. We put the vortices at $v_1(x, y) = (0.5, 0)$ and $v_2(x, y) = (-0.5, 0)$. We let the number of time steps equal 100000 with stepsize 0.001.

The following plot shows our results

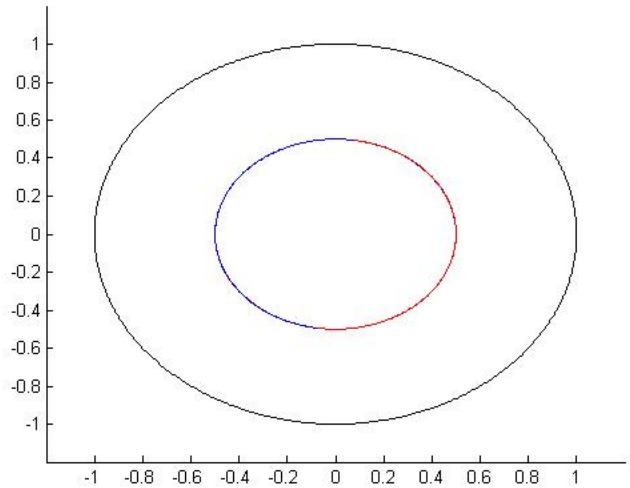

Figure 1: 2 Vortices in symmetrical position

Our results show that the movements of the vortices are indeed purely rotational, and the trajectories are the same circle like we expected. This pattern continues for any observation time frame we've used.

6.1.2 3 Vortices Converging to a Point

Let our 3 vortices be located at

$$v_1(x,y) = (l_2 \frac{(3 + sqrt(3)cos(\theta))}{6}, l2 \frac{(sqrt(3)sin(\theta))}{6})$$

,

$$v_2(x,y) = (l_2 \frac{(-3 + sqrt(3)cos(\theta))}{6}, l2 \frac{(sqrt(3)sin(\theta))}{6})$$

,

$$v_3(x,y) = (l_2 \frac{(2sqrt(3)cos(\theta))}{3}, l2 \frac{(2sqrt(3)sin(\theta))}{3})$$

with intensities $(2, 2, -1)$. 3 vortices given as such in the $\mathbb{R}^2$ plane collapses to a single vortex [2]. In our simulation, we used $l_2 = 0.3$ and $\theta = 0.2$. Our simulation was run over 100000 time steps with stepsize 0.0001.

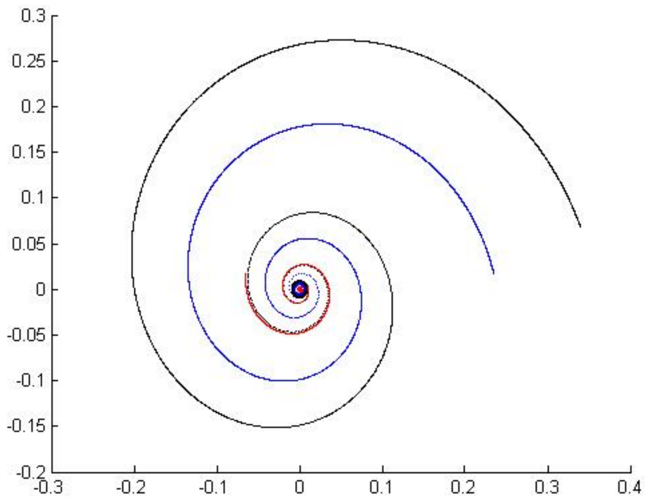

Figure 2: Convergence of 3 vortices

Our results show that the 3 vortices do indeed converge to a single point. However, the rate of convergence seems to be slow as can be seen by the ring around the center point.

6.2 MFE Initial Conditions for fixed β over long time frames for large N

For our simulation, we use $\beta = -10$ and 1000 vortices sampled from the solution of the MFE. Our time step is 0.001 and the number of time steps is 200000. The following plots show the vortex distribution at time step 1, 50000, 100000, 150000, and 200000.

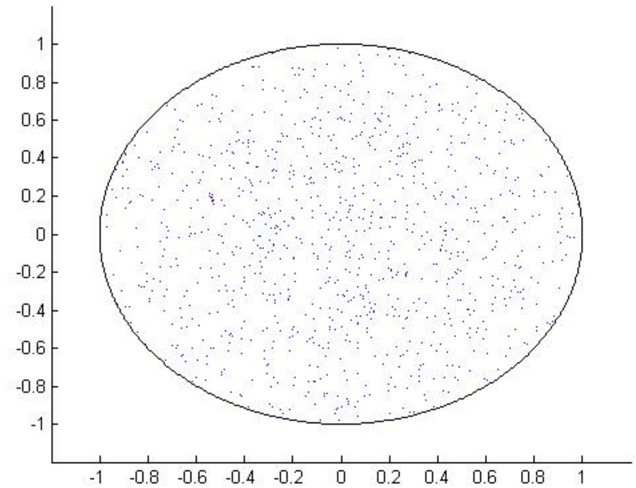

Figure 3: System when t=0.001

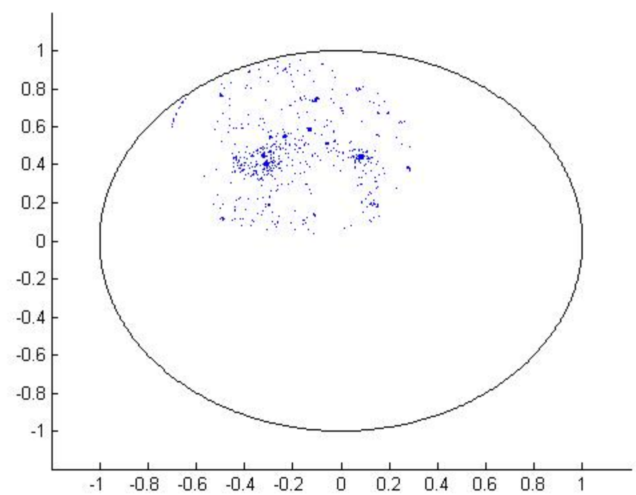

Figure 4: System when t=50

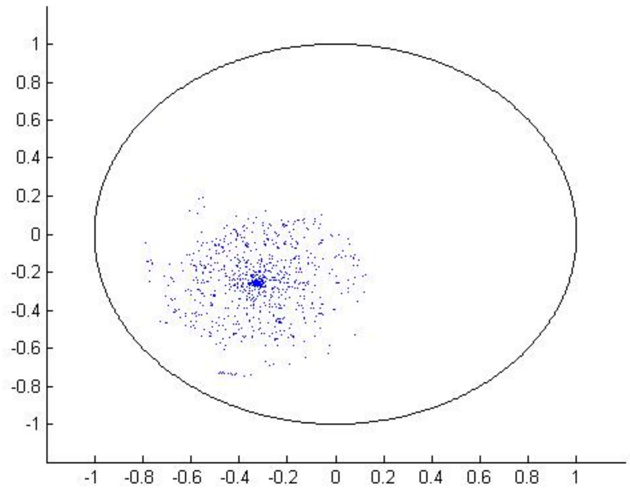

Figure 5: System when t=100

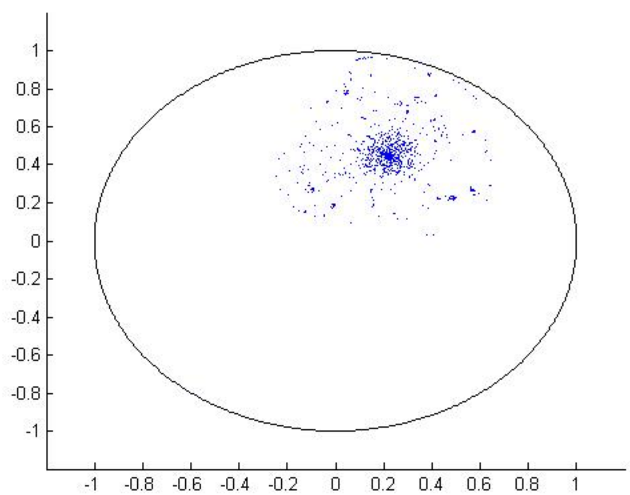

Figure 6: System when t=150

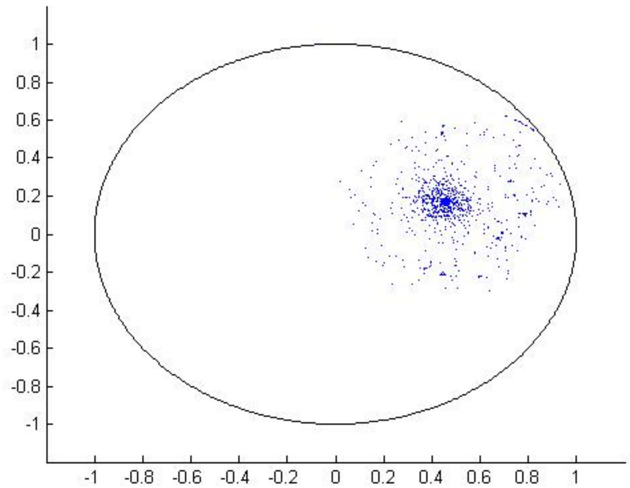

Figure 7: System when t=200

It seems that the vortices form a cluster, and the cluster rotates around the circle. The intial and final distribution compared to the sampling distribution plotted below.

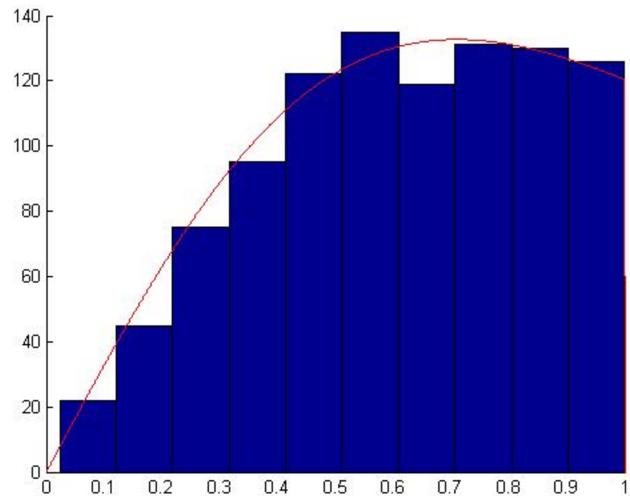

Figure 8: Initial distribution of vortices with MFE as red line.

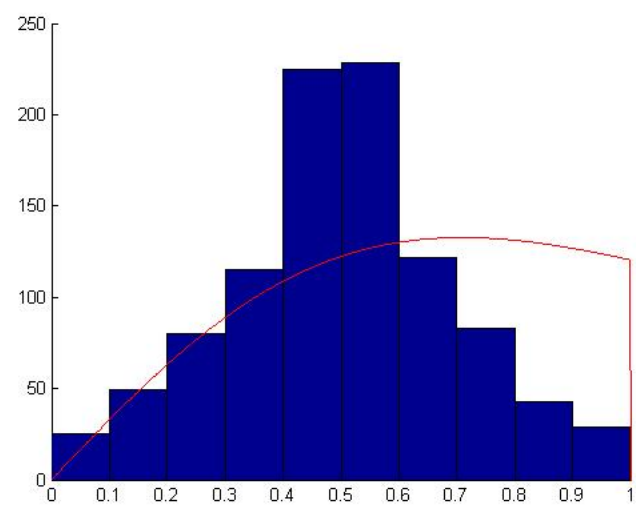

Figure 9: Final distribution of vortices with MFE as red line

The red line indicates the mean field distribution. Clearly, our results are not converging to the distribution given by the MFE. The concentration of vortices near the center is due to the center of cluster formed. The cluster formed is not radially symmetric like the distribution given by the MFE. However, further testing and analysis is needed to verify the numerical integrity of the simulations. The clustering does support our interpretation of the negative temperature states.

7 Conclusions

Although we have obtained preliminary evidence that the distribution given by the MFE does not converge to the behavior from the discrete dynamics, we need to verify the results further by running systematic simulations. We would to establish an explicit time scale for the simulation and obtain further evidence that the error is within tolerable range.

8 Future Works

- Figure out the exact relationship between β and the strength of vortices.
- Verify that the energy of the system remains constant throughout the simulation and determine whether a symplectic integrator is needed.
- Verify numerical stability over fixed time frames using systematic conditions for simulations.
- Work with different domains.
- Possibly use a symplectic integrator rather than RK4 if energy changes are too large.

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