Infinity-harmonic functions on SG

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1 Background

1.1 The infinity Laplacian Δ_∞

In this paper, we seek to explore how the **infinity Laplacian** operator Δ_∞ could be defined on the fractal SG. The infinity Laplacian is defined on $\mathbb{R}^n$ as

$$\Delta_\infty u = |\nabla u|^{-2} \sum_{i,j} u_{x_i} u_{x_i,x_j} u_{x_j},$$

which can be understood informally as "the second derivative of u in the direction of the gradient." It has been studied in connection with absolutely minimizing Lipschitz extensions, game theory, and probably other contexts as well [3]. The connections to the standard Laplacian are not immediately obvious from this definition, but we do not need to concern ourselves with it anymore, anyway. We will focus primarily on the discrete infinity Laplacian. Given a graph G with node set X, the **graph infinity Laplacian** is defined as

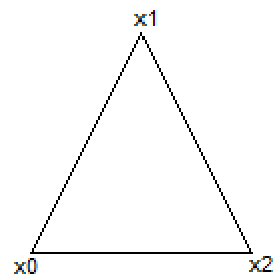
$$\Delta_\infty u(x) = \frac{1}{2}\left(\sup_{y \in N(x)} u(y) + \inf_{y \in N(x)} u(y) \right) - u(x), \tag{1}$$

where $N(x)$ is the neighbor set of x in G. Note how this compares with the definition of the discrete Laplacian,

$$\Delta u(x) = \frac{1}{|N(x)|} \sum_{y \in N(x)} u(y) - u(x).$$

1.2 The construction of SG

The fractal SG and its analytic properties have been studied extensively over the past 30 years, with Strichartz's book [4] a good introduction. One method for constructing SG, described in [4], is as a limit of graph approximations. This series of approximating graphs can be constructed entirely without reference to an ambient space, but it is cleaner and intuitively easier to understand if we consider the graphs as embedded in $\mathbb{R}^2$. Start with a graph Γ_0 with node set $V_0 = \{x_0, x_1, x_2\}$ as vertices of a nondegenerate triangle.



The *boundary* of SG is defined to be these three nodes. For each of these x_i, we define a contraction mapping $F_i : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$F_i(x) = \frac{1}{2}(x - x_i) + x_i.$$

So, we can define vertex sets inductively, starting with V_0 and then

$$V_n = \bigcup_i F_i(V_{n-1}).$$

Note that the contracting mappings can send distinct elements of V_k to the same element of V_{k+1}. For instance, $F_0(x_2) = F_2(x_0)$. Then starting with Γ_0, we inductively define the n^{th}*-level approximation graph* Γ_n as

$$\Gamma_n = \bigcup_i F_i(\Gamma_{n-1}),$$

identifying together nodes of the graphs which map to the same node in V_n. $\{\Gamma_n\}$ approximates SG in that SG is the inverse limit of the sequence $\Gamma_1, \Gamma_2, \ldots$, ordered by graph inclusion. The traditional analytic machinery, starting with the Laplacian, is constructed on SG making use of a relationship between the object on the fractal and discrete objects on $\{\Gamma_n\}$. To understand the infinity Laplacian, we will not require the Laplacian construction. However, we will still proceed by establishing a relationship between the discrete infinity Laplacian and any plausible definition on SG. The first thing we set out to do is defining what could be called Δ_∞*-harmonic* functions on SG: functions $u : SG \to \mathbb{R}$ that are a limit in some sense of functions $u_n : \Gamma_n \to \mathbb{R}$ which satisfy ${\Delta_\infty}^{(n)} u_n = 0$. In order to do so, we must better understand how Δ_∞-harmonic functions on graphs behave, and the primary tool for that is the so-called Lazarus algorithm.

1.3 The Lazarus algorithm

If u is Δ_∞-harmonic (satisfies $\Delta_\infty u = 0$), then we have that

$$u(x) = \frac{1}{2}\left(\sup_{y \in N(x)} u(y) + \inf_{y \in N(x)} u(y)\right). \tag{2}$$

Given a certain subset $\mathcal{B} \subseteq X$, let us call it the *boundary set* of G, a natural question to consider is the existence of a function u which takes prescribed values on $\mathcal{B}$ and satisfies $\Delta_\infty u = f$ on $X \setminus \mathcal{B}$. Such a function can be shown to exist in great generality, but for our purposes it suffices to know that such a u exists as long as $f > 0$ or $f = 0$ everywhere and $u|_B$ is finite [3]. An iterative scheme for computing such a function on a given graph with prescribed boundary values was developed by Adam Oberman in [2]. This is useful for numerical work, but a second interesting algorithm for computing u exists, due to Lazarus [1]. Despite computing the exact values that u takes on X in finite time, for numerical computations Oberman's scheme is more efficient. However, for our theoretical work, understanding the Lazarus algorithm is useful.

Let G be a finite graph with boundary set $B \subseteq G$, and for $x, y \in G$ define $d(x, y)$ to be the number of links in the shortest path connecting x and y in $G \setminus B$, and let $\rho(x, y) \subseteq X$ be this shortest path. For each pair $b, c \in B$, compute the **slope** of $\rho(b, c)$ as

$$S(\rho(b,c)) = \frac{|u(b) - u(c)|}{d(b,c)}. \tag{3}$$

For the pair $b^*, c^* \in B$ which maximizes (3), Lazarus shows that along $\rho(b^*, c^*)$, u is linear [1]. So if $d(b^*, c^*) = \ell$, for each $x \in \rho(b^*, c^*)$ we find that

$$u(x) = \frac{d(x, b^*)}{\ell} u(c^*) + \frac{d(x, c^*)}{\ell} u(b^*).$$

Then, since this is computed correctly, the next step of the Lazarus algorithm is the same as the first, with an expanded boundary set $B_1 = B \cup \rho(x, y)$. Along the path between elements of B_1 through $G \setminus B_1$ with greatest slope, u is again linear. We interpolate again to find the values of u, and repeat this procedure until all nodes are assigned values. The resulting u satisfies the equation $\Delta_\infty u = 0$, where Δ_∞ is the graph infinity Laplacian.

Let us call the path ρ_i of maximum slope during the i^{th} iteration of the Lazarus algorithm the $i^{\mathbf{th}}$ **Lazarus path** through G. One property of these Lazarus paths which will be useful to us is that slopes are nonincreasing. If $i > j$, $S(\rho_i) \leq S(\rho_j)$. The proof can be found in [1].

Actually, we will use a slight variant on the Lazarus algorithm as described here, with a simple class of weighted graph metrics. Instead of the weight of each edge being 1 as above, each edge will have weight c for some $c > 0$, but for any particular graph, each edge will be of the same weight. The results Lazarus proved using the link-counting metric still hold here, since these new metrics differ from the link-counting metric by a constant multiple.

2 Δ_∞-harmonic functions on $\{G_n\}$

With the Lazarus algorithm and the nonincreasing slope property in mind, we start looking at Δ_∞-harmonic functions on the $\{\Gamma_n\}$ graphs. To actually compute these functions, it is best to use a variant of Oberman's scheme [2] which takes advantage of these graphs' self-similarity to speed up convergence. However, we can use insight from the Lazarus algorithm to understand how these functions are built. It is instructive to work through the simple example of determining all possible Δ_∞-harmonic functions on Γ_1.

2.0.1 Example: Γ_1

Since a Δ_∞-harmonic function on Γ_1, or indeed any of the Γ_n, is uniquely determined by its boundary values, there is a three-dimensional space of such functions on Γ_1. However, taking into account two properties of infinity-harmonic functions, we can essentially reduce this to a one-dimensional space. We can verify directly from the definition of the graph infinity Laplacian that, if u defined on G satisfies $\Delta_\infty u = 0$ on $G \setminus B$, then v defined by the affine transformation

$$v(x) = u(x) - K$$

for some constant K satisfies $\Delta_\infty v = 0$ on $G \setminus B$ as well. If we define w by the scaling

$$w = u(x)/K,$$

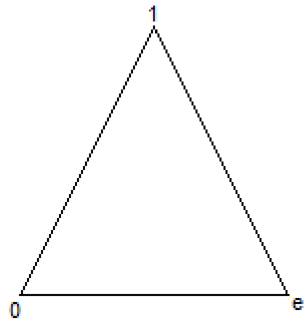
this also satisfies $\Delta_\infty w = 0$ on $G \setminus B$. So, consider an infinity-harmonic function u on Γ_n with $u(x_0) = a$, $u(x_1) = b$, and $u(x_2) = c$. WLOG we can assume $a \leq b \leq c$. If $a = b = c$, u is trivially just a constant function and equivalent to the 0 function by an affine transformation. Otherwise, we can construct

$$v(x) = \frac{1}{c-a}\,(u(x) - a)$$

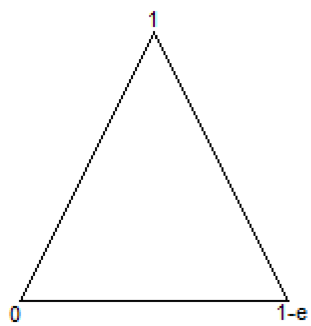
to obtain an infinity-harmonic function v with boundary data $v(x_0) = 0$, $v(x_1) = e$, and $v(x_2) = 1$ for some $e \in [0, 1]$. Thus, if we can determine the behavior of v on the interior of Γ_n, this entirely determines the behavior of u as well, since

$$u(x) = (c-a)v(x) + a.$$

So, we can associate each infinity-harmonic function, up to an scaled affine transformation, with an e value, a parameter we will call **eccentricity**. We can reduce this space even further: suppose $e \in (1/2, 1]$. Our boundary looks like:



We can construct $u'(x) = -u(x)+1$, also infinity-harmonic, with boundary data $u'(x_0) = 1$, $u'(x_1) = 1 - e$, $u'(x_2) = 0$. We can then just reflect this function horizontally to produce a u'' which looks like:

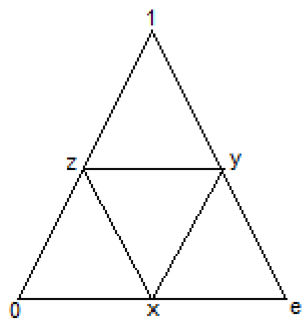


Now, our eccentricity is back in the range $[0, 1/2]$, and the behavior of u'' determines the behavior of u.

Therefore, we can restrict ourselves to determining all infinity-harmonic functions on Γ_1 with boundary data $0, e, 1$ for $e \in [0, 1/2]$. The functions fall into two cases.

Case 1: $e \in [1/3, 1/2]$

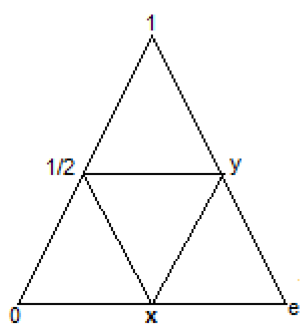
Our function u can be represented as:



The first Lazarus path in this case, and in fact in the $e \in [0, 1/3)$ case as well, is the path

$$\rho_1 = (0, z, 1),$$

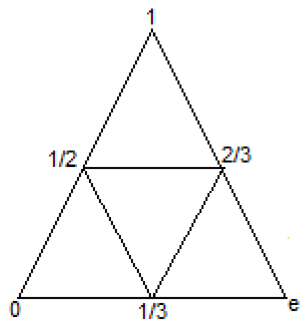
with slope 1/2. Note that, for brevity of notation, we are referring to nodes by their values alone, letting z stand for both the node $z \in V_1$ and $u(z)$. So, by the Lazarus algorithm, we conclude that $z = 1/2$.



The second Lazarus path is

$$\rho_2 = (0, x, y, 1),$$

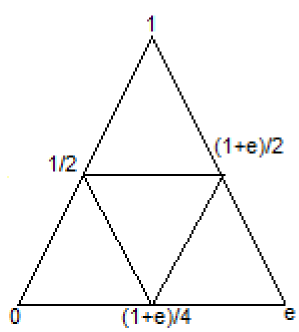
with slope 1/3. So, $x = 1/3$ and $y = 2/3$, giving us the complete description of u.



Note that the resulting function is independent of what value e actually takes, so long as it is within $[1/3, 1/2]$.

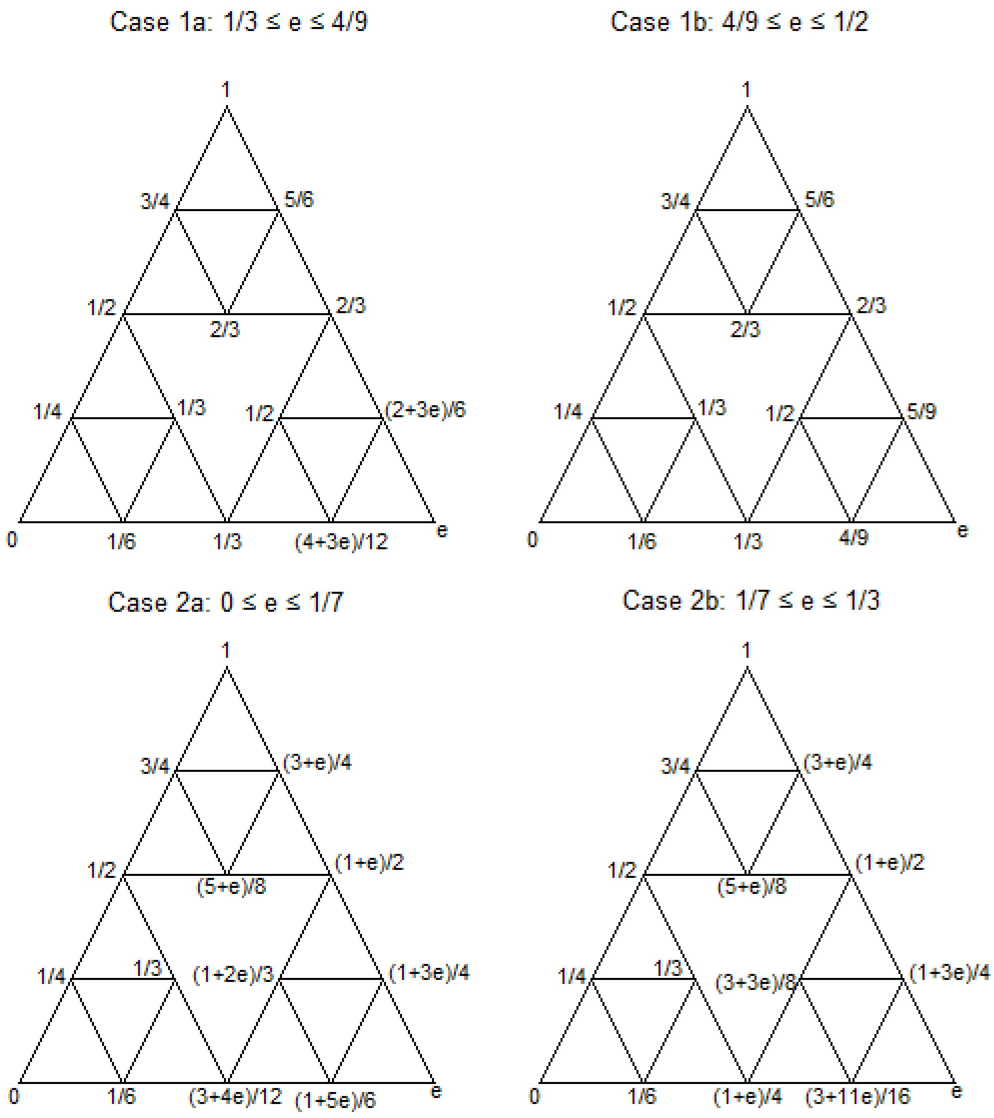
Case 2: $e \in [0, 1/3)$

Γ_1 starts with the same boundary values, and ρ_1 in this case is the same as before. However, when we go to calculate our second Lazarus path, we see that the path $(e, y, 1)$ has slope $(1-e)/2$, greater than the $1/3$ slope of the path $(0, x, y, 1)$ chosen in the previous case. So, $y = (1+e)/2$. This leaves x on the Lazarus path $(0, x, (1+e)/2)$ with slope $(1+e)/4$, so $x = (1+e)/4$:



Note that when $e = 1/3$, both of these paths have the same slopes, giving us the agreement between the two possible sequences of Lazarus paths at $e = 1/3$ which we would hope for.

We can calculate infinity-harmonic functions on higher-level graph approximations as well. This gives us a better picture of how a limiting object ought to behave. On Γ_2, for instance, it turns out that the behavior within each of the ranges $[0, 1/3)$ and $[1/3, 1/2]$ again splits based on smaller ranges. The details of the calculation are similar to those for Γ_1, only with more paths to consider. The final results can be broken down into four cases:



3 Δ_∞-harmonic functions on SG

Note that, in the case $e \in [0, 1/7)$, the node labeled $z \in V_1$ has the property that $u_1(z) \neq u_2(z)$, while for every other node in V_1 and possible e value, u does not change value on V_1. Moreover, numerical evidence suggests that $u|_{\Gamma_1}$ does not change on Γ_n for any $n > 2$. We can perform the same numerical examination for functions with boundary data in $[0, 1/3)$. This suggests that for different values of e, $u|_{\Gamma_1}$ changes from Γ_n to Γ_{n+1} for some range of n, until becoming fixed and then never changing. So, we define a function $u : SG \to \mathbb{R}$ in the following way:

Definition 1. *A function $u : SG \to \mathbb{R}$ is **infinity-harmonic** if*

$$u := \lim_{n\to\infty} u_n : \Gamma_n \to \mathbb{R} \mid \Delta_\infty u_n = 0,$$

with the aim of showing that for any fixed level graph Γ_k, $u_n|_{\Gamma_k}$ eventually is unchanging, and that therefore a limit value exists for u. The number of levels needed before the value is fixed seems to increase as $e \to 0$. This then is what we would like to prove. Note that this immediately would imply that $u_n|_{\Gamma_k}$ is fixed on Γ_n for all n greater than some N depending on e and k. This gives us a stronger-than-pointwise convergence of the sequence $\{u_k\}$ to a limiting object on u, though uniform convergence is not always possible, as a known counterexample shows.

Our primary theorem for infinity-harmonic functions then is

Theorem 1. *For all $k \in \mathbb{N}$ and any boundary data, $u_n|_{\Gamma_k} = u_{n+1}|_{\Gamma_k}$ for all n greater than some $N \in \mathbb{N}$, depending on e and k.*

3.1 A Useful Result

Proving this requires first understanding how the Lazarus paths behave on $\{\Gamma_n\}$, and the relationships between Lazarus paths on successive levels. A first step towards this is the proof of the following proposition.

Proposition 1. *Let $\{\Gamma_n\}, n = 1, \ldots$ be the standard sequence of graph approximations to SG. Then, for fixed boundary data (i.e., for a fixed infinity-harmonic function), the k^{th} Lazarus path is the same for all $m \geq k$.*

Before we get there, a few other things need to be proven, and some notation needs to be set down. Hopefully, this will accomplish it.

Consider the successive stages of the Lazarus algorithm on the nth level graph approximation Γ_n to SG. An obvious fact is that the first path accepted on the 1^{st} level graph approximation is the first path accepted on every level graph approximation. But why is this? This is the path of steepest slope in Γ_1, and the method by which we extend Γ_n to Γ_{n+1} is such that the shortest path between any two points in our boundary set doesn't decrease in length. In fact, these properties are enough to guarantee that this property will hold for arbitrary graphs, as we will now prove:

Consider graphs G_1, G_2 with vertex sets V_1, V_2 respectively such that $V_1 \subset V_2$, and for every pair of nodes $x, y \in V_1$ such that $y \in N_1(x)$, the neighbor set of x in G_1, there exists a path connecting x and y in G_2 which does not contain any other element of V_1. For these graphs, let d_1, d_2 be metrics on G_1 and G_2 respectively. A path $\rho_2 \subseteq G_2$ is a **refinement** of $\rho_1 \subseteq G_1$ if the terminal nodes of ρ_1 and ρ_2 are the same, each $x \in \rho_1$ is also in ρ_2, and if for every sequential $x_i, x_{i+1} \in \rho_1$, the shortest (wrt d_2) path connecting x_i and x_{i+1} does not pass through V_1. If each path $\rho_i \subset G_1$ has such a refinement in G_2, let us say that

$$G_1 < G_2.$$

Moreover, let us also assume that for each $x, y \in V_1$, $d_1(x, y) = d_2(x, y)$. Let us denote two graphs related in this way as as

$$G_1 \prec G_2.$$

It is easy to see that this relation forms a partial order on graphs. Note also that if which d_j is being used is clear from context, we will relegate this to an understood distinction and just refer to d.

Now, we want to consider the relation between the values of infinity-harmonic functions defined on these graphs, u_1 on G_1 and u_2 on G_2. In addition to the assumptions made in the preceding paragraph, let us add another: For the boundary sets $B_1 \subset V_1$ and $B_2 \subset V_2$, let $B_1 = B_2 = B$, and moreover for each $x \in B$, let $u_1(x) = u_2(x)$. We want to consider which path is first accepted by the Lazarus algorithm for both G_1 and G_2. This depends on calculated slopes, so let us adopt the following notation in addition to the previous: define the slope

$$S(x, y) := S(\rho(x, \ldots, y)$$

and $S(x)$ to be $S(x_1, x_2)$ for the x_1, x_2 forming the terminal nodes of the Lazarus path from which $u(x)$ is computed. Lazarus says that for $x^*, y^* \in B$ such that $S(x^*, y^*) \geq S(x, y)$ for all $x, y \in B$, along the path connecting x^* and y^*, u changes linearly. Specifically, for the generalized Lazarus algorithm which takes into account unequal link lengths, at each point x_i on the path ($x^* = x_1, x_2, \ldots, x_n = y^*$),

$$u(x_i) = \frac{d(x_i, x_n)}{d(x_1, x_n)} u(x_1) + \frac{d(x_1, x_i)}{d(x_1, x_n)} u(x_n).$$

So, we see that the first path accepted by Lazarus in G_1 is the same as the first path accepted by Lazarus in G_2: $u_1 = u_2$ on B, and $d(x, y)$ is the same on both G_1 and G_2 for, so $S(x, y)$ is equal for $x, y \in B$ in both G_1 and G_2. So, for each $x \in V_1$ along this path,

$$u_1(x) = u_2(x).$$

The Lazarus algorithm is of course a multi-step process: We now repeat the procedure on G_1 and G_2 with expanded boundary sets, but

$$B_1 \cup \rho_1^1 \neq B_2 \cup \rho_1^2$$

, where ρ_j^i is the j^{th} Lazarus path on G_i. So, we cannot apply the same reasoning as before to conclude that $\rho_2^1 = \rho_2^2$.

We can reinterpret this iterative procedure though: For a graph G with boundary set B, some path $(x_1, \ldots, x_n)$ is first accepted by the Lazarus algorithm We compute u along $(x_1, \ldots, x_n)$. With this information, we create a new graph G^1 with the same vertices as G and the same link set with the following exception: for every link in the path $(x_1, \ldots, x_n) \subset G$, we delete the corresponding link in G^1, and construct our boundary set B^1 in G^1 as $B \cup \{x_1, x_2, \ldots, x_n\}$. The second path accepted under Lazarus in G then is the same as the first path accepted in G^1, and so we can create a graph G^2 analogously from G^1 and repeat this procedure to find the third accepted path in G, and so on. In this interpretation, we need not talk about multiple paths generated by the Lazarus algorithm, as at each G^i, a single path is computed, which induces a new graph G^{i+1}. Therefore, let us talk about "*the* Lazarus path" in reference to the first path computed by the Lazarus algorithm on some G^i.

A minor issue to resolve is the possibility of a Lazarus path ρ on G^i containing elements of B^i besides the terminal nodes of ρ. It is easy to see that no such path can have slope strictly greater than the slope of a path in G^i containing no elements of B^i besides the terminal nodes: For such a pathological ρ, we can decompose this into a union of paths $\rho_1, \rho_2, \ldots, \rho_n$ such that each ρ_k contains no elements of B^i besides the terminal nodes, and each ρ_k has terminal nodes in B^k. Then, we have that

$$S(\rho) = \sum_{k=1}^{n} \frac{d(\rho_k)}{d(\rho)} S(\rho_k),$$

where $d(\rho_i)$ is the length of ρ_i. Either $S(\rho_\ell) = S(\rho_k) = S(\rho)$ for all ℓ and k, or there exists some ρ_{k^*} which has a greater slope than ρ. Either way, we can restrict our choice of Lazarus path to paths containing only terminal nodes in B^i.

Note that this highlights a slight ambiguity in this definition of the Lazarus algorithm. Namely, if in the scenario above, on some G^i it may occur that for two pairs $x, y, x', y' \in B^i$, $S(x, y) = S(x', y')$ is the greatest slope. There is then a question of which path to choose. In the case of the Sierpinski gasket considered below, each G will be some Γ_m approximation to SG. In the following work, this is resolved on an *ad-hoc* basis: It suffices to show that, out of the possible Lazarus paths at each Γ_n, there exists *some* path which satisfies the conditions that we seek.
The preceding paragraphs highlighted interplays between different graphs with respect to the Lazarus algorithm in two different ways, individually. Now, we want to consider a scenario in which both are occurring simultaneously, and try to examine how they relate. Let $\mathcal{G} = \{G_i\}$ be a (possibly infinite) sequence of graphs such that $G_i \prec G_{i+1}$ for all i. For each $G_j \in \mathcal{G}$, let G_j^k be the kth subgraph induced by the Lazarus algorithm, with $G_j = G_j^0$, the entire graph before any links have been removed by Lazarus. The standard sequence of graph approximations $\{\Gamma_n\}$ to SG is a sequence of this type. It would be nice if some

similar relation existed among $\{\Gamma^i_j\}$ for some j and i. Specifically, to prove Proposition 1, we would like to show that if $i \leq j$,

$$\Gamma^i_j \prec \Gamma^i_{j+1}.$$

We can show that this does not hold, sadly, though that is omitted here. However, we can try to construct a new relation which captures the critical properties possessed by $\prec$ which allowed graphs related by it to have the same Lazarus paths. Consider some sequence of Lazarus-induced graphs $\{G^i\}, i = 0, 1, \ldots$, such that $B = B^0$ is the boundary whose data is prescribed prior to the calculation of any Lazarus paths. For SG, this would be V_0, the vertex set of Γ_0. Which Lazarus path is calculated on each G^i is completely determined by the prescribed values on B and our procedure for tiebreaking, since all other elements of B^i are given values by linear interpolation. So, for a given (fixed) tiebreaking procedure such as the one detailed above, let $\mathcal{B}^i \subset B^i$ be the set of all nodes in B^i ever needed to compute a Lazarus path on G^i for all choices of u values on B. $G_i \prec G_{i+1}$ implies graph inclusion and that the set of all shortest paths starting and ending in the vertex set of G_i did not shorten upon inclusion into G_{i+1}. However, in proving that this implied that the Lazarus path on G_i is the same as the Lazarus path on G_{i+1}, we only really need the fact that $\mathcal{B}_i = \mathcal{B}_{i+1}$, *and* $d_i(x, y) = d_{i+1}(x, y)$ *for all* $x, y \in \mathcal{B}_i$. Let us say that

$$G_i \triangleleft G_{i+1}$$

if $G_i < G_{i+1}$ and if G_i and G_{i+1} are related in this way. It is then straightforward to modify the proof that $G \prec H$ implies G and H have the same Lazarus path to show that $G \triangleleft H$ implies the same. It should also be clear that $G_i \prec G_{i+1} \Rightarrow G_i \triangleleft G_{i+1}$. Then, we can rephrase Proposition 1 in this language as follows:

Proposition 2. *Let* $\{\Gamma_n\}, n = 1, \ldots$ *be the standard sequence of graph approximations to SG. For all* $j \geq 0$ *and all* $m \geq j$, $\Gamma^j_m \triangleleft \Gamma^j_{m+1}$.

In a slightly more graphical form, we are saying the following: Without much consideration being given to the structure of this complicated doubly-indexed set of graph approximations, we know the following relations hold true:

$$
\begin{array}{ccccccccc}
\Gamma_0^0 & \prec & \Gamma_1^0 & \prec & \Gamma_2^0 & \prec & \Gamma_3^0 & \prec & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\Gamma_0^1 & & \Gamma_1^1 & & \Gamma_2^1 & & \Gamma_3^1 & & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\Gamma_0^2 & & \Gamma_1^2 & & \Gamma_2^2 & & \Gamma_3^2 & & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\Gamma_0^3 & & \Gamma_1^3 & & \Gamma_2^3 & & \Gamma_3^3 & & \dots \\
 & & \vee & & \vee & & \vee & & \vee \\
 & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}
$$

What we would like to prove, via Proposition 2, is that the following is also true:

$$
\begin{array}{ccccccccc}
\Gamma_0^0 & \prec & \Gamma_1^0 & \prec & \Gamma_2^0 & \prec & \Gamma_3^0 & \prec & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\Gamma_0^1 & & \Gamma_1^1 & \lhd & \Gamma_2^1 & \lhd & \Gamma_3^1 & \lhd & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\Gamma_0^2 & & \Gamma_1^2 & & \Gamma_2^2 & \lhd & \Gamma_3^2 & \lhd & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\Gamma_0^3 & & \Gamma_1^3 & & \Gamma_2^3 & & \Gamma_3^3 & \lhd & \dots \\
 & & \vee & & \vee & & \vee & & \vee \\
 & & \vdots & & \vdots & & \vdots & \lhd & \ddots
\end{array}
$$

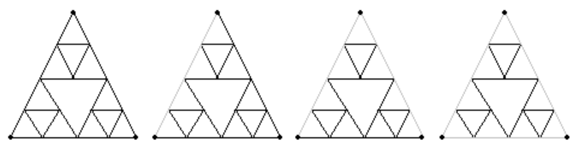
From here on out, let V_m be the set of vertices in Γ_m, which is also the set of vertices in Γ_m^j for all j. Let B_m^j be the set of boundary nodes in Γ_m along with all nodes in the first j accepted paths on Γ_m under the Lazarus algorithm, and $\mathcal{B}_m^j \subset B_m^j$ as above. Moreover, for convenience, let us identify SG with its standard embedding into $\mathbb{R}^2$, so that we can use the restriction of the Euclidean metric on $\mathbb{R}^2$ onto SG as a metric d on this space, and likewise create a family of metrics d_m^j for the graph approximations in the same way. This can be turned into an embedding-independent family of metrics on $\{\Gamma_n\}$, but the details are unnecessary for us here. A proof of Proposition 2 depends on each graph approximation Γ_m^j having the following two properties for $m \geq j$:

P1. $\mathcal{B}_m^j = B_{j-1}^j$

P2. If the shortest path between $x, y \in B_m^j$ in Γ_{m+1}^i is not a refinement of the shortest path between $x, y \in B_m^j$ in Γ_m, then x or y is in $B_m^j \setminus B_{m-1}^j$.

At a high level, the proof proceeds as follows: It is given that the proof that $\Gamma_0^0 \prec \Gamma_1^0 \prec \dots$ is trivial, that this implies $\Gamma_0^0 \lhd \Gamma_1^0 \lhd \dots$, and that P1 and P2 hold for these graphs. We want to use this inductively to prove that, if all of these properties (besides the $\prec$ relation) hold for all Γ_m^i, $m \geq i$ for all i less than some j, then P1 and P2 hold for Γ_j^j as well. Then, with this established, we conclude that $\Gamma_n^j \lhd \Gamma_{n+1}^j$ for all $n \geq j$.

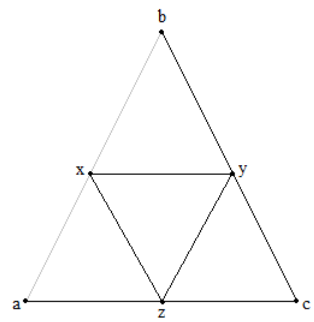
Before doing so however, let us introduce a bit more terminology. Consider a k-cell $C_k \subseteq \Gamma_m$, $m > k$. That is, a cell in Γ_m with at least one boundary point in $V_k \setminus V_{k-1}$, with boundary set $\{c_1, c_2, c_3\}$. It is easy to show that for such a k-cell, in fact two elements of the boundary set are in $V_k \setminus V_{k-1}$. Then, for any i, consider the subgraph of Γ_m^i with the same vertex set as C_k, call this C_k^i. Let us say that C_k^i is **type 0** if C_k had no links removed by the first i iterations of the Lazarus algorithm on Γ_m, so that each link in C_k is contained in C_k^i. C_k^i is **type 1** if exactly one of the straight paths connecting $c_1 - c_2$, $c_1 - c_3$, or $c_2 - c_3$ has been entirely removed under the first i iterations of the Lazarus algorithm on Γ_m, and no other links have been removed. C_k^i is **type 2** if two such paths have been entirely removed, and no other links have been removed, and **type 3** if all three paths have been removed.



With that in mind, we can prove the following:
Lemma 1. *For all $j > 0$, if $\Gamma_m^i \triangleleft \Gamma_{m+1}^i$ for all $i < j$ and for all $m \geq i$, and P1 and P2 hold for all such graphs, then P1 and P2 hold for all Γ_m^j, $m \geq j$.*

Proof. For $j = 0$, this same result is clear, so consider some $j > 0$. First, we will show that P1 holds on Γ_j^j, and extend this result to all Γ_m^j, $m \geq j$. Suppose for contradiction that there were a path ρ computed on Γ_j^j whose terminal or initial node lies in $B_j^j \setminus V_{j-1}$ and whose slope is strictly greater than that of any path with terminal and initial nodes in B_{j-1}^j. Since by hypothesis, the Lazarus path on Γ_j^{j-1} is the same as the Lazarus path Γ_{j-1}^{j-1}, it cannot change direction at any node in $V_j \setminus V_{j-1}$. So, any node $x \in B_j^j \setminus V_{j-1}$ used to compute ρ lies in the interior of a $(j-1)$-cell of type 1 or 2, since only elements of V_0 can be contained in type 0 cells. Let us knock out the type 1 case first:

Case: Type 1
Suppose x is contained in a type 1 $(j-1)$-cell, as below:

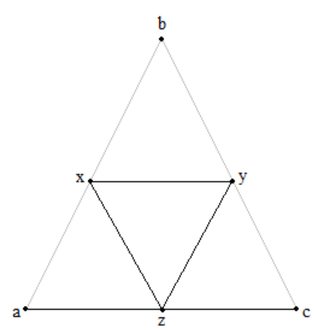


There are three paths that ρ could take from x inside this cell *a priori*: (x, z, a), (x, y, b), or $(x, y/z, c)$. Neither (x, z, a) nor (x, y, b) can be part of ρ though, since

$$S(a, z, y, b) \geq S(x, z, a) = S(x, z, b).$$

$(x, y/z, c)$ cannot be a part of ρ either: Since $x = \frac{a+b}{2}$, either the path (a, z, c) or (b, y, z) will have a greater slope than $(x, y/z, c)$. So, ρ cannot contain a node x in $B_j^j \setminus V_{j-1}$ lying in a type 1 $(j-1)$-cell. We will show that same is true for x in a type 2 $(j-1)$-cell:

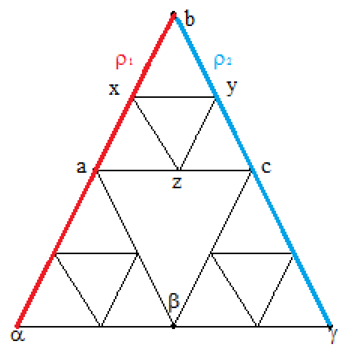
Case: Type 2



For x in this situation, we must pull back one layer and consider the $(j-2)$-cell C_{j-2}^j in which this $(j-1)$-cell lies. There will be a few subcases here: First we want to note that in the preceding picture, the accepted paths (b, x, a) and (b, y, c) cannot be part of the same Lazarus path, since (a, z, c) has strictly greater slope for nonconstant boundary data (in which case the entire function is trivial). So, WLOG we will assume that (b, x, a) is part of a Lazarus path ρ_1 accepted prior to the path ρ_2 containing (b, y, c) and then prove that no ρ with terminal node x or y exists. To do so, we note that ρ_1 was at latest the

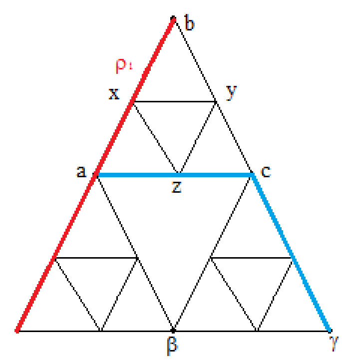
$(j-1)$st Lazarus path, and so cannot change direction on any nodes in $V_j \setminus V_{j-1}$. Likewise, ρ_2 was then at latest the $(j-2)$nd Lazarus path, and so cannot change direction on any nodes in $V_j \setminus V_{j-2}$. So, ρ_2 does not change direction on C^j_{j-2}, and there are only three possible paths ρ_1 can take through C^j_{j-2}.

Subcase 1:



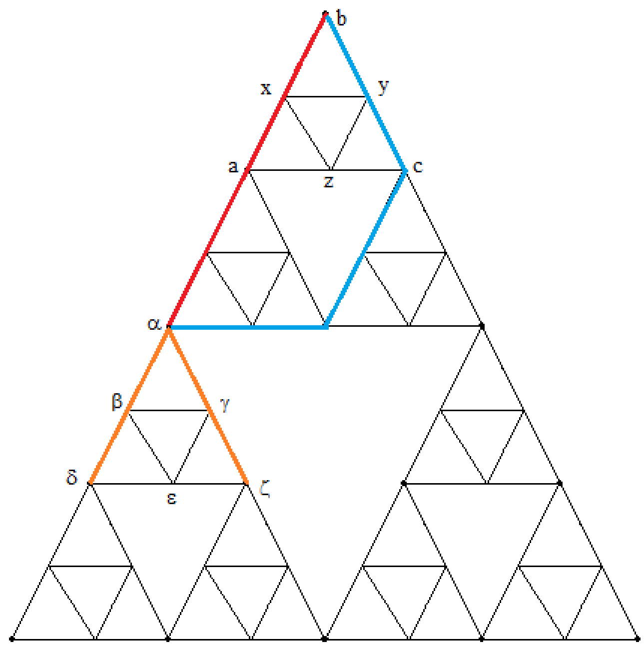
Here, note that $S(x,z,a) = S(a,\beta,\alpha)$, $S(x,z,c) = S(a,\beta,\gamma)$, $S(y,z,a) = S(c,\beta,\alpha)$, and $S(y,z,c) = S(c,\beta,\gamma)$, so no strictly dominating path exists starting from x or y.

Subcase 2:



This case is knocked out easily: $S(a,\beta,\gamma) = S(\rho_1)$, so no path in C^j_{j-2} dominates.

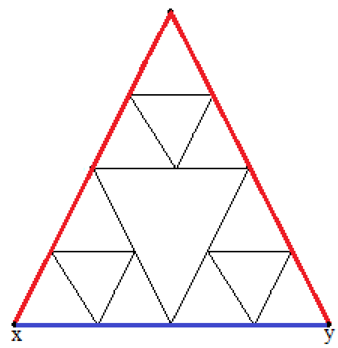
Subcase 3:



If ρ_2 stays in the $(j-2)$-cell, then we must consider how ρ_1 behaves in the $(j-3)$-cell C^j_{j-3} containing C^j_{j-2}. Note that since 2 paths have been accepted already, $j \geq 3$, so C^j_{j-3} exists. The orange lines in the picture above indicate which paths ρ_1 could take through this $(j-1)$ cell. ρ_1 of course must continue beyond this $(j-1)$-cell, but this is the only part we're interested in. If the path in this cell is (α, β, δ), then $S(\alpha, \gamma, \epsilon, \delta) = S(\rho_2)$, and since Lazarus produces monotonically decreasing slopes, no path in C^j_{j-1} can beat this. Similarly if the path through this new $(j-1)$-cell is (α, γ, ζ), then $S(\alpha, \beta, \epsilon, \zeta) = S(\rho_2)$, so we get the same conclusion.

This proves that P1 holds for Γ^j_j. Before proving that P1 holds for all Γ^j_m, $m \geq j$, let's show that P2 holds for all such Γ^j_m. Let x, y be a pair of points in B^j_m such that $d^j_{m+1}(x,y) < d^j_m(x,y)$. Call the shortest path in Γ^j_{m+1}, ρ . Suppose that both x and y are in V_{m-1}. We can decompose ρ into a union of subpaths $\rho_1, \rho_2, \ldots, \rho_k$ through the k $(m-1)$-cells $C^j_{m-1,1}, C^j_{m-1,2}, \ldots, C^j_{m-1,k}$ containing ρ. In order for ρ to be strictly shorter than the shortest path connecting x and y in Γ^j_m, it follows that in at least one of these $(m-1)$-cells, the shortest path through the cell considered as a part of Γ^j_m must be longer than ρ restricted to the cell. Let us show that this does not happen.

So, we want to consider what sort of $(m-1)$ cells could contain x and y. First, suppose that one (and therefore the other) is contained in a type 2 $(m-1)$-cell C^j_{m-1} on both levels m and $m+1$. Since by assumption $x, y \in V_{m-1}$, this implies that they are boundary points of C^j_{m-1}, and we are in the situation in this picture:

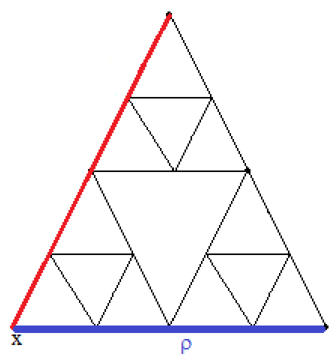


where the red paths are accepted, and the blue path is the shortest between x and y. This is present in both Γ^j_m and Γ^j_{m+1} and has the same length in both graphs, contradicting the lengthening assumption.

We can therefore assume that x and y lie in type 1 $(m-1)$-cells. They cannot lie in the same $(m-1)$-cell for the same reason that they can they could not lie in a type 2 $(m-1)$-cell: they will be on the boundary of the cell, and the straight line path connecting them is the shortest on both levels m and $m+1$. So, they lie in different $(m-1)$-cells, and we can look at how ρ behaves in each such cell. Let us start with the interior cells, if any exist. By interior, I mean a $(m-1)$-cell through which ρ passes, but which does not contain x or y. Any such cell must be type 0, since any Lazarus path entering a type 1 $(m-1)$-cell must terminate in that cell, and no Lazarus path can even enter a type 2 or 3 cell. But a type 0 $(m-1)$-cell in Γ^j_m is isomorphic as a graph to Γ^0_1, and in Γ^j_{m+1} it is isomorphic to Γ^0_2. We already know that P2 holds for these graphs, so ρ cannot lengthen on interior cells. Therefore, this lengthening must occur in one of the cells containing x or y. Again, however, we see that by assumption x and y lie on the boundary of their respective $(m-1)$-cells as in the following picture, and the shortest path exiting the cell is the same on both levels.

Therefore, there is no $(m-1)$ cell which ρ passes through, in which the shortest path on Γ^j_m is longer than ρ. So, *P2 holds for all* Γ^j_m, $m \geq j$.

Using this fact, we can finish off the proof, by showing that P2 holding for all such Γ^j_m and P1 holding for Γ^j_j implies that P1 holds for all such Γ^j_m. We'll induct on m: given that P1 holds for all Γ^j_n with $j \leq n < m$, consider a path ρ as before with at least one boundary node in $B^j_m \setminus V_{j-1}$ whose slope is strictly greater than that of any path with both boundary nodes in V_{j-1}. Since no such path exists lying in $B^j_n \setminus V_{j-1}$ for any $n < m$, and by P2 no shortest



paths shorten for elements of B^j_n with $n < m-1$, we conclude that our new pathological points must lie in $B^j_{m-1} \setminus V_{j-1}$. Our proof then proceeds exactly as it did for the base case, using $(m-1)$ cells instead of $(j-1)$ cells, along which previous paths have also not changed direction, and likewise for $(m-2)$ and $(m-3)$ cells. Given this, we are done. Proposition 1 is proven. □

The last step in the proof is to show that if P1 and P2 hold for Γ^j_m and Γ^j_{m+1}, then $\Gamma^j_m \triangleleft \Gamma^j_{m+1}$. This is pretty straightforward.

Lemma 2. *Suppose P1 and P2 hold for Γ^j_m and Γ^j_{m+1}. Then, $\Gamma^j_m \triangleleft \Gamma^j_{m+1}$.*

Proof. By P1, $\mathcal{B}^j_m = \mathcal{B}^j_{m+1} = B^j_{j-1}$, and by P2 $d^j_m(x,y) = d^j_{m+1}(x,y)$ for all $x, y \in B^j_{j-1}$, and so it follows immediately from the definition that

$$\Gamma^j_m \triangleleft \Gamma^j_{m+1}.$$

□

The proof of Proposition 2 then follows immediately from Lemmas 1 and 2, and therefore Proposition 1 is correct.

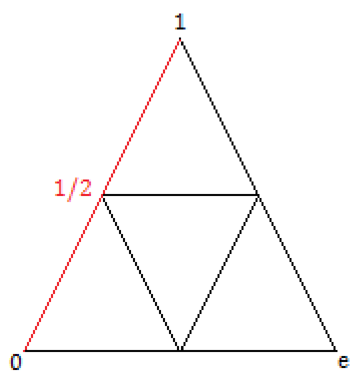
From Proposition 1, we can get some results immediately. If $e \in [1/3, 1/2]$, the second Lazarus path on Γ_2 passes through all of the V_1 vertices, so Theorem 1 is true for e in this range. However, Proposition 1 does not immediately help us for $e \in [0, 1/3)$. In fact, just calculating Lazarus paths for this e range suggests that none of these always-accepted Lazarus paths pass through $z \in V_1$ for any Γ_n.

What is needed is a generalization of Proposition 2.

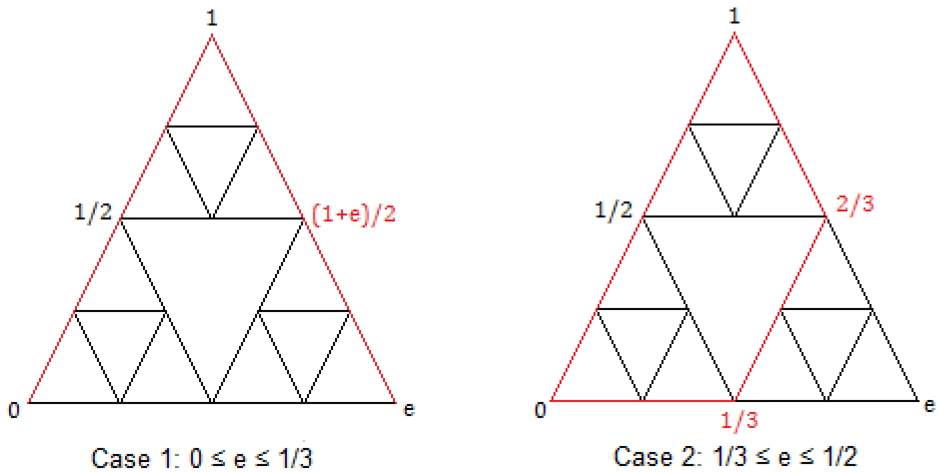
3.2 A generalization of Proposition 2

The overall goal is to show that each Lazarus path, under certain conditions at least, partitions the sequences of graphs into sequence of unions of subgraphs, each of which is amenable to the same techniques used on the original graph. In doing so, we are then able to calculate Lazarus paths on each subgraph at each step, enlarging the set of points which we can show to be computed correctly at each level of approximation to SG.

Recall Theorem 1 on page six. Proposition 1 gives us some results in proving this. Consider as before the following level 1 graph:



Since the V_0 nodes are always correct trivially we're only concerned with the nodes $a, b, c \in V_1 \setminus V_0$. We know the first Lazarus path is the left path from 0 to 1 for any choice of e, so $a = 1/2$ is given. The second Lazarus path on Γ_2 does depend on e, but in a simple way. We have two choices:



Regardless, for $e \in [1/3, 1/2]$, the first and second Lazarus paths on Γ_2 run through all nodes in V_1, and therefore will do so for all Γ_k, $k \geq 2$. So, Theorem 1 is true for $e \in [1/3, 1/2]$. For $e \in (0, 1/3)$ however, we do not get the same behavior. But then, consider Γ_k for $k \geq 2$. For each such Γ_k, we know that the first two Lazarus paths are a refinement of the first two Lazarus paths on Γ_2. Moreover, we can calculate that for any e in this range, the third Lazarus path on Γ_k is fixed, as drawn below. So, the problem of running the Lazarus algorithm on Γ_k^j for $j, k > 3$ splits into running it on three separated domains,

$$\Gamma_{k,1}, \Gamma_{k,2}, \Gamma_{k,3},$$

pictured on the next page in Figure 1. As an aside: the doubly-indexed subscripts will come up a few times here, so a bit more notation might be handy here. Just as $SG = \lim_{k\to\infty} \Gamma_k$, let

$$\Gamma_{\cdot,i} := \lim_{k\to\infty} \Gamma_{k,i}\,.$$

It would be nice if we could say that the sequence of graphs $\Gamma_{k,3}$ in the above picture, with appropriate choice of initial boundary data, had the same properties possessed by $\{\Gamma_k\}$ which allowed for Proposition 2 to hold true. Then, we could restrict our focus to $\Gamma_{k,3}$ and apply the procedure to see what information is gained about the nodes of Γ_k therein. It would be nicer to generalize this result to a wider class of graphs than just $\Gamma_{k,3}$, say all of the "regular finite fractafolds," Actually, let us try to do just that.

Definition 1: A *regular finite fractafold* is the quotient space

$$\left(\bigsqcup_{i=1}^{n} SG \right) / \varphi,$$

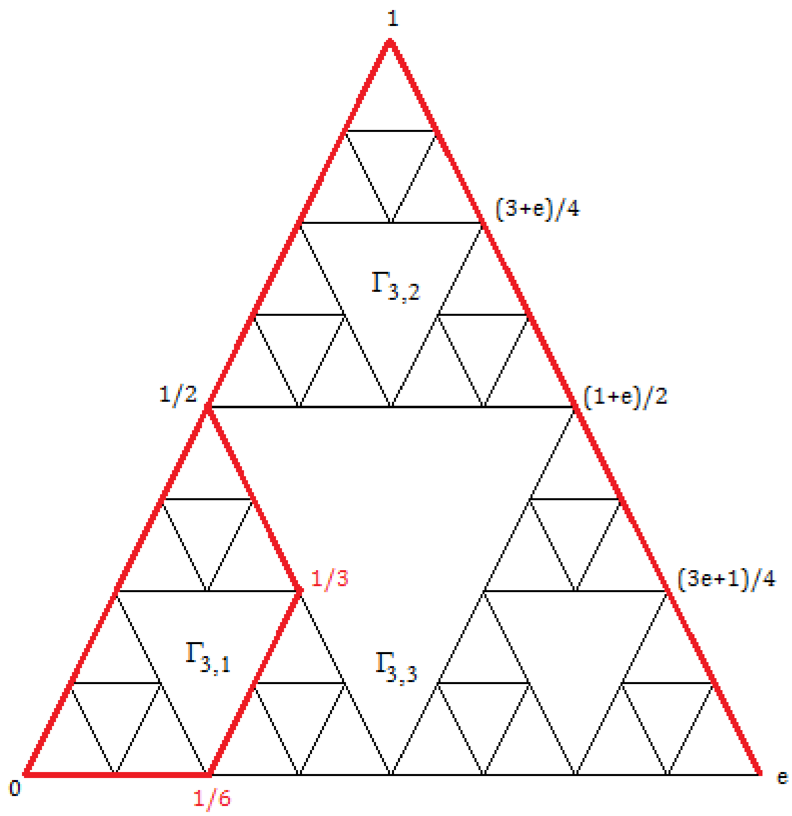


Figure 1: The $e \in (0, 1/3)$ case

where φ is a quotient map such that in the ith copy of SG (SG_i), φ identifies each boundary point of SG_i with at most one boundary point of at most one SG_j, $i \neq j$, and φ never identifies two different boundary points in SG_i with boundary points in one other SG_j.

For instance, each of the $\Gamma_{\cdot,i}$, $i \in [1, 3]$ constructed above is a regular finite fractafold. In fact, any of the subgraphs created by partitioning SG along Lazarus paths should be a regular finite fractafold, if the appropriate initial boundary data is chosen. But that is irrelevant here, as long as we have the $\Gamma_{\cdot,i}$.

Given this definition of regular finite fractafold, let F be such an object. Then, analogously to the way in which $\Gamma_k := SG|_{V_k}$, let

$$F_k := F|_{V_k}.$$

If it is not clear, V_k in this setting is just the union of the V_k sets of the copies of SG comprising F, with appropriate identifications made. Moreover, it will sometimes be useful to refer to the different copies of SG making up F, as well as their graph approximations. So, just as we index the copies of SG as $\{SG_i\}$, let the copies of the kth-level graph approximations to SG be indexed as $\{\Gamma_{k,i}\}$. Moreover, we would like some boundary data on F_0 from which to run our Lazarus algorithm. Just using V_0 here will not work, we would like to use just those elements which remain unidentified, having only 2 neighbors in F. This will be B_0 (and also $\mathcal{B}_0$). So, with such boundary data, let us say the following. It is easy to show that $F_k \prec F_{k+1}$, with $\prec$ defined as in the previous writeup. Also analogously to that writeup, let us define F_k^j to be the jth Lazarus subgraph of F_k. Then, is is also easy to show that $F_k^j < F_k^{j-1}$, too. This gives us the following picture:

$$\begin{array}{ccccccccc}
F_0^0 & \prec & F_1^0 & \prec & F_2^0 & \prec & F_3^0 & \prec & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
F_0^1 & & F_1^1 & & F_2^1 & & F_3^1 & & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
F_0^2 & & F_1^2 & & F_2^2 & & F_3^2 & & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
F_0^3 & & F_1^3 & & F_2^3 & & F_3^3 & & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\vdots & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}$$

Again, we want to show that the following is also true:

$$\begin{array}{ccccccccc}
F_0^0 & \prec & F_1^0 & \prec & F_2^0 & \prec & F_3^0 & \prec & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
F_0^1 & & F_1^1 & \lhd & F_2^1 & \lhd & F_3^1 & \lhd & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
F_0^2 & & F_1^2 & & F_2^2 & \lhd & F_3^2 & \lhd & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
F_0^3 & & F_1^3 & & F_2^3 & & F_3^3 & \lhd & \dots \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\vdots & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}$$

Lemma 1: Let F be a regular finite fractafold. Then, for all j and $m \geq j$,

$$F_m^j \lhd F_{m+1}^j.$$

Proof. Again, this will depend solely on the sequences $\{F_m^j\}, m \geq j > 0$ for all $j \in N$ satisfying P1 and P2. The proof is much simpler here though, since we know the same is true for $F = SG$.

P1: For all $m \geq j > 0$, $\mathcal{B}_m^j = \mathcal{B}_{j-1}^j$.

Proof. Suppose that for some $\Gamma_{m,i}^j$ comprising F_m^j, there exists some $b \in B_m^j \setminus \mathcal{B}_{j-1}^j$ necessary for computing the next Lazarus path on F_m^j. Just as in the proof of P1 for SG, we can look at the $(j-1)$ and $(j-2)$ cells in which b lies. These are still inside $\Gamma^j_{m,i}$, and so the proof that no such b exists follows directly from the proof for the graph approximations to SG, since these cells are identical.

□

P2: If the shortest path between $x, y \in B^j_m$ in F^j_{m+1} is not a refinement of the shortest path between $x, y \in B^j_m$ in Γ^j_m, then x or y is in $B^j_m \setminus B^j_{m-1}$

Proof. Suppose this were not true. Let x, y be such elements of B^j_m and let ρ_m be the path connecting them in F^j_m, while ρ_{m+1} is the path connecting them in F^j_{m+1}. If ρ_{m+1} is not a refinement of ρ_m, then there is some copy SG_i of SG in which it first shortens. So, we can restrict ourself to looking at how the path behaves in SG_i, in which case we are back to the conditions of the original proof, and know that no such shortening occurs.

□

So, the sequences $\{F^j_m\}$, $j \geq m$, satisfy P1 and P2. The results of Lemma 1 and Lemma 2 from the previous writeup transfer immediately over to the regular finite fractafold setting, and so we conclude that

$$F^j_m \lhd F^j_{m+1}$$

for all j and all $m \geq j$.

□

This is good news. Going back to the situation illustrated in Figure 1, Lemma 1 tells us that we can run this compute-Lazarus-path-then-refine procedure on each subgraph individually. For now, let us focus on $F = \Gamma_{3,3}$, as that is where all the interesting things are happening. If we let it inherit the boundary data from the computed values on Γ^3_3, the graph of F^0_0 as depicted in Figure 2 on the next page.

Note that, in calling this F_0, we are going to introduce some confusion that should be clarified. This is level 0 for this particular fractafold, but included into Γ_2 it is a level 2 portion of the copy of Γ_2 from which it was taken. A result about F_k will translate into a result about Γ_{k+2}. Let us leave this at the level of tacit understanding where there is no chance for confusion, and otherwise, call the level of refinement on the graph F the *local level*, and the level of refinement if F were to be considered as included in the original copy of SG the *inherited level*. This causes some problems for our slope renormalization scheme, as well. Which level do we use when calculating path slopes? For consistency, let us always use the inherited level, not the local level. Another subtlety arises here that the $F = SG$ case hides: There are nodes born on level 0 here that are not boundary data. Yet, Lazarus paths computed on this level are not necessarily going to be correct on future levels. Informally, the graph is too coarse at this

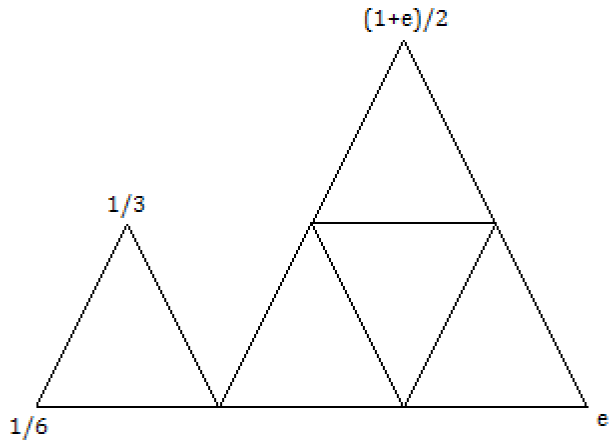
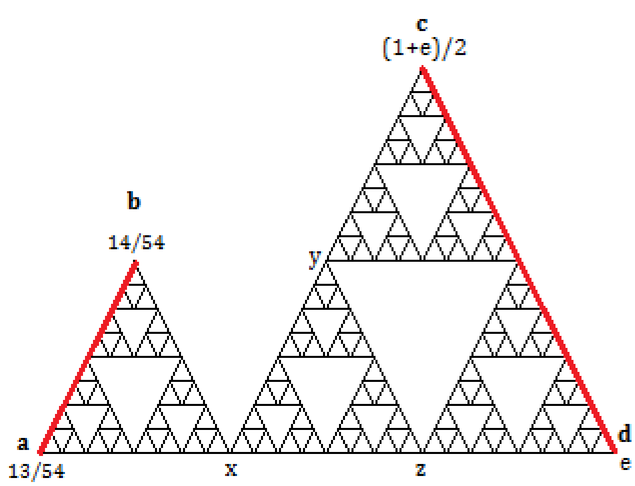


Figure 2: F_0^0

level in comparison with the spacing of boundary data. That in mind, we now know we can compute Lazarus paths on F, but will they agree with the paths computed in Γ_3 on the nodes that are not boundary data for F? Hopefully, the answer is "Yes for all regular finite fractafolds obtained in this way, so long as the boundary data agrees", but we shall see. In any case, for this particular F we can directly verify that the first Lazarus path on F_1 connects e and $(1+e)/2$, and the second Lazarus path on F_2 connects $1/6$ and $1/3$. This gives us the following graph for F_3^2:



So, what is the next Lazarus path on this domain? It depends on e. Again, we can verify that it will be one of two paths,

$$\rho_1 = (a, x, y, c)$$

or

$$\rho_2 = (c, y, z, e).$$

We are on local level 3, so our inherited level is 5. Therefore, our slope renormalization factor is 2^5. The shortest path connecting a and b has renormalized slope of $\frac{1/12}{12} \cdot 2^5 = 4/9$ and the shortest path b and d has slope $\frac{4-12e}{9}$, while

$$S(\rho_1) = \frac{4+6e}{9}$$

and

$$S(\rho_2) = \frac{2-2e}{3}.$$

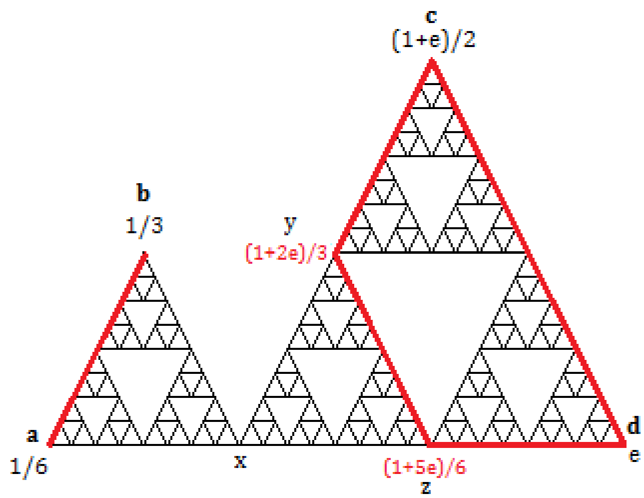
For any $e \in (0, 1/3)$, either slope strictly dominates the slopes of the $a, \ldots, b$ and $b, \ldots, d$ paths (as well as the rest of the possible paths if you care to check). However, $S(\rho_1) > S(\rho_2)$ for $e \in (1/6, 1/3)$, while $S(\rho_2) > S(\rho_1)$ for $e \in (0, 1/6)$. Mixed news, but mostly good! This means that x is computed correctly with value

$$x = \frac{5+3e}{18}$$

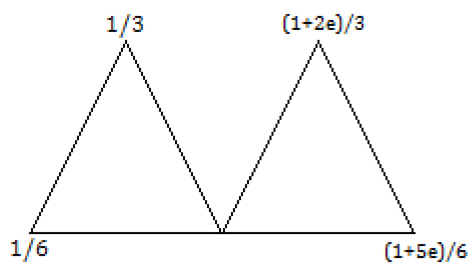
at local level 3 (and each thereafter) if $e \in (1/6, 1/3)$. This corresponds to x being computed correctly on Γ_5. But note: $x \in V_1$ (here, V_1 understood as the vertex set of the original copy of Γ_1), and the other elements of V_1 were computed correctly by Γ_2. What does this mean?

For $e \in (1/6, 1/3)$, $u_j^e(x)$ is computed correctly for all $x \in V_1$ for all $j \geq 5$.

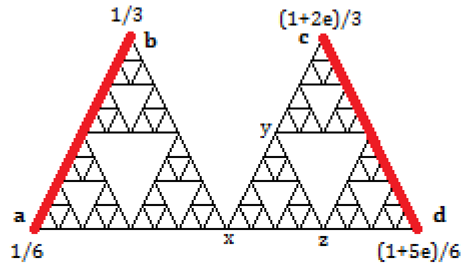
Before this, we had that Theorem 1 was true for $e \in [1/3, 1/2]$ easily. This last bit extends that, we now have that Theorem 1 is true for $e \in [1/6, 1/2]$. Moreover, the same techniques used to extend the result here can be used to greater advantage. Suppose $e \in (0, 1/3)$. Then, on F_3^2 our Lazarus path is ρ_2, and we have this situation:



Our node x is now contained in a different regular finite fractafold (rff), call it G, with boundary nodes a, b, y, z. Let it inherit boundary data on these nodes as before, so that G_0 is the following:



Keeping in mind that we compute no paths on this level, we can refine one level to see that the first Lazarus path on G_1 is the one connecting $1/6$ and $1/3$, with renormalized slope $2/3$ again (our renormalization constant on local G level 1 is 2^3), and the second Lazarus path on G_2 is the line connecting $(1+5e)/6$ and $(1+2e)/3$, with slope $(2-2e)/3$, putting us here at local G level 3:



As before, the renormalized slope of the shortest path connecting a and b (call it ρ_0) is 4/9. The slope of the path γ connecting b to d is $\frac{1-5e}{3}$ The other contender paths are $\rho_1 = (a, x, y, c)$ and $\rho_2 = (d, z, y, c)$. Here,

$$S(\rho_1) = \frac{1+4e}{3}$$

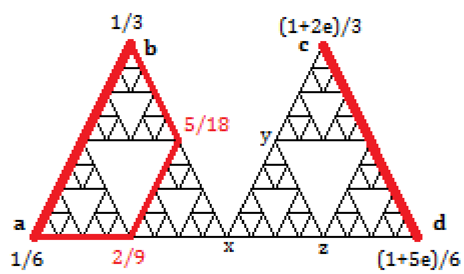
and

$$S(\rho_2) = \frac{4-4e}{9}.$$

Note that γ is still out of the running but, $S(\rho_2) < S(\rho_0)$, so our competition is between ρ_0 and ρ_1. A little algebra shows us that ρ_0 wins out when $e \in (0, 1/12)$, and ρ_1 wins out when $e \in (1/12, 1/6)$. So, for $e \in (1/12, 1/6)$, once we get to this stage (inherited level 5 again), x is still computed correctly as

$$x = \frac{3+4e}{12},$$

just via a different route. For the $e \in (0, 1/12)$ range, however, we find ourself with the following picture:



Now, x is contained in yet another subgraph/rff, let us call this one H. Something interesting has happened here though: H is identical to F, scaled down by a factor of 2. The (renormalized) slopes along the boundaries of H are 2/3 of the corresponding slopes for F. On H, the first Lazarus path connects $(1+23)/3$ and $(1+5e)/6$, and the second connects 2/9 and 5/18, so that at local H level 3 we find ourselves in a familiar situation:

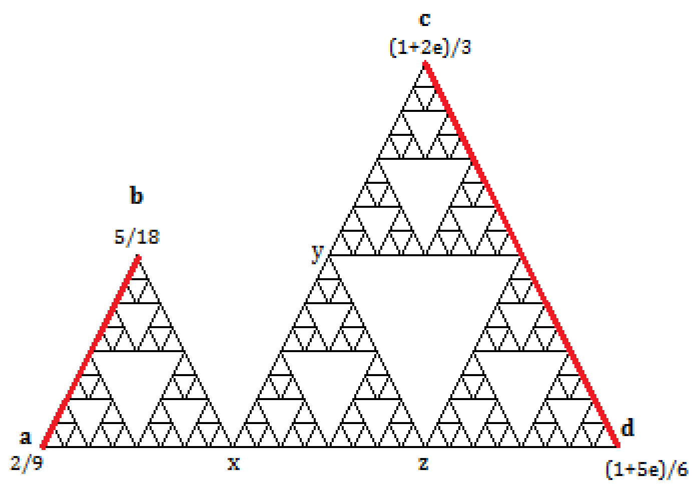


Figure 3: H_3^2 this time, not F_3^2

Let's see what happens this time with the third Lazarus path. Here, our inherited level is 6, so our scaling is 2^6. The shortest path ρ_0 connecting a and b has slope

$$S(\rho_0) = 8/27.$$

The path $\rho_1 = (a, x, y, c)$ has slope

$$S(\rho_1) = \frac{8 + 48e}{27}$$

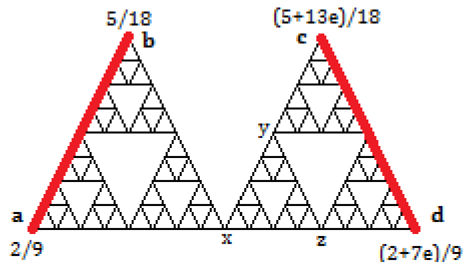
and the path $\rho_2 = (d, z, y, c)$ has slope

$$S(\rho_2) = \frac{4 - 4e}{9}.$$

Again, we can just use a little algebra to find that ρ_1 dominates for $e \in (1/15, 1/12)$. So, for e in this range, u^e is computed correctly on the V_1 vertices by level 6, with

$$x = \frac{7 + 6e}{27}.$$

For $e \in (0, 1/15)$ however, we must go deeper still. Let us call the next subgraph/rff J:



Again we can go through the same calculations to show that the two competing paths are ρ_0 connecting a and b, and ρ_1 connecting a and c. We have

$$S(\rho_0) = \frac{8}{27}$$

and

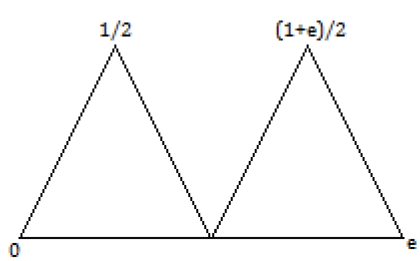
$$S(\rho_1) = \frac{6 + 78e}{27}.$$

ρ_1 is preferred when $e > 1/39$, so again by level 6, for $e \in (1/39, 1/15)$ x is computed correctly with value

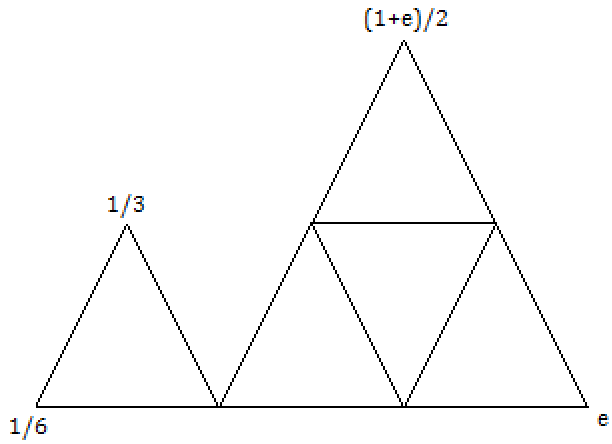
$$x = \frac{9 + 13e}{36}.$$

If $e \in (0, 1/39)$ we continue on.

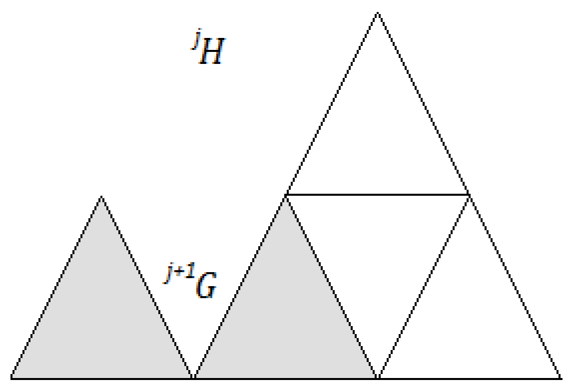
However, by this point we notice a pattern is forming. We seem to be alternating between scaled copies of two different graphs/rff's, one consisting of two cells of the same size joined together at a point, and the other consisting of two cells of different sizes joined together at a point. If we can show that this is indeed the case in general, and get closed-form expressions for the boundary data on each copy of these graphs, we can hope to get the results we want. We will want some terminology to talk about these things. Despite the confusion with previous names and general ugliness, let us call the former graph, two same-sized cells glued together, ${}^jG$, and the latter as ${}^kH$, where the j and k work as follows: ${}^jG$ is the rff produced by the gluing together of two $(j+1)$-cells, and ${}^kH$ is the gluing together of a $(k+1)$-cell and a $(k+2)$-cell. So for instance, ${}^0G_0$ is

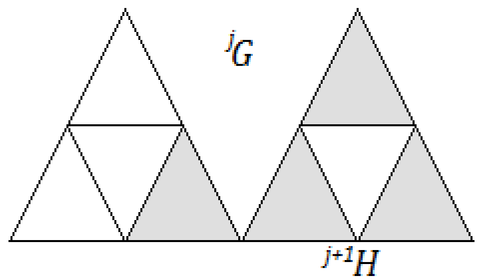


and ${}^{1}H_0$ is

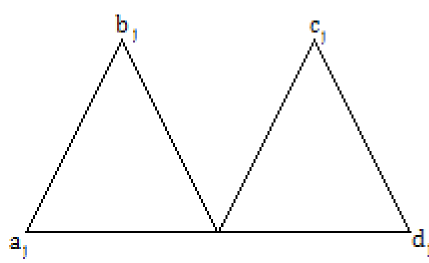


The remaining graphs are defined inductively as: ${}^{j+1}G$ is ${}^{j}H$ minus the upper and right $(j+2)$-cells contained within the $(j+1)$-cell that comprises part of ${}^{j}H$. ${}^{j+1}H$ is formed from ${}^{j+1}G$ by removing the upper and left $(j+3)$-cells from the left $(j+2)$-cell of ${}^{j+1}G$.

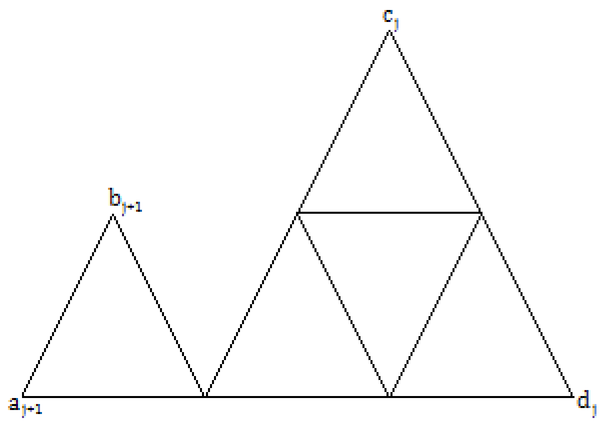




With this in mind, we want to show the following: Given that ${}^1G$ is as above, we want to show that the either the Lazarus path on ${}^jG_3^2$ partitions the graph into ${}^jH$ and its complement or passes through the remaining V_1 node for all j, and likewise that the Lazarus path on ${}^jH_3^2$ partitions the graph into ${}^{j+1}G$ and its complement or passes through the V_1 node. In the process, we will also derive an expression for the boundary data inherited from Lazarus paths on each level j for both of these graphs, and therefore the range of e values for which this process hits the V_1 node for each j, ultimately giving an upper bound for the level of the correct computation of the V_1 set as a function of e. First, let us name the boundary data values as a function of the level. The boundary data for ${}^jG$ is



and similarly for ${}^{j}H$:



By looking at the numbers we have gotten for the first few j values by hand, we can conjecture that

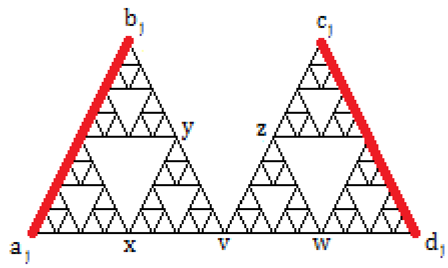
$$\begin{aligned} a_j &= \frac{1-(1/3)^j}{4} \\ b_j &= \frac{1+(1/3)^j}{4} \\ c_j &= \frac{1+(1/3)^j}{4} + \frac{3-(1/3)^j}{4}e \\ d_j &= \frac{1-(1/3)^j}{4} + \frac{3+(1/3)^j}{4}e \end{aligned}$$

Moreover, it is easy to prove this by induction, assuming that the procession of graphs we conjectured holds true, since

$$a_{j+1} = \frac{2}{3}a_j + \frac{1}{3}b_j,$$

and similarly for the others.

Given this, let us look at ${}^{j}G_1$. I don't want to keep track of slope renormalization constants here, and anyway doing so is irrelevant for the purposes of deciding which path will be the Lazarus path on any particular graph, since everything is renormalized equally. To make the distinction clear though, when any of these slopes are referenced I'll denote them as $S'(\rho)$, for some path ρ. We compute that the first two Lazarus paths on ${}^{0}G$ are the left side of the left 1-cell and the right side of the right 1-cell. For $j > 0$, we use the fact that the boundary data has (by inductive hypothesis) been inherited from previous Lazarus paths on ${}^{j-1}H$. Since Lazarus paths are decreasing in slope, their restriction to this graph (i.e. the left side of the left $(j+1)$-cell and right side of the right $(j+1)$-cell again) must again be the first two Lazarus paths here, as anything with steeper slope through the interior would have been part of a Lazarus path previously. So, let us jump right to ${}^jG_3^2$.



There are essentially four possible Lazarus paths (since the rest are just contractions of these with the same slope or clearly worse), $\rho_0 = (a_j, x, v, z, c_j)$, $\rho_1 = (b_j, y, v, w, d_j)$, $\rho_2 = (a_j, x, y, b_j)$, and $\rho_3 = (c_j, z, w, d_j)$. These slopes can be calculated as

$$\begin{aligned}
S'(\rho_0) &= \frac{3(1/3)^j + e[9/2 - 1/2(1/3)^{j-1}]}{96} \\
S'(\rho_1) &= \frac{3(1/3)^j + e[-9/2 - 1/2(1/3)^{j-1}]}{96} \\
S'(\rho_2) &= \frac{4(1/3)^j}{96} \\
S'(\rho_3) &= \frac{4(1/3)^j - e[4(1/3)^j]}{96}.
\end{aligned}$$

ρ_0 beats out ρ_1 and ρ_2 beats out ρ_3, so we compare ρ_0 and ρ_2. The point of equality is

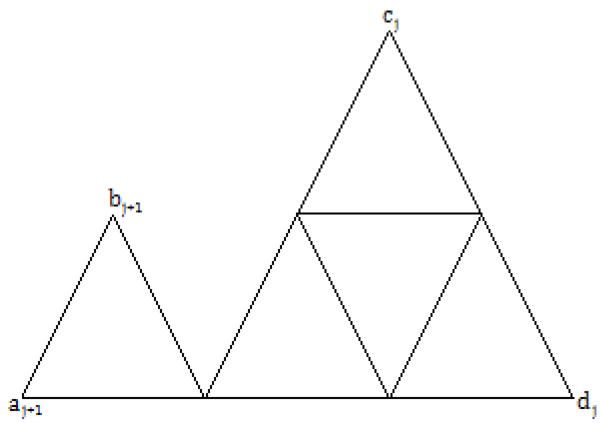
$$e = \frac{2}{3^{j+2} - 3}.$$

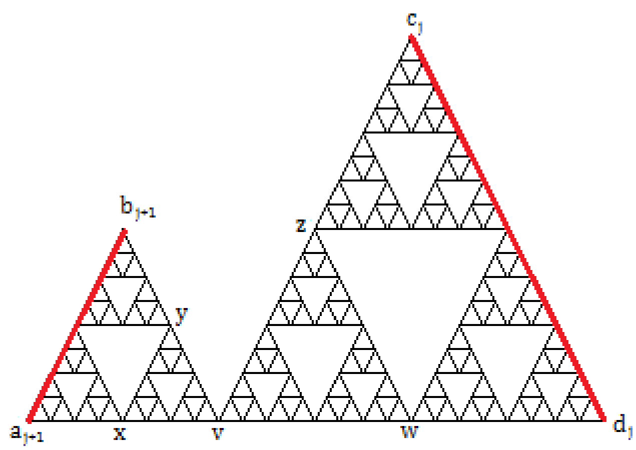
Above this, ρ_0 hits v and so we take care of V_1, with

$$v = \frac{2 + (3 - (1/3)^j)e}{8}.$$

Below this, and we go onto the ${}^jH$ graph. This shows that at least the first of our putative transitions happen the way we would like them to. For ${}^jH_0$ we have:

Note that, since Lazarus paths have decreasing slope, the first two Lazarus paths on ${}^jH_0$ must be the same as the restriction of the previous Lazarus paths on ${}^jG$ to this subgraph, since any path through the interior must have smaller slope. So, we again have





in the general case, and essentially four paths to choose from again: $\rho_0 = (a_{j+1}, x, v, z, c_j)$, $\rho_1 = (b_{j+1}, y, v, w, d_j)$, $\rho_2 = (a_{j+1}, x, y, b_{j+1})$, and $\rho_3 = (c_j, z, w, d_j)$. We can calculate these slopes as

$$S'(\rho_0) = \frac{(1/3)^j + (1/3)^{j+1} - e[(1/3)^j - 3]}{96}$$
$$S'(\rho_1) = \frac{(1/3)^j + (1/3)^{j+1} - e[(1/3)^j + 3]}{96}$$
$$S'(\rho_2) = \frac{4(1/3)^{j+1}}{96}$$
$$S'(\rho_3) = \frac{6(1/3)^{j+1} - 6(1/3)^{j+1}}{96}$$

Again we see that ρ_0 beats ρ_1, and now ρ_3 beats ρ_2 for $e < 1/3$, with ρ_3 beating

ρ_0 for e small. Again we can calculate the point of equality as

$$e = \frac{2}{3^{j+2} + 3}$$

with

$$v = \frac{3 + (1/3)^j + (3 - (1/3)^j)e}{12}.$$

Finally, we notice that if we are below this e cutoff, our next subgraph containing v is indeed ${}^{j+1}G$, so the induction is basically complete. All that remains now is to relate these j values back to levels on the original graph. First, v is computed correctly by level 2 for $e > 1/3$, as we know. On ${}^{0}G$ we get nothing, since our formula would only give us a cutoff of $e = 1/3$ and we are already in this case. On ${}^{0}H$, we get v in the third Lazarus path for $e \in (1/6, 1/3)$, on local level 3, which corresponds with an inherited level of 5, so we compute v correctly for

$$e \in (1/6, 1/3).$$

Then, on local level 3 of ${}^{1}G$ we compute v correctly for $e \in (1/12, 1/6)$, which corresponds to inherited level 5 as well. Since we are always computing the correct v (in the appropriate range) on local level 3 for all ${}^{j}G$ and ${}^{j}H$, an easy induction argument shows that this corresponds to inherited level $j+4$ for ${}^{j}G$, $j > 0$, and inherited level $j+5$ for ${}^{j}H$. Thus, on any (global) level k, we compute v correctly from ${}^{k-5}H$ for

$$e \in \left(\frac{2}{3^{k-3} + 3}, \frac{2}{3^{k-3} - 3} \right)$$

and we compute v correctly from ${}^{k-4}G$ for

$$e \in \left(\frac{2}{3^{k-2} - 3}, \frac{2}{3^{k-3} + 3} \right).$$

Since the latter subsumes the former, if all we care about is computing an upper bound for the level at which v is correctly computed as a function of e, we can lump these results together, and say that by level k,

$$v \text{ is computed correctly for all } e > \frac{2}{3^{k-2} - 3}, \quad k > 4.$$

So, the lower bound for the e values for which V_1 has been computed correctly by level k is $O(1/3^k)$, giving us definite convergence to 0 as $k \to \infty$. Since we know how to deal with the $e = 0$ case already, this is everything we need to show that Theorem 1 is true, completing our proof.

4 Conclusion

Theorem 1 proves the existence of a function u on SG which is a limit of functions u_k on each Γ_k satisfying $\Delta_\infty u_k = 0$. Naturally, we would like to get a much more general result, proving the existence of a function $u : SG \to \mathbb{R}$ as a limit of functions $u_k : \Gamma_k \to \mathbb{R}$ such that $-c^k \Delta_\infty u_k = f_k$, where c is a possible renormalization constant and $f_k = f|_{V_k}$ for some suitably wide class of functions f, so that we could define

$$-\Delta_\infty u = f$$

on SG. The proof of existence of infinity-harmonic functions depended on the use of the Lazarus algorithm, which has no immediate analogue for non-harmonic functions. One could try to construct some sort of similar procedure for non-zero f on the right-hand side of the previous equation, and that is one direction for possible future exploration.

This aside, we can take stock of what has been accomplished. We have a plausible definition for infinity-harmonic functions on SG, and a proof of their existence. Perhaps equally important is the new understanding of how the discrete infinity-harmonic functions behave on the graph approximations $\{\Gamma_k\}$. It suggests that extensions of these results could be found for other, similarly defined fractal sets.

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