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Some New Results on the Combinatorial Laplacian

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Abstract

In this thesis we discuss some new results concerning the combinatorial Laplace operator of a simplicial complex. The combinatorial Laplacian of a simplicial complex encodes information about the relationships between adjacent simplices in the complex. This thesis is divided into two relatively disjoint parts. In the first portion of the thesis, we derive a relationship between the Laplacian spectrum of a simplicial complex and the Laplacian spectra of its covering complexes. A covering of a simplicial complex is built from many copies of simplices of the original complex, maintaining the adjacency relationships between simplices. We show that for dimension at least one, the Laplacian spectrum of a simplicial complex is contained inside the Laplacian spectrum of any of its covering complexes.

In the second part of the thesis, we discuss the spectral recursion formula for computing the eigenvalues of the Laplacian. Almost all simplicial complexes do not satisfy the spectral recursion formula. We look to find a general set of criteria for determining if a given simplicial complex satisfies the spectral recursion. We find that all one-dimensional simplicial complexes that do not contain the three-edge graph P_4 as an induced subcomplex satisfy the spectral recursion formula, and state some preliminary results regarding two-dimensional complexes.

Acknowledgements

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Contents

1 Introduction

The combinatorial Laplacian of a simplicial complex has been extensively studied both in geometry and combinatorics. Combinatorial Laplacians were originally studied on graphs, beginning with Kirchhoff and his study of electrical networks in the mid-1800s. Simplicial complexes can be viewed as generalizations of graphs, and the graph Laplacian was likewise generalized to the combinatorial Laplacian of simplicial complexes, which is studied here. The study of the Laplacian of simplicial complexes is relatively recent, beginning in the mid-1970s, when Dodziuk and Patodi [3] found that by triangulating a Riemannian manifold, the eigenvalues of the combinatorial Laplacian of the triangulation approached the eigenvalues of the continuous Laplacian of the manifold as the mesh size of the triangulation went to zero. In a similar vein, Forman [8] used the combinatorial Laplacian to define combinatorial Ricci curvature, an analogue to the classical Ricci curvature of a Riemannian manifold.

The combinatorial Laplacian has been shown to have many interesting properties. Most intriguing is the fact that some types of simplicial complexes, namely chessboard complexes [9], matching complexes [4], matroid complexes [13], and shifted complexes [7], all have integer Laplacian spectra. Matroid and shifted complexes also satisfy a recursion formula for calculating the Laplacian spectrum in terms of certain subcomplexes [6]. We will study this recursion formula in this thesis.

This thesis is divided into two relatively disjoint parts. In the first part we discuss the relationship between the Laplacian and covering complexes. The motivation for this part of the thesis comes from the search for relationships between the Laplacians of different simplicial complexes. It also comes out of the study of covering complexes, the simplicial analogue of topological covering spaces, and their properties. Covering complexes have many applications outside of combinatorial theory; for example, the theory of covering complexes can be used to show the famous result that subgroups of free groups are free. Covering complexes also can be used to study Galois theory. See [17] for more information about covering complexes.

The combinatorial Laplace operator is not a topological invariant; thus even simplicial maps that preserve the underlying topological structure of a simplicial complex might change the Laplacian. Like topological spaces, complexes can have coverings. Our goal in this part of the thesis is to determine the relationship between the Laplacian of a simplicial complex and the Laplacians of its coverings. Our main theorem is the following relationship between the spectra of the two Laplacians.

Theorem. *Let $(\widetilde{K}, p)$ be a covering complex of simplicial complex K, and let $\widetilde{\Delta}_d$ and Δ_d be the d^{th} Laplacian operators of $\widetilde{K}$ and K, respectively. Then for all $d \geq 1$, $\mathrm{Spec}(\Delta_d) \subseteq \mathrm{Spec}(\widetilde{\Delta}_d)$.*

It is not true that $\mathrm{Spec}(\Delta_0) \subseteq \mathrm{Spec}(\widetilde{\Delta}_0)$, due to the presence of the empty set $\emptyset$, which is considered a (-1)-dimensional face of all simplices of K. We can define an unreduced simplicial complex $\overline{K}$ which does not include $\emptyset$, and a corresponding unreduced Laplacian $\overline{\Delta}_d$, which corresponds to Δ_d for all $d > 0$, but does have the advantage that $\mathrm{Spec}(\overline{\Delta}_0) \subseteq \mathrm{Spec}(\widetilde{\overline{\Delta}}_0)$.

In the second part of the thesis we analyze the spectrum polynomial of a simplicial complex, which encodes information about the eigenvalues of the Laplacian, and the spectral recursion formula of Duval [6]. This recursion formula determines the spectrum polynomial of a simplicial complex in terms of the spectrum polynomials of smaller complexes. Duval drew on inspiration from the Tutte polynomial of a matroid, which satisfies a similar recursion relation as the spectrum polynomial. Kook, Reiner, and Stanton [13] conjectured a Tutte polynomial-like recursion formula for the spectrum polynomial of a matroid, and Kook [12] found such a recursion formula. Duval generalized this formula to arbitrary simplicial complexes and showed that both matroid complexes and shifted complexes satisfy the formula.

Duval admitted that the fact that both matroid complexes and shifted complexes satisfy the spectral recursion formula is somewhat of a coincidence, as the methods of proving that they satisfy the formula are quite different for the two types of complexes. We attempt to find a general characteristic of simplicial complexes that will guarantee that the spectral recursion formula is satisfied.

While we have not found such a general characteristic, we have made strides in that direction. The case of one-dimensional complexes is fairly well understood. Duval shows in [6] that the three-edge graph P_4 does not satisfy spectral recursion. We build on his statement, showing that all one-dimensional complexes that are P_4-free satisfy spectral recursion. We conjecture that the converse of this statement is true also, and that P_4-free graphs are the *only* one-dimensional complexes that satisfy spectral recursion.

The case for two-dimensional complexes is not nearly as well studied. Our goal when beginning our study of two-dimensional complexes was to find two-dimensional complexes that failed to satisfy spectral recursion for a "two-dimensional reason," by which we mean the one-skeleton of the complex satisfied spectral recursion, yet the complex itself did not. We have found some two-dimensional analogues of the three-edge graph, i.e. minimal two-dimensional complexes that do not satisfy spectral recursion. However, at this time it is unknown as to whether there are a finite number of such minimal complexes, or if there are an infinite number.

We give an outline of how the paper is structured. All calculations of combinatorial Laplacians were done using GAP, and all calculations of eigenvalues were done using Mathematica. In Section 2, we give the basic definitions of simplicial complexes, including many operations for forming new simplicial complexes from given ones. We define the boundary operator, and state a formula for computing its adjoint. Using the boundary operator, we define the Homology groups $H_d(K)$ and the Laplace operator $\Delta_d(K)$, and prove the Combinatorial Hodge Theory, which says $H_d(K) \cong \ker(\Delta_d(K))$.

In Section 3 we introduce the notion of a covering complex. Covering complexes are the simplicial complex analogue of covering spaces in topology. We prove our first main theorem, that for dimension at least one, the Laplacian spectrum of a simplicial complex is contained in the Laplacian spectrum of any of its covering complexes.

Section 4 is a survey of results focusing on the spectral recursion formula for calculating the eigenvalues of the combinatorial Laplacian. We begin with a discussion of simplicial pairs, a generalization of simplicial complexes. We then define the spectrum polynomial and state the spectral recursion formula. Duval showed that the P_4 graph does not satisfy spectral recursion. We generalize his result by showing that all cographs, i.e. graphs without P_4 as an induced subgraph, satisfy the spectral recursion formula. Our attempts to prove the converse of this statement led to the following conjecture:

Conjecture. *Let K be a simplicial complex, let $V \subseteq K^{(0)}$, and let $K[V]$ be the induced subcomplex on V. If $K[V]$ does not satisfy spectral recursion, then K does not satisfy spectral recursion.*

The Tutte polynomial has the advantageous property that the matroids produced in the recursion relation also satisfy the Tutte polynomial. Analogous to this, we introduce the notion of strong spectral recursion, and show that a one-dimensional complex satisfies strong spectral recursion if and only if it is P_4-free.

We conclude the thesis with some simple results concerning two-dimensional simplicial complexes. We attempted to find two-dimensional complexes K with the property that the 1-skeleton $K^{(1)}$ satisfied spectral recursion, yet K itself did not. Our reason for doing this was to find a two-dimensional property of a complex, similar to the one-dimensional property of containing an induced P_4, that would prohibit the complex from satisfying spectral recursion.

2 Background

2.1 Simplicial Complexes

This section is devoted to definitions and basic facts about simplicial complexes. The definitions here will be used throughout the thesis. See [14,16] for more information about abstract simplicial complexes.

Definition. *An* ***abstract simplicial complex*** *K is a collection of finite sets that is closed under set inclusion, i.e. if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$.*

We will usually drop the word "abstract," and occasionally the word "simplicial," and just use the term "simplicial complex" or "complex." In addition, in this thesis we will only deal with finite abstract simplicial complexes, i.e. only the case where $|K| < \infty$. A set $\sigma \in K$ is called a *simplex* of K. The *dimension* of a simplex σ is one less than the number of elements of σ. The *dimension* of K is the largest dimension of all of the simplices in K, or is infinite if there is no largest simplex. Since we will only discuss finite complexes, all complexes in this thesis will have finite dimension. We call σ a *d-simplex* if it has dimension d.

The *p-skeleton* of K, written $K^{(p)}$, is the set of all simplices of K of dimension less than or equal to p. The non-empty elements of the set $K^{(0)}$ are called the *vertices* of K. Occasionally we will refer to the 1-simplices as *edges*. We require that the empty set $\emptyset$ be in K for all K, so that $\emptyset \subseteq \sigma$ for all $\sigma \in K$. We say that $\emptyset$ has dimension -1.

A simplicial complex K is *connected* if, for every pair of vertices $u, v \in K^{(0)}$, there is a sequence of vertices $\{v_1, \ldots, v_n\}$ in K such that $v_1 = u$, $v_n = v$, and $\{v_i, v_{i+1}\}$ is an edge in K for all $i = 1, \ldots, n-1$.

Definition. *Let K and L be two abstract simplicial complexes. A map $f : K^{(0)} \to L^{(0)}$ is called a* ***simplicial map*** *if whenever $\{v_0, \ldots, v_d\}$ is a simplex in K, then $\{f(v_0), \ldots, f(v_d)\}$ is a simplex in L.*

While a simplicial map f maps the vertices of K to the vertices of L, we will often speak of f as mapping K to L and write $f : K \to L$; thus if $\sigma \in K$ is a simplex, we will write $f(\sigma)$. Notice that if σ is a d-simplex in K, then $f(\sigma)$ need not be a d-simplex in L, as $f(\sigma)$ might in fact be of lower dimension.

There are multiple ways of creating new simplicial complexes from given ones. If K is a simplicial complex, then any subset $K' \subseteq K$ that is also a simplicial complex (i.e. if $\sigma \in K'$, then $\tau \in K'$ if $\tau \subseteq \sigma$) is called a *subcomplex* of K. If $V \subseteq K^{(0)}$ is a set of vertices, then the *induced subcomplex* $K[V]$ on V is defined by $K[V] = \{\sigma \in K \mid \sigma^{(0)} \subseteq V\}$, i.e. $K[V]$ is the collection of all simplices which have all of their vertices in V.

Given two disjoint complexes K and L (i.e. $K^{(0)} \cap L^{(0)} = \emptyset$), define the *union* $K \cup L$ and the *join* $K \star L$ by

$$K \cup L = \{\sigma \mid \sigma \in K \text{ or } \sigma \in L\}$$
$$K \star L = \{\sigma \cup \tau \mid \sigma \in K, \tau \in L\}$$

Sometimes we will say "simplicial join" in order to distinguish $K \star L$ from other types of joins. It is easy to see that both $K \cup L$ and $K \star L$ are simplicial complexes whenever both K and L are.

Given a d-simplex σ, there are $(d+1)!$ ways of ordering (i.e., listing) the $d+1$ vertices composing σ. We want a way to distinguish between possible orderings. Recall that a permutation is a bijection from a set to itself. A permutation of a finite set is called even if it consists of an even number of transpositions, i.e. interchanges of pairs of elements (see [5] for more information on permutations).

We define an equivalence relation on the set of orderings of σ as follows: we say that two orderings are equivalent if there is an even permutation sending one to the other. It is easy to check (see [16]) that this is an equivalence relation, and for $d > 0$, there are exactly two equivalence classes for each simplex σ.

We call each of these equivalence classes an *orientation* of σ, and a simplex with an orientation is called an *oriented simplex*. An *oriented simplicial complex* K is one for which we have chosen an orientation for each of its simplices.

Given an oriented simplicial complex K, let $C_d(K)$ be the set of all formal $\mathbb{R}$-linear combinations of oriented d-simplices of K. The set $C_d(K)$ is then a vector space over $\mathbb{R}$ with the oriented d-simplices as a basis. Each element of $C_d(K)$ is called a *d-chain* (see [16] for a more formal construction of the d-chains). We will write C_d instead of $C_d(K)$ when the simplicial complex K is clear. In particular, note that $C_{-1}(K) = \mathbb{R}$ for all K, as $\emptyset$ is the only (-1)-simplex of K.

Since the oriented d-simplices form a basis for C_d, we can define linear functions on C_d by defining how they act on the oriented d-simplices. We now define perhaps the most important linear function on C_d, the boundary operator.

Definition. *The* ***boundary operator*** $\partial_d : C_d(K) \to C_{d-1}(K)$ *is the linear function defined for each oriented d-simplex* $\sigma = [v_0, \ldots, v_d]$ *by*

$$\partial_d(\sigma) = \partial_d[v_0, \ldots, v_d] = \sum_{i=0}^{d} (-1)^i [v_0, \ldots, \widehat{v}_i, \ldots, v_d] \tag{1}$$

where $[v_0, \ldots, \widehat{v}_i, \ldots, v_d]$ *is the subset of* $[v_0, \ldots, v_d]$ *obtained by removing the vertex* v_i.

Notice that if $\sigma = [v] \in C_0(K)$, then $\partial_0(\sigma) = \emptyset \in C_{-1}(K)$, and since there are no simplices of dimension -2, $\partial_{-1}(\emptyset) = 0$. If $f : K \to L$ is a simplicial map, we define a homomorphism $f_\# : C_d(K) \to C_d(L)$ by defining it on basis elements (i.e. oriented simplices) as follows:

$$\begin{aligned} f_\#(\sigma) &= f_\#([v_0, \ldots, v_d]) \\ &= \begin{cases} [f(v_0), \ldots, f(v_d)], & \text{if } f(v_0), \ldots, f(v_d) \text{ are distinct} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

We then require $f_\#$ to be a homomorphism by setting $f_\# \left(\sum r_i \sigma_i\right) = \sum r_i f_\#(\sigma_i)$. We call the family $\{f_\#\}$ the *chain map induced by the simplicial map* f.

Technically speaking, each $f_\#$ acts only on one d-chain C_d. When we want to specify which dimension we are working with, we shall write f_d instead of $f_\#$. Chain maps have the special property that they commute with the boundary operator.

Lemma 2.1. *The homomorphism* $f_\#$ *commutes with the boundary operator* ∂*, that is,*

$$f_{d-1} \circ \partial_d = \partial_d \circ f_d.$$

Proof. Since both $f_\#$ and ∂ are linear, we need only look at their action on basis elements. A simple computation gives

$$\begin{aligned} f_{d-1}(\partial_d[v_0, \ldots, v_d]) &= f_{d-1}\left(\sum_{i=0}^{d} (-1)^i [v_0, \ldots, \widehat{v}_i, \ldots, v_d] \right) \\ &= \sum_{i=0}^{d} (-1)^i f_{d-1}[v_0, \ldots, \widehat{v}_i, \ldots, v_d] \\ &= \partial_d(f_d[v_0, \ldots, v_d]). \end{aligned}$$

$\square$

Since we are assuming that $|K| < \infty$, $C_d(K)$ is a finite dimensional vector space for all d, so we can define an inner product $\langle\ \rangle_d$ on $C_d(K)$ as follows: Let $\sigma_1, \ldots, \sigma_n$ be the oriented d-simplices of simplicial complex K, and let $a, b \in C_d$ be arbitrary elements of C_d, which we can write as:

$$a = \sum_{i=1}^{n} a_i \sigma_i \qquad\qquad b = \sum_{i=1}^{n} b_i \sigma_i$$

where the $a_i, b_i \in \mathbb{R}$ (this is possible since the σ_i form a basis for C_d). Then the inner product of a and b is given by

$$\langle a, b \rangle_d = \sum_{i=1}^{n} a_i b_i.$$

It is easy to check that this definition satisfies the properties of an inner product.

Since each boundary operator $\partial_d : C_d \to C_{d-1}$ is a linear map, we can associate to it its adjoint operator $\partial_d^* : C_{d-1} \to C_d$ as the unique linear operator that satisfies

$$\langle \partial_d(a), b \rangle_{d-1} = \langle a, \partial_d^*(b) \rangle_d$$

where $\langle\ \rangle_{d-1}$ and $\langle\ \rangle_d$ are the inner products on C_{d-1} and C_d, respectively, and $a \in C_d, b \in C_{d-1}$. Since ∂_d and ∂_d^* are both linear, they both have associated matrices, which we call $\mathcal{B}_d$ and $\mathcal{B}_d^T$, respectively (where here, $\mathcal{B}_d^T$ is the transpose of $\mathcal{B}_d$, as ∂_d^* is the adjoint of ∂_d).

We now give a way of calculating ∂_d^*. Let $S_d(K)$ be the set of all oriented d-simplices of the simplicial complex K (i.e. the set of basis elements of $C_d(K)$), and let $\tau \in S_{d-1}(K)$. Then define the two sets

$$S_d^+(K, \tau) = \{\sigma \in S_d(K) | \text{the coefficient of } \tau \text{ in } \partial_d(\sigma) \text{ is } +1\}$$
$$S_d^-(K, \tau) = \{\sigma \in S_d(K) | \text{the coefficient of } \tau \text{ in } \partial_d(\sigma) \text{ is } -1\}.$$

Notice that S_d^+ and S_d^- are only defined for $d \geq 0$, and since $S_{-1}(K) = \{\emptyset\}$ with $\partial_0(\sigma) = \emptyset$ for all $\sigma \in S_0(K)$, when $d = 0$ we have $S_0^+(K, \emptyset) = S_0(K)$ and $S_0^-(K, \emptyset) = \emptyset$. We now give an explicit formula for calculating ∂_d^*, which we will use later on in proving Theorem 3.4.

Theorem 2.2. *Let ∂_d^* be the adjoint of the boundary operator ∂_d, and let $\tau \in S_{d-1}(K)$. Then*

$$\partial_d^*(\tau) = \sum_{\sigma' \in S_d^+(K,\tau)} \sigma' \; - \sum_{\sigma'' \in S_d^-(K,\tau)} \sigma'' \; .$$

Proof. Let $f : C_{d-1} \to C_d$ be defined on basis elements by

$$f(\tau) = \sum_{\sigma' \in S_d^+(K,\tau)} \sigma' \; - \sum_{\sigma'' \in S_d^-(K,\tau)} \sigma'' \; .$$

We show that $\langle \partial_d(a), b \rangle_{d-1} = \langle a, f(b) \rangle_d$, for $a \in C_d, b \in C_{d-1}$, for then the function f will satisfy the requirements for the adjoint operator, and since the adjoint is unique, we will have $f = \partial_d^*$. First observe that f is linear, so (since ∂_d is also linear) we only need to show $\langle \partial_d(\sigma), \tau \rangle_{d-1} = \langle \sigma, f(\tau) \rangle_d$ for σ, τ basis elements, i.e. $\sigma \in S_d(K), \tau \in S_{d-1}(K)$.

Look at the term $\langle \partial_d(\sigma), \tau \rangle_{d-1}$. As τ is a single simplex, $\langle \partial_d(\sigma), \tau \rangle_{d-1} \neq 0$ if and only if τ is in the sum $\partial_d(\sigma)$, that is, if and only if $\tau \subseteq \sigma$. Since the coefficient of every term of ∂_d is ± 1, we see that

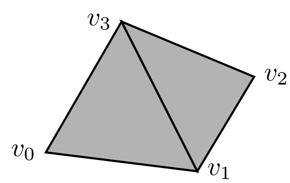
$$\langle \partial_d(\sigma), \tau \rangle_{d-1} = \begin{cases} 1, & \tau \subseteq \sigma \text{ and the coefficient of } \tau \text{ in } \partial_d(\sigma) \text{ is } +1 \\ -1, & \tau \subseteq \sigma \text{ and the coefficient of } \tau \text{ in } \partial_d(\sigma) \text{ is } -1 \\ 0, & \tau \not\subseteq \sigma \end{cases}$$

Now look at the term $\langle \sigma, f(\tau) \rangle_d$. Since σ is a single simplex, $\langle \sigma, f(\tau) \rangle_d \neq 0$ if and only if σ is in the sum $f(\tau)$, that is, if and only if $\sigma \supseteq \tau$. Since the coefficient of every term of $f(\tau)$ is ± 1, we see that

$$\langle \sigma, f(\tau) \rangle_d = \begin{cases} 1, & \sigma \supseteq \tau \text{ and the coefficient of } \sigma \text{ in } f(\tau) \text{ is } +1 \\ -1, & \sigma \supseteq \tau \text{ and the coefficient of } \sigma \text{ in } f(\tau) \text{ is } -1 \\ 0, & \sigma \not\supseteq \tau \end{cases}$$

By the definition of f, however, we have that the coefficient of τ in $\partial_d(\sigma)$ is $+1$ if and only if the coefficient of σ in $f(\tau)$ is $+1$, and the coefficient of τ in $\partial_d(\sigma)$ is -1 if and only if the coefficient of σ in $f(\tau)$ is -1. Thus we have that $\langle \partial_d(\sigma), \tau \rangle_{d-1} = \langle \sigma, f(\tau) \rangle_d$, so by definition of the adjoint, $f = \partial_d^*$. □

Example 2.3. Let K be the following simplicial complex:



So K has four vertices (0-simplices), five edges (1-simplices), and two triangles (2-simplices). Order the 0-simplices $[v_0]$, $[v_1]$, $[v_2]$, $[v_3]$, order the 1-simplices $[v_0, v_1]$, $[v_0, v_3]$, $[v_1, v_2]$, $[v_1, v_3]$, $[v_2, v_3]$, and order the 2-simplices $[v_0, v_1, v_3]$, $[v_1, v_2, v_3]$. Writing the boundary and adjoint boundary operators in matrix form, we get

$$\mathcal{B}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathcal{B}_0^T = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathcal{B}_1 = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \qquad \mathcal{B}_1^T = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\mathcal{B}_2 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad \mathcal{B}_2^T = \begin{bmatrix} 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

Notice that, as claimed, $\mathcal{B}_d^T$ is in fact the transpose of $\mathcal{B}_d$.

With all of this information at hand, we can define the Homology groups of a simplicial complex. First we need a lemma:

Lemma 2.4. *If K is a simplicial complex, the composition $\partial_{d-1} \circ \partial_d = 0$.*

Proof. We need only show that the equation holds for basis elements, as the boundary map is linear. A simple computation gives

$$\begin{aligned}
\partial_{d-1}(\partial_d(\sigma)) &= \partial_{d-1}\left(\sum_{i=0}^{d}(-1)^i[v_0,\ldots,\widehat{v_i},\ldots,v_d]\right) \\
&= \sum_{j<i}(-1)^i(-1)^j[v_0,\ldots,\widehat{v_j},\ldots,\widehat{v_i},\ldots,v_d] \\
&\qquad\qquad + \sum_{j>i}(-1)^{j-1}(-1)^i[v_0,\ldots,\widehat{v_i},\ldots,\widehat{v_j},\ldots,v_d] \\
&= \sum_{j<i}(-1)^i(-1)^j[v_0,\ldots,\widehat{v_j},\ldots,\widehat{v_i},\ldots,v_d] \\
&\qquad\qquad + \sum_{i>j}(-1)^{i-1}(-1)^j[v_0,\ldots,\widehat{v_j},\ldots,\widehat{v_i},\ldots,v_d] \\
&= 0.
\end{aligned}$$

□

As a result, we see that $\mathrm{im}(\partial_{d+1}) \subseteq \ker(\partial_d)$. Thus if we think of $\ker(\partial_d)$ and $\mathrm{im}(\partial_{d+1})$ as groups (they are both abelian groups, since they are vector spaces), we can define the d^{th} *homology group* $H_d(K)$ as the quotient group $H_d(K) = \ker(\partial_d)/\,\mathrm{im}(\partial_{d+1})$. The homology groups of a complex are a topological invariant, that is, if K and K' are homeomorphic as topological spaces, then $H_d(K) = H_d(K')$; see [11] for a proof. We will see how the homology groups are related the the Laplace operator in the next section.

2.2 The Combinatorial Laplacian

We now define the combinatorial Laplace operator and the Laplacian spectrum for a simplicial complex.

Definition. *Let K be a finite oriented complex. The* ***d^{th} combinatorial Laplacian*** *is the linear operator $\Delta_d : C_d(K) \to C_d(K)$ given by*

$$\Delta_d = \partial_{d+1} \circ \partial_{d+1}^* + \partial_d^* \circ \partial_d.$$

The d^{th} *Laplacian matrix* of K, denoted $\mathcal{L}_d$, with respect to the standard bases for C_d and C_{d-1}, is the matrix representation of Δ_d, given by

$$\mathcal{L}_d = \mathcal{B}_{d+1}\mathcal{B}_{d+1}^T + \mathcal{B}_d^T\mathcal{B}_d.$$

Note that the combinatorial Laplacian is actually a set of operators, one for each dimension in the complex. Since the product of a matrix and its transpose is symmetric, both $\mathcal{B}_d^T\mathcal{B}_d$ and $\mathcal{B}_{d+1}\mathcal{B}_{d+1}^T$ are symmetric, and thus so is $\mathcal{L}_d$. As a result, $\mathcal{L}_d$ is real diagonalizable, so the Laplacian Δ_d has a complete set of real eigenvalues. The d^{th} *Laplacian spectrum* of a finite oriented simplicial complex K, denoted $\mathrm{Spec}(\Delta_d(K))$, is the multiset of eigenvalues of the Laplacian $\Delta_d(K)$.

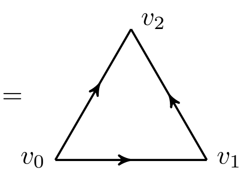
The Laplacian acts on an oriented simplicial complex. However, simplicial complexes are not naturally oriented. Notice that when we constructed the boundary operator, and thus the Laplacian, we gave the simplicial complex an arbitrary orientation. This might lead one to believe that the same simplicial complex could produce different Laplacian spectra for different orientations of its simplices. However, this is not the case, as is shown in the following theorem. See [10] for the proof.

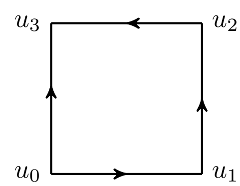
Theorem 2.5. *Let K be a finite simplicial complex. Then Spec$(\Delta_d(K))$ is independent of the choice of orientation of the d-simplices of K.*

As a result, we can speak of the Laplacian spectrum of a simplicial complex without regard to its orientation.

Every simplicial complex can be embedded in $\mathbb{R}^n$ for some n, and thus can be considered a topological space (see [16] for the proof). It is possible for two different simplicial complexes to embed in $\mathbb{R}^n$ as the same topological space; any cycle, for example, is homeomorphic to a circle. A natural question to ask, then, is whether the Laplacian is a topological invariant; that is, whether different simplicial complexes that are homeomorphic as topological spaces have the same Laplacian. The answer is no, the Laplacian is not a topological invariant. We show this with an example.

Example 2.6. Let K_1 and K_2 be the following one-dimensional oriented simplicial complexes:

K_1= 

K_2= 

Notice that both K_1 and K_2 are graphs, and the edges of the graphs are exactly the 1-simplices of the complexes. Clearly K_1 and K_2 are topologically equivalent; they are both cycles, and thus both homeomorphic to the circle. However, it is easily seen that

$$\mathcal{L}_1(K_1) = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}, \qquad \mathcal{L}_1(K_2) = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 2 & 0 & 1 \\ -1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \end{bmatrix}.$$

Thus the Laplacian operators on K_1 and K_2 are not the same. Even the spectra of the two Laplacians are not the same, as $\text{Spec}(\Delta_1(K_1)) = \{0, 3, 3\}$ and $\text{Spec}(\Delta_1(K_2)) = \{0, 2, 2, 4\}$.

It should be noted, however, that the kernel of the Laplacian is a topological invariant, which we now show. The following argument is inspired by [19].

Lemma 2.7. *The kernel of the Laplacian can be characterized by*

$$\ker(\Delta_d) = \{a \in C_d \mid \partial_d(a) = \partial^*_{d+1}(a) = 0\}.$$

Proof. First assume that $\partial_d(a) = \partial^*_{d+1}(a) = 0$. Then by definition

$$\Delta_d(a) = \partial_{d+1}(\partial^*_{d+1}(a)) + \partial^*_d(\partial_d(a)) = \partial_{d+1}(0) + \partial^*_d(0) = 0,$$

so $a \in \ker(\Delta_d)$. Now suppose $a \in \ker(\Delta_d)$. Then since $\partial_{d+1}(\partial^*_{d+1}(a)) + \partial^*_d(\partial_d(a)) = 0$, we have

$$\begin{aligned} 0 &= \langle \partial_{d+1}(\partial^*_{d+1}(a)) + \partial^*_d(\partial_d(a)), a\rangle \\ &= \langle \partial_{d+1}(\partial^*_{d+1}(a)), a\rangle + \langle \partial^*_d(\partial_d(a)), a\rangle \text{ , since the inner product is bilinear} \\ &= \langle \partial^*_{d+1}(a), \partial^*_{d+1}(a)\rangle + \langle \partial_d(a), \partial_d(a)\rangle \text{ , since } \partial_d, \partial^*_d \text{ are adjoint operators.} \end{aligned}$$

Since $\langle b, b\rangle > 0$ for all $b \neq 0$, this means that $\partial_d(a) = \partial^*_{d+1}(a) = 0$, completing the proof. □

Recall that if V is a vector space with inner product $\langle\,\rangle$ and U is a subspace of V, then the subspace $U^\perp = \{v \in V \mid \langle v, u\rangle = 0 \text{ for all } u \in U\}$ is called the *orthogonal complement* of U. We can always decompose V as the internal direct sum $V = U \oplus U^\perp$. Since $\ker(\Delta_d) \subseteq C_d(K)$, it too has an orthogonal complement $(\ker(\Delta_d))^\perp$ such that $C_d(K) = \ker(\Delta_d) \oplus (\ker(\Delta_d))^\perp$.

Lemma 2.8. *The orthogonal complement of* $\ker(\Delta_d) \subseteq C_d(K)$ *is* $(\ker(\Delta_d))^\perp = \operatorname{im}(\Delta_d)$.

Proof. First we show $\operatorname{im}(\Delta_d)) \subseteq (\ker(\Delta_d))^\perp$. To do this, we must show that if $a \in \operatorname{im}(\Delta_d)$, then $\langle a, b\rangle = 0$ for all $b \in \ker(\Delta_d)$. Since $a \in \operatorname{im}(\Delta_d)$, there is a $c \in C_d$ such that $\Delta_d(c) = a$. Thus for all $b \in \ker(\Delta_d)$ we have

$$\begin{aligned} \langle a, b\rangle = \langle \Delta_d(c), b\rangle &= \langle c, \Delta_d(b)\rangle \text{ , since } \Delta_d \text{ is symmetric} \\ &= \langle c, 0\rangle \text{ , since } b \in \ker(\Delta_d) \\ &= 0. \end{aligned}$$

Thus $\operatorname{im}(\Delta_d) \subseteq (\ker(\Delta_d))^\perp$. Now we show the opposite inclusion. Since Δ_d is a symmetric linear map on C_d, by the Spectral Theorem (see [15], Theorem 15.7.1) there is a complete set of orthonormal eigenvectors of Δ_d, i.e. there exist $v_1, \ldots, v_n \in C_d$ such that $\langle v_i, v_j\rangle = \delta_{ij}$ and $\Delta_d(v_i) = \lambda_i v_i$ for some eigenvalue $\lambda_i \in \mathbb{R}$. Without loss of generality we can assume $\lambda_1 = \cdots = \lambda_k = 0$ and $\lambda_i \neq 0$ for $i = k+1, \ldots, n$, i.e. $v_1, \ldots, v_k$ form a basis for $\ker(\Delta_d)$ and $v_{k+1}, \ldots, v_n$ form a basis for $(\ker(\Delta_d))^\perp$.

For any $a \in C_d$, we can write $a = \sum \alpha_i v_i$ with $\alpha_i \in \mathbb{R}$. If $a \in (\ker(\Delta_d))^\perp$, then $\langle a, b\rangle = 0$ for all $b \in \ker(\Delta_d)$. In particular, $\langle a, v_i\rangle = 0$ for all $i = 1, \ldots, k$. But $\langle a, v_i\rangle = \alpha_i$, so this means $\alpha_i = 0$ for all $i = 1, \ldots, k$, so we can write

$$a = \sum_{i=k+1}^{n} \alpha_i v_i.$$

We claim that $a \in \operatorname{im}(\Delta_d)$. Define $c \in C_d$ by

$$c = \sum_{i=k+1}^{n} \frac{\alpha_i}{\lambda_i} v_i.$$

The vector c is well-defined, since $\lambda_i \neq 0$ for all $i = k+1, \ldots, n$. Then

$$\begin{aligned} \Delta_d(c) = \Delta_d\left(\sum_{i=k+1}^{n} \frac{\alpha_i}{\lambda_i} v_i\right) &= \sum_{i=k+1}^{n} \Delta_d\left(\frac{\alpha_i}{\lambda_i} v_i\right) \\ &= \sum_{i=k+1}^{n} \frac{\alpha_i}{\lambda_i} \Delta_d(v_i) = \sum_{i=k+1}^{n} \frac{\alpha_i}{\lambda_i} \lambda_i v_i \\ &= \sum_{i=k+1}^{n} \alpha_i v_i = a. \end{aligned}$$

Thus $(\ker(\Delta_d))^\perp \subseteq \operatorname{im}(\Delta_d)$, so they are equal. □

As a result, we can decompose C_d as the internal direct sum $C_d = \ker(\Delta_d) \oplus \operatorname{im}(\Delta_d)$. In fact, we can go further, as is seen in the following lemma:

Lemma 2.9. *The space of d-chains $C_d(K)$ can be decomposed as $C_d(K) = \ker(\Delta_d) \oplus \operatorname{im}(\partial_{d+1}) \oplus \operatorname{im}(\partial_d^*)$.*

Proof. We have just shown that we can decompose C_d as $C_d = \ker(\Delta_d) \oplus \operatorname{im}(\Delta_d)$. Thus it suffices to show that we can decompose $\operatorname{im}(\Delta_d)$ as $\operatorname{im}(\Delta_d) = \operatorname{im}(\partial_{d+1}) \oplus \operatorname{im}(\partial_d^*)$. First we show that $\operatorname{im}(\partial_{d+1})$ and $\operatorname{im}(\partial_d^*)$ are orthogonal, so that $\operatorname{im}(\partial_{d+1}) \oplus \operatorname{im}(\partial_d^*)$ is well defined; i.e. we must show that if $a \in \operatorname{im}(\partial_{d+1})$ and $b \in \operatorname{im}(\partial_d^*)$, then $\langle a, b\rangle = 0$. Since $a \in \operatorname{im}(\partial_{d+1})$, there is an $a' \in C_{d+1}$ such that $\partial_{d+1}(a') = a$. Similarly, since $b \in \operatorname{im}(\partial_d^*)$, there is a $b' \in C_{d-1}$ such that $\partial_d^*(b') = b$. We then have

$$\begin{aligned}\langle a, b\rangle &= \langle \partial_{d+1}(a'), \partial_d^*(b')\rangle \\ &= \langle \partial_d(\partial_{d+1}(a')), b'\rangle \text{ , since } \partial_d, \partial_d^* \text{ are adjoint operators} \\ &= \langle 0, b'\rangle \text{ , by Lemma 2.4} \\ &= 0.\end{aligned}$$

Thus the direct sum $\operatorname{im}(\partial_{d+1}) \oplus \operatorname{im}(\partial_d^*)$ is well-defined. Now we show $\operatorname{im}(\Delta_d) \subseteq \operatorname{im}(\partial_{d+1}) \oplus \operatorname{im}(\partial_d^*)$. Let $a \in \operatorname{im}(\Delta_d)$, so there is a $b \in C_d$ such that $\Delta_d(b) = a$. By definition $\Delta_d(b) = \partial_{d+1}(\partial_{d+1}^*(b)) + \partial_d^*(\partial_d(b))$. Setting $\alpha = \partial_{d+1}^*(b)$ and $\beta = \partial_d(b)$, this becomes $a = \Delta_d(b) = \partial_{d+1}(\alpha) + \partial_d^*(\beta)$, so $a = u + v$ for $u \in \operatorname{im}(\partial_{d+1})$ and $v \in \operatorname{im}(\partial_d^*)$, so $\operatorname{im}(\Delta_d) \subseteq \operatorname{im}(\partial_{d+1}) \oplus \operatorname{im}(\partial_d^*)$.

Now we show that $\ker(\Delta_d)$ is orthogonal to both $\operatorname{im}(\partial_{d+1})$ and $\operatorname{im}(\partial_d^*)$, i.e. if $v \in \ker(\Delta_d)$, $a \in \operatorname{im}(\partial_{d+1})$, and $b \in \operatorname{im}(\partial_d^*)$, then $\langle v, a\rangle = \langle v, b\rangle = 0$. Since $a \in \operatorname{im}(\partial_{d+1})$, there is an $a' \in C_{d+1}$ such that $\partial_{d+1}(a') = a$, and since $b \in \operatorname{im}(\partial_d^*)$, there is a $b' \in C_{d-1}$ such that $\partial_d^*(b') = b$. Thus

$$\begin{aligned}\langle v, a\rangle &= \langle v, \partial_{d+1}(a')\rangle \\ &= \langle \partial_{d+1}^*(v), a'\rangle \text{ , since } \partial_{d+1}, \partial_{d+1}^* \text{ are adjoint operators} \\ &= \langle 0, a'\rangle \text{ , by Lemma 2.7} \\ &= 0 \\ \langle v, b\rangle &= \langle v, \partial_d^*(b')\rangle \\ &= \langle \partial_d(v), b'\rangle \text{ , since } \partial_d, \partial_d^* \text{ are adjoint operators} \\ &= \langle 0, b'\rangle \text{ , by Lemma 2.7} \\ &= 0\end{aligned}$$

As a result, $\operatorname{im}(\partial_{d+1})$ and $\operatorname{im}(\partial_d^*)$ are both contained in the orthogonal complement of $\ker(\Delta_d)$, and thus so is their direct sum, i.e. $\operatorname{im}(\partial_{d+1}) \oplus \operatorname{im}(\partial_d^*) \subseteq (\ker(\Delta_d))^\perp = \operatorname{im}(\Delta_d)$, the last equality by Lemma 2.8. Thus $\operatorname{im}(\Delta_d) = \operatorname{im}(\partial_{d+1}) \oplus \operatorname{im}(\partial_d^*)$, so we have $C_d = \ker(\Delta_d) \oplus \operatorname{im}(\Delta_d) = \ker(\Delta_d) \oplus \operatorname{im}(\partial_{d+1}) \oplus \operatorname{im}(\partial_d^*)$. □

Lemma 2.10. *The kernel of the map ∂_d can be decomposed as $\ker(\partial_d) = \ker(\Delta_d) \oplus \operatorname{im}(\partial_{d+1})$.*

Proof. First observe that both $\ker(\Delta_d)$ and $\operatorname{im}(\partial_{d+1})$ are contained in $\ker(\partial_d)$, the former by Lemma 2.7 and the latter by Lemma 2.4, so $\ker(\Delta_d) \oplus \operatorname{im}(\partial_{d+1}) \subseteq \ker(\partial_d)$ (note that this is a valid direct sum by the previous lemma).

We now must show the the opposite inclusion, that $\ker(\partial_d) \subseteq \ker(\Delta_d) \oplus \operatorname{im}(\partial_{d+1})$. We do this by showing that $\ker(\partial_d)$ is orthogonal to $\operatorname{im}(\partial_d^*)$. Let $v \in \ker(\partial_d)$ and $u \in \operatorname{im}(\partial_d^*)$, so $\partial_d(v) = 0$ and there is a $w \in C_{d-1}$ with $\partial_d^*(w) = u$. Then $\langle v, u\rangle = \langle v, \partial_d^*(w)\rangle = \langle \partial_d(v), w\rangle = \langle 0, w\rangle = 0$. Thus $\ker(\partial_d) \subseteq (\operatorname{im}(\partial_d^*))^\perp = \ker(\Delta_d) \oplus \operatorname{im}(\partial_{d+1})$, completing the proof. □

We can now prove the Combinatorial Hodge Theory. With all that we have done to this point, the proof is now trivial.

Theorem 2.11. *If K is a simplicial complex, then* $\ker(\Delta_d(K)) \cong H_d(K)$.

Proof. By definition $H_d(K) = \ker(\partial_d)/\operatorname{im}(\partial_{d+1})$. Thus by Lemma 2.10, we have

$$H_d(K) = \ker(\partial_d)/\operatorname{im}(\partial_{d+1}) = (\ker(\Delta_d) \oplus \operatorname{im}(\partial_{d+1}))/\operatorname{im}(\partial_{d+1}) = \ker(\Delta_d).$$

□

Since the homology groups $H_d(K)$ are a topological invariant, as a result of Theorem 2.11 this means that $\ker(\Delta_d(K))$ is a topological invariant as well. In light of Example 2.6, this is most likely the only topological invariant of the Laplacian.

Occasionally, we will want to work with a modified definition of a simplicial complex, in which $\emptyset \notin K$. Then $C_{-1}(K) = \emptyset$, so $\partial_0(\sigma) = 0$ for all $\sigma \in C_0(K)$. We call such complexes without the empty set *unreduced simplicial complexes* and denote them by $\overline{K}$. Everything we have done so far holds for unreduced simplicial complexes. We denote the homology groups of an unreduced simplicial complex by $\overline{H}_d$, and call them *unreduced homology groups*. Similarly, we denote by $\overline{\Delta}_d$ the *unreduced Laplacian operator.*[1]

Any simplicial complex can be turned into an unreduced simplicial complex by simply removing the empty set, and vice-versa. For a given simplicial complex, it is clear that for $d \geq 1$, $H_d = \overline{H}_d$ and $\Delta_d = \overline{\Delta}_d$. Observe that $\overline{\Delta}_0 = \partial_1 \circ \partial_1^*$, since $\partial_0 = 0$.

3 Covering Complexes

We can think of simplicial complexes as topological spaces. As we have just shown, however, the Laplacian is not a topological invariant. Thus, while two complexes might be topologically homeomorphic, they could have very different Laplacian spectra. Our goal in this section is to show that if two simplicial complexes are related by a covering map, then their Laplacian spectra are also related. We begin with the definition of a covering complex. This definition comes from [17], and is similar to the definition of a topological covering space.

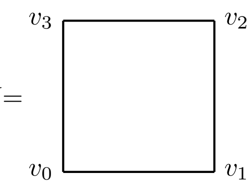
Definition. *Let K be a simplicial complex. A pair $(\widetilde{K}, p)$ is a* ***covering complex*** *of K if:*

1. *$\widetilde{K}$ is a connected simplicial complex.*
2. *$p : \widetilde{K} \to K$ is a simplicial map.*
3. *For every simplex $\sigma \in K$, $p^{-1}(\sigma)$ is a union of pairwise disjoint simplices, $p^{-1}(\sigma) = \bigcup \widetilde{\sigma}_i$, with $p|_{\widetilde{\sigma}_i} : \widetilde{\sigma}_i \to \sigma$ a bijection for each i.*

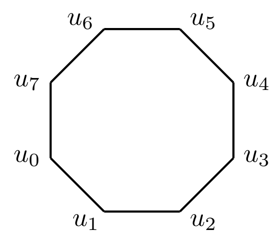
[1] This is very non-standard terminology. In fact, most people would call what we have defined as the homology groups the reduced homology groups, and would call what we have defined as the unreduced homology groups simply the homology groups; similarly for the Laplacian. However, as we work almost exclusively with simplicial complexes with $\emptyset$, we have decided to use the terminology given.

Example 3.1. Let K and $\widetilde{K}$ be the following simplicial complexes:

$K=$



$\widetilde{K}=$



We see that $\widetilde{K}$ is connected. Define the map $p : \widetilde{K}^{(0)} \to K^{(0)}$ by

$$p(u_i) = \begin{cases} v_i & 0 \leq i < 4 \\ v_{i-4} & 4 \leq i \leq 7 \end{cases}$$

One can easily check that p is a simplicial map and that condition (3) above is satisfied, so that $(\widetilde{K}, p)$ is a covering complex of K.

Covering complexes are the simplicial complex equivalent of the covering space of a topological space. The reason we require $\widetilde{K}$ to be connected is to exclude the trivial case where $\widetilde{K}$ is the disjoint union of some number of copies of K.

Since a covering is a simplicial map, there is a chain map associated to it. Let $(\widetilde{K}, p)$ be a covering of an oriented complex K. Define the *chain covering map* $p_\# : C_d(\widetilde{K}) \to C_d(K)$ to be the chain map induced by the covering map p. Notice that by definition of p, if $\sigma = \{v_0, \ldots, v_d\} \in S_d(\widetilde{K})$, then $p(\sigma) = \{p(v_0), \ldots, p(v_d)\} \in S_d(K)$ (i.e. the $p(v_i)$ are distinct), so we can define $p_\#$ on basis elements by

$$p_\#(\sigma) = p_\#([v_0, \ldots, v_d]) = [p(v_0), \ldots, p(v_d)].$$

Again, if we want to specify which dimension the chain covering acts on, we will write p_d instead of $p_\#$. By Lemma 2.1, we see that $p_\#$ commutes with the boundary operator ∂. Normally, a chain map will not commute with the adjoint boundary operator. We now show, however, that for the chain covering they do commute. We will use the following lemma to show that the Laplacian Δ_d commutes with the chain covering $p_\#$ when $d \geq 1$.

Lemma 3.2. *If $d \geq 1$, the adjoint boundary operator ∂_d^* commutes with the chain covering $p_\#$, that is,*

$$p_d \circ \partial_d^* = \partial_d^* \circ p_{d-1}.$$

Proof. Since both ∂^* and $p_\#$ are linear, we only need to look at one basis element $\tau \in S_{d-1}(\widetilde{K})$, that is we must show

$$p_d \circ \partial_d^*(\tau) = \partial_d^* \circ p_{d-1}(\tau).$$

Using the formula for ∂^* from Theorem 2.2, we see that

$$\begin{aligned} p_d \circ \partial_d^*(\tau) &= p_d \left(\sum_{\sigma' \in S_d^+(\widetilde{K}, \tau)} \sigma' - \sum_{\sigma'' \in S_d^-(\widetilde{K}, \tau)} \sigma'' \right) \\ &= \sum_{\sigma' \in S_d^+(\widetilde{K}, \tau)} p_d(\sigma') - \sum_{\sigma'' \in S_d^-(\widetilde{K}, \tau)} p_d(\sigma''), \end{aligned}$$

the last step because $p_\#$ is linear. In addition, we see that

$$\partial_d^* \circ p_{d-1}(\tau) = \sum_{\eta' \in S_d^+(K, p_{d-1}(\tau))} \eta' - \sum_{\eta'' \in S_d^-(K, p_{d-1}(\tau))} \eta''.$$

First we show that for every $\sigma \in S_d(\widetilde{K})$ such that $\sigma \supseteq \tau$, there is exactly one $\eta \in S_d(K)$ such that $\eta \supseteq p(\tau)$ and $p(\sigma) = \eta$, and conversely (i.e. for every $\eta \in S_d(K)$ with $\eta \supseteq p(\tau)$ there is exactly one $\sigma \in S_d(\widetilde{K})$ with $\sigma \supseteq \tau$ and $p(\sigma) = \eta$).

Pick a $\sigma \in S_d(\widetilde{K})$ with $\sigma \supseteq \tau$. Since $p|_\sigma$ is a bijection, $p(\sigma)$ is unique and is in $S_d(K)$. But $\tau \subseteq \sigma$, so $p(\tau) \subseteq p(\sigma)$, so $\eta = p(\sigma)$ is the unique d-simplex satisfying the requirements.

Now pick an $\eta \in S_d(K)$ with $\eta \supseteq p(\tau)$. Look at $p^{-1}(\eta) = \bigcup \sigma_i$ with $\sigma_i \cap \sigma_j = \emptyset$ if $i \neq j$ and $p|_{\sigma_i}$ a bijection. Since $p(\tau) \subseteq \eta$, $p^{-1}(p(\tau)) \subseteq p^{-1}(\eta)$. But $\tau \subseteq p^{-1}(p(\tau))$, so $\tau \subseteq p^{-1}(\eta)$. Since τ is a simplex and $\dim(\tau) \geq 0$ (as $\tau \in S_{d-1}(\widetilde{K})$ and $d \geq 1$), τ is connected, so it lies in exactly one of the σ_i in the inverse image of η. Call this unique simplex $\sigma \in S_d(\widetilde{K})$. Then $p(\sigma) = \eta$ and $\tau \subseteq \sigma$, with σ clearly unique by construction.

Now observe that since $p_\#$ simply assigns an orientation to each simplex in addition to performing the action of p, the above statement also holds for $p_\#$, i.e. if $\tau \in C_{d-1}(\widetilde{K})$ a basis element, then for every $\sigma \in C_d(\widetilde{K})$ a basis element such that $\sigma \supseteq \tau$, there is exactly one $\eta \in C_d(K)$ a basis element such that $\eta \supseteq p_{d-1}(\tau)$ and $p_d(\sigma) = \eta$, and for every $\eta \in C_d(K)$ a basis element with $\eta \supseteq p_{d-1}(\tau)$ there is exactly one $\sigma \in C_d(\widetilde{K})$ a basis element with $\sigma \supseteq \tau$ and $p_d(\sigma) = \eta$. Thus we see that for every term in $p_d \circ \partial_d^*(\tau)$, there is exactly one term in $\partial_d^* \circ p_{d-1}(\tau)$, and vice-versa; we now show that these terms are equal.

First, pick a $\sigma \in S_d^+(\widetilde{K}, \tau)$. We know that $p(\sigma) = \eta$ for some $\eta \in C_d(K)$. There are two cases, corresponding to $p_\#$ either preserving the orientation of σ or reversing it:

1. $p_d(\sigma) = \eta$
2. $p_d(\sigma) = -\eta$

First we show the case for (1). By definition, $\sigma \in S_d^+(\widetilde{K}, \tau)$ if and only if $p_d(\sigma) \in S_d^+(K, p_{d-1}(\tau))$, which is true if and only if $\eta \in S_d^+(K, p_{d-1}(\tau))$. Now η is a basis element of $C_d(K)$, so $p_d(\sigma)$ is also. Thus the coefficient of the basis element η in $\partial_d^* \circ p_{d-1}(\tau)$ is $+1$, and the coefficient of basis element $p_d(\sigma) = \eta$ in $p_d \circ \partial_d^*(\tau)$ is $+1$, proving case (1).

Now we prove the case for (2). By definition, $\sigma \in S_d^+(\widetilde{K}, \tau)$ if and only if $p_d(\sigma) \in S_d^+(K, p_{d-1}(\tau))$, which is true if and only if $-\eta \in S_d^+(K, p_{d-1}(\tau))$, which is true if and only if $\eta \in S_d^-(K, p_{d-1}(\tau))$ (this last equivalence is because switching the orientation switches the sign). Now η is a basis element of $C_d(K)$, so $-p_d(\sigma)$ is a basis element of $C_d(K)$. Thus the coefficient of basis element η in $\partial_d^* \circ p_{d-1}(\tau)$ is -1, and the coefficient of non-basis element $p_d(\sigma)$ in $p_d \circ \partial_d^*(\tau)$ is $+1$. But we want everything in terms of basis elements, so the coefficient of basis element $-p_d(\sigma) = \eta$ in $p_d \circ \partial_d^*(\tau)$ is -1, proving case (2).

Now pick a $\sigma \in S_d^-(\widetilde{K}, \tau)$. We know that $p(\sigma) = \eta$ for some $\eta \in C_d(K)$. There are two cases, corresponding to $p_\#$ either preserving the orientation of σ or reversing it:

1. $p_d(\sigma) = \eta$
2. $p_d(\sigma) = -\eta$

First we show the case for (1). By definition, $\sigma \in S_d^-(\widetilde{K}, \tau)$ if and only if $p_d(\sigma) \in S_d^-(K, p_{d-1}(\tau))$, which is true if and only if $\eta \in S_d^-(K, p_{d-1}(\tau))$. Now η is a basis element of $C_d(K)$, so $p_d(\sigma)$ is also. Thus the coefficient of the basis element η in $\partial_d^* \circ p_{d-1}(\tau)$ is -1, and the coefficient of basis element $p_d(\sigma) = \eta$ in $p_d \circ \partial_d^*(\tau)$ is -1, proving case (1).

Now we prove the case for (2). By definition, $\sigma \in S_d^-(\widetilde{K}, \tau)$ if and only if $p_d(\sigma) \in S_d^-(K, p_{d-1}(\tau))$, which is true if and only if $-\eta \in S_d^-(K, p_{d-1}(\tau))$, which is true if and only if $\eta \in S_d^+(K, p_{d-1}(\tau))$ (this last equivalence is because switching the orientation switches the sign). Now η is a basis element of $C_d(K)$, so $-p_d(\sigma)$ is a basis element of $C_d(K)$. Thus the coefficient of basis element η in $\partial_d^* \circ p_{d-1}(\tau)$ is $+1$, and the coefficient of non-basis element $p_d(\sigma)$ in $p_d \circ \partial_d^*(\tau)$ is -1. But we want everything in terms of basis elements, so the coefficient of basis element $-p_d(\sigma) = \eta$ in $p_d \circ \partial_d^*(\tau)$ is $+1$, proving case (2).

Thus we have that $p_d \circ \partial_d^*(\tau) = \partial_d^* \circ p_{d-1}(\tau)$, completing the proof. □

With Lemmas 2.1 and 3.2, we can now prove that the Laplacian and the chain covering commute.

Theorem 3.3. *If $d \geq 1$, the Laplacian Δ_d commutes with the chain covering $p_\#$, that is,*

$$\Delta_d \circ p_d = p_d \circ \Delta_d.$$

Proof. By definition, we have that $\Delta_d = \partial_{d+1} \circ \partial_{d+1}^* + \partial_d^* \circ \partial_d$. By Lemma 2.1, we know that $\partial_d \circ p_d = p_{d-1} \circ \partial_d$, and by Lemma 3.2, since $d \geq 1$ we know that $\partial_d^* \circ p_{d-1} = p_d \circ \partial_d^*$. Thus we have that

$$\begin{aligned}
(\partial_{d+1} \circ \partial_{d+1}^* + \partial_d^* \circ \partial_d) \circ p_d &= (\partial_{d+1} \circ \partial_{d+1}^*) \circ p_d + (\partial_d^* \circ \partial_d) \circ p_d \\
&= \partial_{d+1} \circ (\partial_{d+1}^* \circ p_d) + \partial_d^* \circ (\partial_d \circ p_d) \\
&= \partial_{d+1} \circ (p_{d+1} \circ \partial_{d+1}^*) + \partial_d^* \circ (p_{d-1} \circ \partial_d) \\
&= (\partial_{d+1} \circ p_{d+1}) \circ \partial_{d+1}^* + (\partial_d^* \circ p_{d-1}) \circ \partial_d \\
&= (p_d \circ \partial_{d+1}) \circ \partial_{d+1}^* + (p_d \circ \partial_d^*) \circ \partial_d \\
&= p_d \circ (\partial_{d+1} \circ \partial_{d+1}^*) + p_d \circ (\partial_d^* \circ \partial_d) \\
&= p_d \circ (\partial_{d+1} \circ \partial_{d+1}^* + \partial_d^* \circ \partial_d).
\end{aligned}$$

Thus $\Delta_d \circ p_d = p_d \circ \Delta_d$, completing the proof. □

We can now prove that the spectrum of a covering complex contains the spectrum of the original complex. Recall that the d^{th} *Laplacian spectrum* of a simplicial complex K, denoted Spec($\Delta_d(K)$), is the multiset of eigenvalues of the Laplacian $\Delta_d(K)$.

Theorem 3.4. *Let $(\widetilde{K}, p)$ be a covering complex of simplicial complex K, and let $\widetilde{\Delta}_d$ and Δ_d be the Laplacian operators of $\widetilde{K}$ and K, respectively. Then for all $d \geq 1$, Spec(Δ_d) $\subseteq$ Spec($\widetilde{\Delta}_d$).*

Proof. By definition the map $p_\#$ is surjective. Let $\ker(p_\#)$ be the kernel of $p_\#$. Then $\widetilde{\Delta}_d$ carries $\ker(p_\#)$ to itself, for if $\sigma \in \ker(p_\#)$, then $p_\#(\sigma) = 0$, which implies $\Delta_d(p_\#(\sigma)) = 0$, which by Theorem 3.3 (since $d \geq 1$) implies that $p_\#(\widetilde{\Delta}_d(\sigma)) = 0$, which implies that $\widetilde{\Delta}_d(\sigma) \in \ker(p_\#)$.

Choose a basis $v_1, \ldots, v_k$ for $\ker(p_\#)$, and choose $u_1, \ldots, u_j \in C_d(\widetilde{K})$ so that $v_1, \ldots, v_k, u_1, \ldots, u_j$ is a basis for $C_d(\widetilde{K})$. Then $p_\#(u_1), \ldots, p_\#(u_j)$ is a basis for $C_d(K)$. (To see why, suppose not; then we can write $\sum \alpha_i p_\#(u_i) = 0$ with the α_i not all 0. But then since $p_\#$ is linear, that means $p_\#(\sum \alpha_i u_i) = 0$, which means $\sum \alpha_i u_i \in \ker(p_\#)$, a contradiction, since we chose the u_i so that this would not be true.) Let M be the matrix for Δ_d with respect to the basis $p_\#(u_1), \ldots, p_\#(u_j)$, and let N be the matrix for $\widetilde{\Delta}_d$ restricted to $\ker(p_\#)$, with respect to $v_1, \ldots, v_k$. Then (by [15] Theorem 14.3.7) the matrix for $\widetilde{\Delta}_d$ with respect to the basis $v_1, \ldots, v_k, u_1, \ldots, u_j$ is in the block form

$$\Gamma = \begin{bmatrix} N & * \\ 0 & M \end{bmatrix}.$$

We then see that

$$\det(\Gamma - \lambda I) = \det \begin{bmatrix} N - \lambda I & * \\ 0 & M - \lambda I \end{bmatrix} = \det(N - \lambda I)\det(M - \lambda I).$$

As a result, the characteristic polynomial of M divides the characteristic polynomial of Γ, so we have $\mathrm{Spec}(\Delta_d) \subseteq \mathrm{Spec}(\widetilde{\Delta}_d)$. □

Example 3.5. Let K and $\widetilde{K}$ be the two complexes from Example 3.1. We want to verify Theorem 3.4 with K and $\widetilde{K}$. For dimension greater than one the result is trivial, as neither K nor $\widetilde{K}$ have any simplices of dimension greater than one, so their Laplacian spectra will both be empty. Thus the only case we have to consider is dimension 1. One can easily compute that

$$\mathrm{Spec}(\Delta_1) = \{0, 2, 2, 4\}$$
$$\mathrm{Spec}(\widetilde{\Delta}_1) = \{0, 2 - \sqrt{2}, 2 - \sqrt{2}, 2, 2, 2 + \sqrt{2}, 2 + \sqrt{2}, 4\}$$

We see that, as predicted, $\mathrm{Spec}(\Delta_1) \subseteq \mathrm{Spec}(\widetilde{\Delta}_1)$. Notice that if we compute Δ_0 and $\widetilde{\Delta}_0$, we will se that $\mathrm{Spec}(\Delta_0)$ contains two 4's, while $\mathrm{Spec}(\widetilde{\Delta}_0)$ contains only one 4, so Theorem 3.4 does not necessarily hold for dimension 0.

It is disappointing that Theorem 3.4 does not hold for dimension 0. The reason it does not hold is because in dimension 0 the adjoint boundary does not commute with the chain covering, i.e. $p_0(\partial_0^*(\emptyset)) \neq \partial_0^*(p_{-1}(\emptyset))$. This is because $\emptyset$ is contained in every 0-simplex, so the key step in the proof of Lemma 3.2, that for a given $\eta \in S_0(K)$ we can find a unique $\sigma \in S_0(\widetilde{K})$ with $p(\sigma) = \eta$ and $\emptyset \subseteq \sigma$, fails.

Since the unreduced Laplacian $\overline{\Delta}_d$ does not include the empty set, $\overline{\Delta}_0 = \partial_1 \circ \partial_1^*$, we get the following lemma:

Lemma 3.6. *The unreduced Laplacian $\overline{\Delta}_d$ commutes with the chain covering $p_\#$ for all d, that is,*

$$\overline{\Delta}_d \circ p_d = p_d \circ \overline{\Delta}_d.$$

Proof. When $d > 0$, this is exactly Lemma 3.2, so we only need to look at the case $d = 0$. By definition, $\overline{\Delta}_0 = \partial_1 \circ \partial_1^*$. We know $\partial_1 \circ p_1 = p_0 \circ \partial_1$ and $\partial_1^* \circ p_0 = p_1 \circ \partial_1^*$ by Lemma 2.1, so we get

$$\begin{aligned} \overline{\Delta}_0 \circ p_0 &= (\partial_1 \circ \partial_1^*) \circ p_0 = \partial_1 \circ (\partial_1^* \circ p_0) \\ &= \partial_1 \circ (p_1 \circ \partial_1^*) = (\partial_1 \circ p_1) \circ \partial_1^* \\ &= (p_0 \circ \partial_1) \circ \partial_1^* = p_0 \circ (\partial_1 \circ \partial_1^*) \\ &= p_0 \circ \overline{\Delta}_0. \end{aligned}$$

□

Notice we do not need to specify what happens when $d = -1$, as $C_{-1}(\overline{K}) = \emptyset$. We thus see that for the unreduced Laplacian $\overline{\Delta}_d$, Theorem 3.4 holds for all d, not just $d \geq 1$:

Theorem 3.7. *Let $(\widetilde{K}, p)$ be a covering complex of simplicial complex K, and let $\widetilde{\overline{\Delta}}_d$ and $\overline{\Delta}_d$ be the unreduced Laplacian operators of $\widetilde{K}$ and K, respectively. Then for all d, Spec($\overline{\Delta}_d$) $\subseteq$ Spec($\widetilde{\overline{\Delta}}_d$).*

Proof. The proof is exactly the same as the proof of Theorem 3.4. □

4 Spectral Recursion

We now move away from the topic of covering complexes and begin our study of spectral recursion. There are many avenues of research in this area, and we are only able to focus on a small portion of the topic.

4.1 Simplicial Pairs and the Spectral Recursion Formula

Before we can discuss the spectral recursion formula, we must first generalize the notion of a simplicial complex. This discussion is based on [6]. Instead of a single complex, we talk about pairs of complexes, in the following manner. Let V be a set of points, and let K, K' be two simplicial complexes with $K^{(0)}, K'^{(0)} \subseteq V$. We define an equivalence relation on the set of ordered pairs of complexes (K, K') with vertices contained in V as follows. We say two ordered pairs (K, K') and (L, L') are equivalent if $K \setminus K' = L \setminus L'$. This is clearly an equivalence relation. Then define a *simplicial pair* to be an equivalence class of pairs of complexes (K, K'). Note that under this definition, every simplicial complex K is in one-to-one correspondence with the simplicial pair $(K, \emptyset)$. While (K, K') is technically an equivalence class of pairs of complexes, we will often work with a single representative of the equivalence class when doing calculations. We can define unions and joins of pairs by $(K, K') \cup (L, L') = (K \cup L, K' \cup L')$ and $(K, K') \star (L, L') = (K \star L, K' \star L')$; it is easy to check that these definitions are well-defined.

We want to extend the notion of a d-chain of a simplicial complex to one for simplicial pairs. We have to make sure that any definition we give is well defined, i.e. if $(K, K') = (L, L')$, then we want the d-chains of (K, K') to equal the d-chains of (L, L'). Define $C_d(K, K') = C_d(K)/C_d(K')$. It is easy to verify that $(K, K') = (L, L')$ implies $C_d(K, K') = C_d(L, L')$, and that $C_d(K, K')$ is the set of all formal $\mathbb{R}$-linear combinations of d-simplices of $K \setminus K'$; see [11] for the proof.

We can also define the boundary operator $\partial_d : C_d(K, K') \to C_{d-1}(K, K')$ by

$$\partial_d[v_0, \ldots, v_d] = \sum_{i=0}^{d} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_d],$$

where the sum is restricted to the simplices in $C_{d-1}(K, K')$. As is the case for a simplicial complex, we can define an inner product on $C_d(K, K')$ in which all of the basis elements are orthonormal. This gives us an adjoint boundary operator $\partial_d^* : C_{d-1}(K, K') \to C_d(K, K')$, so we can define the d^{th} combinatorial Laplacian $\Delta_d : C_d(K, K') \to C_d(K, K')$ by

$$\Delta_d = \partial_{d+1} \circ \partial_{d+1}^* + \partial_d^* \circ \partial_d.$$

Notice that all of these definitions are equivalent to the ones given for a simplicial complex when the simplicial pair is $(K, \emptyset)$.

As is the case for a simplicial complex, the simplicial pair Laplacian is symmetric, and thus diagonalizable with a complete set of real eigenvalues. Let $\mathrm{Spec}(\Delta_d)$ be the multiset of eigenvalues of $\Delta_d(K, K')$. For $\lambda \in \mathrm{Spec}(\Delta_d)$, let m_λ be the multiplicity of λ in $\mathrm{Spec}(\Delta_d)$. For a given simplicial pair (K, K'), we can define a polynomial in two variable t, q that enumerates all eigenvalues of $\Delta_d(K, K')$ for all dimensions d. The following definition comes from [6]

Definition. *Let (K, K') be a simplicial pair and let Spec(Δ_d) be the multiset of eigenvalues of $\Delta_d(K, K')$. The* ***spectrum polynomial*** *$S_{(K,K')}(t, q)$ of (K, K') is defined to be*

$$S_{(K,K')}(t, q) = \sum_{d \geq 0} t^d \sum_{\lambda \in Spec(\Delta_{d-1})} q^\lambda. \tag{2}$$

To see how this definition works, if $m_\lambda t^d q^\lambda$ is a term in $S_{(K,K')}(t,q)$, then $\lambda \in \mathrm{Spec}(\Delta_{d-1}(K,K'))$ and λ has multiplicity m_λ. Notice that we can also define the spectrum polynomial for a simplicial complex by $S_K(t,q) = S_{(K,\emptyset)}(t,q)$. As equation 2 is rather difficult to grasp, we give an example.

Example 4.1. Let K be the complex from Example 2.3. If we compute $\Delta_d(K) = \Delta_d(K,\emptyset)$, we get the spectrum of $\Delta_d(K)$ is given by

$$\begin{aligned}\mathrm{Spec}(\Delta_{-1}) &= \{4\}\\ \mathrm{Spec}(\Delta_0) &= \{2,4,4,4\}\\ \mathrm{Spec}(\Delta_1) &= \{2,2,4,4,4\}\\ \mathrm{Spec}(\Delta_2) &= \{2,4\}\end{aligned}$$

Now we will construct the spectrum polynomial $S_K(t,q)$ Pick a dimension d. For every eigenvalue $\lambda \in \mathrm{Spec}(\Delta_d)$, add a term $t^{d+1}q^\lambda$. We get the following polynomial:

$$S_K(t,q) = q^4 + tq^2 + 3tq^4 + 2t^2q^2 + 3t^2q^4 + t^3q^2 + t^3q^4.$$

Technically, the spectrum polynomial is only a polynomial when the eigenvalues of the Laplacian are non-negative integers. As seen in Example 3.5, this is not always the case; however, we will continue to use the term "spectrum polynomial" for convenience.

We want to use the spectrum polynomial to determine the eigenvalues of the Laplacian of a simplicial complex (so we will mainly be dealing with the case $S_K(t,q)$). While the spectrum polynomial is a convenient way of listing the eigenvalues of the Laplacian, it cannot help us compute the eigenvalues if we do not already know them. Thus we want a way of computing the spectrum polynomial recursively. We define two subcomplexes of a simplicial complex K.

Definition. *Let K be a simplicial complex and let $v \in K^{(0)}$. The **deletion** of K with respect to v is defined to be the simplicial complex*

$$K \setminus v = \{\sigma \in K \mid v \notin \sigma\}.$$

*The **contraction** of K with respect to v is the simplicial complex*

$$K/v = \{\sigma \setminus v \mid \sigma \in K, v \in \sigma\}.$$

It is clear that both $K \setminus v$ and K/v are subcomplexes of K. The contraction of K with respect to v is sometimes called the *link* of v in K. We would hope that we can write S_K in terms of $S_{K\setminus v}$ and $S_{K/v}$. It turns out that in some special cases, we can, providing we add an error term $S_{(K\setminus v, K/v)}$. We give the following definition.

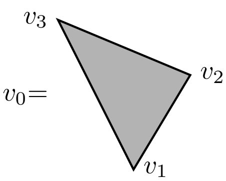
Definition. *We say a simplicial complex K **satisfies spectral recursion with respect to v** if $v \in K^{(0)}$ and*

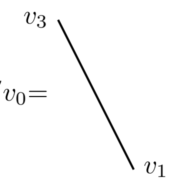
$$S_K(t,q) = qS_{K\setminus v}(t,q) + tqS_{K/v}(t,q) + (1-q)S_{(K\setminus v, K/v)}(t,q). \tag{3}$$

*We say K **satisfies spectral recursion** if K satisfies spectral recursion with respect to all $v \in K^{(0)}$.*

We call equation 3 the *spectral recursion formula*. It is well-known that not all simplicial complexes satisfy spectral recursion. We give two examples, one of a complex that does satisfy spectral recursion, and one of a complex that does not satisfy spectral recursion.

Example 4.2. Let K be the complex from Example 2.3. We will show that K satisfies spectral recursion with respect to v_0. Observe that $K \setminus v_0$ and K/v_0 are the following:

$K \setminus v_0=$ 

$K/v_0=$ 

One can easily compute the spectrum polynomials of $K \setminus v_0$, K/v_0, and $(K \setminus v_0, K/v_0)$; they are as follows:

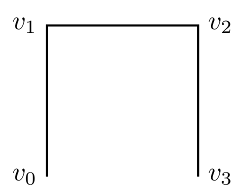
$$S_{K\setminus v_0}(t,q) = q^3 + 3tq^3 + 3t^2q^3 + t^3q^3$$
$$S_{K/v_0}(t,q) = q^2 + 2tq^2 + t^2q^2$$
$$S_{(K\setminus v_0, K/v_0)}(t,q) = tq^2 + 2t^2q^2 + t^3q^2$$

Thus applying equation 3, we get

$$qS_{K\setminus v_0}(t,q) + tqS_{K/v_0}(t,q) + (1-q)S_{(K\setminus v_0,K/v_0)}(t,q) = q^4 + tq^2 + 3tq^4 + 2t^2q^2 + 3t^2q^4 + t^3q^2 + t^3q^4$$

which is $S_K(t,q)$, as we computed in Example 4.1. It is an easy exercise to verify that K satisfies spectral recursion with respect to the other three vertices as well, meaning that K satisfies spectral recursion.

Example 4.3. We give perhaps the most important example of a complex that does not satisfy spectral recursion, namely, the three-edge graph P_4:

$P_4=$ 

The easiest way to see that P_4 does not satisfy spectral recursion is to observe that $\mathrm{Spec}(\Delta_0(P_4))$ and $\mathrm{Spec}(\Delta_1(P_4))$ contain non-integer values, but for all $v \in P_4^{(0)}$, the complexes $P_4 \setminus v$, P_4/v, and $(P_4 \setminus v, P_4/v)$ all have integer Laplacian spectra, so equation 3 cannot hold. In the next section we will see why P_4 is such an important complex.

Duval [6] showed that matroid complexes and shifted complexes satisfy spectral recursion. He also proved the following two lemmas, which we will use in the next section:

Lemma 4.4. *If K and L are disjoint simplicial complexes and each satisfies spectral recursion, then so do their union $K \cup L$ and their join $K \star L$.*

Lemma 4.5. *Let K be a simplicial complex with $\dim K \leq d$. If K satisfies spectral recursion, so does the $d-1$ skeleton $K^{(d-1)}$.*

For the rest of this thesis we will attempt to find general criteria for determining whether or not a given simplicial complex satisfies the spectral recursion. For one-dimensional complexes, the situation has been fairly well studied. Some insights have been made with two-dimensional complexes. For dimension three and higher, we have no definite information.

4.2 One-Dimensional Complexes: Cographs

A one-dimensional simplicial complex is called a *graph*. Graph theory has been studied extensively in its own right, and we will be using some results from that field in our analysis here. We use the following conventions when studying one-dimensional complexes. If G is a one-dimensional simplicial complex (i.e. a graph), then we denote $G^{(0)}$, the set of vertices of G, by $V(G)$, and we call 1-simplices *edges* and denote the set of edges by $E(G)$.

Technically speaking, what we have defined is a *simple graph*, those graphs without loops (edges that connect a vertex to itself) and multiple edges (two or more edges that share the same vertices). Arbitrary graphs can have both loops and multiple edges, but the graphs we will study, being one-dimensional simplicial complexes, will not (it is clear from the definition of a simplicial complex that a one-dimensional simplicial complex must be a simple graph).

In order to discuss graphs in relation to the spectral recursion formula, we first must establish some graph theory vocabulary. As with simplicial complexes, we can create new graphs through the process of graph unions and graph joins. If G_1, G_2 are disjoint graphs, we define the *graph union* $G_1 \cup G_2$ by defining its vertex and edge sets as follows: $V(G_1 \cup G_2) = \{v \mid v \in V(G_1) \text{ or } v \in V(G_2)\}$ and $E(G_1 \cup G_2) = \{e \mid e \in E(G_1) \text{ or } e \in E(G_2)\}$. Notice that the graph union of G_1 and G_2 is exactly the same as the union of G_1 and G_2 as simplicial complexes, so we drop the prefix "graph" and just say the union $G_1 \cup G_2$.

The graph analogue of the simplicial join, however, is different. If G_1, G_2 are disjoint graphs, we define the *graph join* $G_1 * G_2$ by defining its vertex and edge sets as follows:

$$V(G_1 * G_2) = \{v \mid v \in V(G_1) \text{ or } v \in V(G_2)\}$$
$$E(G_1 * G_2) = \{e \mid e \in E(G_1), e \in E(G_2), \text{ or } e = \{v_1, v_2\} \text{ with } v_1 \in V(G_1) \text{ and } v_2 \in V(G_2)\}$$

Notice how this is different from the simplicial join $G_1 \star G_2$, as the graph join, being a graph, is of dimension one, but in most cases the simplicial join will be of dimension greater than one, even if the two complexes being joined are one-dimensional. The graph and simplicial joins are related, however, through the following lemma:

Lemma 4.6. *Let G_1, G_2 be disjoint graphs, let $*$ be the graph join and let $\star$ be the simplicial join. Then $G_1 * G_2 = (G_1 \star G_2)^{(1)}$, that is, $G_1 * G_2$ is the 1-skeleton of $G_1 \star G_2$.*

Proof. The vertex set of $G_1 \star G_2$ is $V(G_1 \star G_2) = V(G_1) \cup V(G_2) = V(G_1 * G_2)$, so we have to show that $e \in E(G_1 * G_2)$ if and only if e is a one-dimensional simplex in $G_1 \star G_2$. By definition $G_1 \star G_2 = \{\sigma \cup \tau \mid \sigma \in G_1, \tau \in G_2\}$, so a one-dimensional simplex $e \in G_1 \star G_2$ is one of the following: either $e \in G_1$, $e \in G_2$, or $e = \{v, u\}$ where $v \in G_1$ and $u \in G_2$. But these are precisely the elements of $E(G_1 * G_2)$, so $G_1 * G_2 = (G_1 \star G_2)^{(1)}$. □

If G is a graph (i.e. a one-dimensional simplicial complex), any subcomplex H of G is called a *subgraph*. If $V \subseteq V(G)$, the induced subcomplex $G[V]$ is called the *induced subgraph* on V.

Given a graph G, we can define the *complement graph* $\overline{G}$ of G by defining its vertices and edges as follows: $V\left(\overline{G}\right) = V(G)$ and $E\left(\overline{G}\right) = \{e \mid e = \{v_1, v_2\}, v_1, v_2 \in V(G) \text{ and } e \notin E(G)\}$. In other words, $\overline{G}$ is the graph with the same set of vertices as G such that $e \in E\left(\overline{G}\right)$ if and only if $e \notin E(G)$.

The *complete graph on n vertices* K_n is the graph with vertices $V(K_n) = \{v_1, \ldots, v_n\}$ and edges $E(K_n) = \{\{v_i, v_j\} \mid 1 \leq i < j \leq n\}$. The complete graph on n vertices has edges connecting every vertex to every other vertex. For example, K_1 is a single vertex, K_2 is a line, K_3 is the edges of a triangle, K_4 is the edges of a tetrahedron, etc. It is clear that K_n is a simplicial complex for all n.

Finally, we define *path on n vertices* P_n to be the graph with with vertices $V(P_n) = \{v_1, \ldots, v_n\}$ and edges $E(P_n) = \{\{v_i, v_{i+1}\} \mid i = 1, \ldots, n-1\}$. Each P_n is a simplicial complex. By Example 4.3, the path on 4 vertices, P_4, does not satisfy spectral recursion. Since P_4 does not satisfy the spectral recursion formula, we want to study those graphs that do not contain P_4 as an induced subgraph; we call such graphs *P_4-free graphs*. Instead of studying P_4-free graphs as they are defined, we study what will turn out to be an equivalent definition, namely cographs.

Definition. *A **cograph** is defined by the following:*

1. *K_1 is a cograph.*
2. *If G is a cograph, then its complement graph $\overline{G}$ is also a cograph.*
3. *If G_1 and G_2 are disjoint cographs, then so is their union $G_1 \cup G_2$.*

It turns out that cographs and P_4-free graphs are one and the same. The following comes from [1].

Lemma 4.7. *A graph G is a cograph if and only if G is P_4-free.*

As a result of Lemma 4.7, any statement we make about cographs will also be true about P_4-free graphs, and vice-versa. We study cographs instead of P_4-free graphs because we have an algorithmic way of "building" cographs from smaller cographs. We will see how this is done in Theorem 4.10. First we need the following two lemmas, which come from [2] and [1], respectively.

Lemma 4.8. *Every induced subgraph of a cograph is also a cograph.*

Lemma 4.9. *The complement of every nontrivial connected induced subgraph of a cograph is disconnected.*

Now we show how we can decompose any cograph into smaller cographs. This will allow us to inductively create any cograph, starting with the smallest one K_1.

Theorem 4.10. *If G is a cograph on $n > 1$ vertices, then there exist disjoint cographs G_1, G_2 such that $G = G_1 \cup G_2$ or $G = G_1 * G_2$, where $*$ is the graph join.*

Proof. First assume G is disconnected, so we can write $G = G_1 \cup G_2$ with $G_1 \cap G_2 = \emptyset$ and both G_1, G_2 nontrivial. Since clearly G_1 and G_2 are induced subgraphs of G and G is a cograph, by Lemma 4.8 both G_1 and G_2 are cographs, so the theorem is proven.

Now suppose that G is connected. By Lemma 4.9, $\overline{G}$ is disconnected, so we can write $\overline{G} = G' \cup G''$ where $G' \cap G'' = \emptyset$. Since G is a cograph, so is $\overline{G}$, and thus by Lemma 4.8, so are G' and G''. Let $\overline{G'} = G_1$ and $\overline{G''} = G_2$. Since G' and G'' are cographs, so are G_1 and G_2. Also $G_1 \cap G_2 = \emptyset$, since they are complement graphs of disjoint graphs G' and G''.

We claim that $G = G_1 * G_2$. Clearly $V(G_1 * G_2) = V(G)$, since $V(G_1 * G_2) = V(G_1) \cup V(G_2) = V(G') \cup V(G'') = V(\overline{G}) = V(G)$. By definition $E(G_1 * G_2)$ consists of all edges in G_1, all edges in G_2, and all edges between vertices $v \in G_1$ and $u \in G_2$. Thus since an edge $e \in E(\overline{G_1 * G_2})$ if and only if $e \notin E(G_1 * G_2)$, we see that no edges between vertices $v \in G_1$ and $u \in G_2$ are in $\overline{G_1 * G_2}$, so $E(\overline{G_1 * G_2}) = E(\overline{G_1}) \cup E(\overline{G_2}) = E(G') \cup E(G'') = E(\overline{G})$. Thus $\overline{G} = \overline{G_1 * G_2}$, so $G = G_1 * G_2$. □

We will be able to use Theorem 4.10 to prove that any cograph satisfies the spectral recursion formula.

Lemma 4.11. *Let K be a simplicial complex and suppose* $\dim K = d$. *Then if K satisfies the spectral recursion, so does $K^{(p)}$ for all $p < d$.*

Proof. The proof is by induction on d. The base case is $d = 0$, i.e. K is a collection of points. Then the only $p < d$ is $p = -1 = d - 1$, so by Lemma 4.5 if K satisfies the spectral recursion so does $K^{(p)}$ for all $p < d$.

Now we prove the inductive step. Let $\dim K = d$ and let K satisfy the spectral recursion. By Lemma 4.5, $K^{(d-1)}$ also satisfies the spectral recursion. Since $K^{(d-1)}$ is itself a simplicial complex satisfying the spectral recursion and $\dim K^{(d-1)} = d - 1$, by our inductive hypothesis $(K^{(d-1)})^{(p)}$ satisfies the spectral recursion for all $p < d - 1$. But clearly if $p < d - 1$, then $(K^{(d-1)})^{(p)} = K^{(p)}$, so this means that $K^{(p)}$ satisfies the spectral recursion for all $p < d$. □

Now we prove our main theorem in this section.

Theorem 4.12. *If G is a cograph, then G satisfies spectral recursion.*

Proof. We prove by induction on $|V(G)|$. The base case is $|V(G)| = 1$, i.e. G is a single point, which clearly satisfies spectral recursion.

Now suppose $|V(G)| = n$. By Theorem 4.10, we can either write $G = G_1 \cup G_2$ or $G = G_1 * G_2$ where G_1 and G_2 are themselves cographs. Since G_1 and G_2 are cographs, we know $|V(G_1)|, |V(G_2)| \geq 1$. Thus since $|V(G)| = |V(G_1)| + |V(G_2)|$, it must be that both $|V(G_1)|, |V(G_2)| < n$, so by our inductive hypothesis, both G_1 and G_2 satisfy spectral recursion. If $G = G_1 \cup G_2$, then by Lemma 4.4, G satisfies spectral recursion. Now suppose $G = G_1 * G_2$. By Lemma 4.6, $G = (G_1 \star G_2)^{(1)}$. By Lemma 4.4, since G_1 and G_2 satisfy spectral recursion, so does $G_1 \star G_2$, and so by Lemma 4.11, since $\dim G_1 \star G_2 \geq 1$, $(G_1 \star G_2)^{(1)} = G$ satisfies spectral recursion. □

Corollary 4.13. *Every induced subgraph of a cograph satisfies the spectral recursion.*

Proof. Apply Lemma 4.8 and Theorem 4.12. □

Lemma 4.7 tells us that a graph G is a cograph if and only if G is P_4-free, i.e. G does not contain P_4 as an induced subgraph. Thus by Theorem 4.12 we know that all P_4-free graphs satisfy the spectral recursion formula. We would like the converse to be true as well, i.e. if a graph G satisfies the spectral recursion formula, then G is P_4-free. In other words, we hope that if G contains P_4 as an induced subgraph, then G does not satisfy the spectral recursion. We have the following, more generalized conjecture.

Conjecture 4.14. *Let K be a simplicial complex, let $V \subseteq K^{(0)}$, and let $K[V]$ be the induced subcomplex of V. If $K[V]$ does not satisfy the spectral recursion, then K does not satisfy the spectral recursion.*

If Conjecture 4.14 is true (we have yet to find a counterexample), then we have that a one-dimensional complex G satisfies the spectral recursion if and only if G is P_4-free.

4.3 Strong Spectral Recursion

In this section we draw inspiration from the Tutte polynomial of a matroid [18]. The Tutte polynomial $T(M)$ for any matroid M satisfies the recursion relation $T(M) = T(M \setminus e) + T(M/e)$, as long as e is not a loop or an isthmus. Since this recursion relation holds for all matroids, it in particular holds for $M \setminus e$ and M/e, so $T(M \setminus e) = T((M \setminus e) \setminus e') + T((M \setminus e)/e')$, and $T(M/e) = T((M/e) \setminus e'') + T((M/e)/e'')$, where $e' \in M \setminus e$ and $e'' \in M/e$ are not loops or isthmuses. We can continue this process, producing smaller and smaller matroids, and the recursion relation for the Tutte polynomial will hold at every step. In other words, every matroid that can be produced from M through a series of deletions and contractions will satisfy the recursion relation for the Tutte polynomial.

Since spectral recursion is a similar recursion relation as that which the Tutte polynomial satisfies, the question arises as to whether or not we can, like the Tutte polynomial, continue spectral recursion on $K \setminus v$

and K/v, and so on. To make this precise, we introduce some terminology. Define a *deletion-contraction sequence* to be a finite set of complexes $\{K_0, K_1, K_2, \ldots, K_n\}$ such that for all $i > 0$, $K_i = K_{i-1} \setminus v$ or $K_i = K_{i-1}/v$ for some $v \in K_{i-1}$. A subcomplex $L \subseteq K$ is called a *deletion-contraction subcomplex* if there exists a deletion-contraction sequence $\{K_0, \ldots, K_n\}$ such that $K_0 = K$ and $K_n = L$. In other words, $L \subseteq K$ is a deletion-contraction subcomplex when it can be produced from K through a finite sequence of deletions and contractions. We say that a complex K satisfies *strong spectral recursion* when all deletion-contraction subcomplexes of K satisfy spectral recursion.

The question of whether or not we can apply spectral recursion to $K \setminus v$, K/v, and all complexes we get as a result, can be rephrased as asking whether or not K satisfies strong spectral recursion. Satisfying strong spectral recursion is most likely a much stronger condition that just satisfying spectral recursion, for saying that a complex K satisfies strong spectral recursion means that many subcomplexes of K satisfy spectral recursion along with K. We should point out, however, that we have yet to find any simplicial complexes that satisfy spectral recursion but do not satisfy strong spectral recursion.

Proposition 4.15. *If K is a finite simplicial complex and K satisfies strong spectral recursion, then all induced subcomplexes of K satisfy strong spectral recursion.*

Proof. Let K' be an induced subcomplex of K, so there is a set $V = \{v_1, \ldots, v_s\} \subseteq K^{(0)}$ such that $K' = K[V]$. Let $U = \{u_1, \ldots, u_r\} = K^{(0)} \setminus V$ (U is finite since we are assuming that K is finite), and let $\{K_0, \ldots, K_r\}$ be a sequence of complexes with $K_0 = K$ and $K_i = K_{i-1} \setminus u_i$. Since each K_i is the deletion of K_{i-1}, the sequence $\{K_0, \ldots, K_r\}$ is a deletion-contraction sequence. Notice that $\sigma \in K_r$ if and only if $u_i \notin \sigma$ for all u_i, i.e. $K_r = \{\sigma \in K \mid \sigma^{(0)} \subseteq V\} = K[V]$. Thus $K' = K[V]$ is a deletion-contraction subcomplex of K.

Now let L be a deletion-contraction subcomplex of K', so there is a deletion-contraction sequence $\{L_0, \ldots, L_m\}$ with $L_0 = K'$ and $L_m = L$. To prove the proposition, we have to show that L satisfies spectral recursion. Consider the sequence of complexes $\{K_0, \ldots, K_r, L_1, \ldots, L_m\}$. We know that $\{K_0, \ldots, K_r\}$ is a deletion-contraction sequence, and (since $K_r = K' = L_0$), $\{K_r, L_1, \ldots, L_m\}$ is a deletion-contraction sequence, so $\{K_0, \ldots, K_r, L_1, \ldots, L_m\}$ is a deletion-contraction sequence with $K_0 = K$ and $L_m = L$, so L is a deletion-contraction subcomplex of K. Since K satisfies strong spectral recursion, we then have that L satisfies spectral recursion, completing the proof. □

Strong spectral recursion says more, in fact. If K satisfies strong spectral recursion, then a large number of subcomplexes of K which are not induced subcomplexes satisfy spectral recursion, namely, all deletion-contraction subcomplexes that contain a contraction in their formation. Note that, if $v \in K^{(0)}$ and V is the set of vertices that share an edge with v, then $K/v = K[V]$ if and only if $\sigma \cup \{v\} \in K$ for all $\sigma \in K[V]$. Since this is usually not the case, most contractions are not induced subcomplexes, yet for a complex satisfying strong spectral recursion, all of these subcomplexes produced by contractions will still satisfy spectral recursion.

In the proof of Proposition 4.15 we proved the following.

Corollary 4.16. *If K is a simplicial complex and $L \subseteq K$ is an induced subcomplex, then L is a deletion-contraction subcomplex.*

With Corollary 4.16 we now know a large class of complexes that do not satisfy strong spectral recursion.

Proposition 4.17. *If K is a simplicial complex and P_4 is an induced subcomplex of K, then K does not satisfy strong spectral recursion.*

Proof. We know that P_4 does not satisfy spectral recursion. Since P_4 is an induced subcomplex of K, by Corollary 4.16, P_4 is a deletion-contraction subcomplex. Thus K contains a deletion-contraction subcomplex that does not satisfy spectral recursion, so K does not satisfy strong spectral recursion. □

In general we do not have a set of criteria which guarantees when a complex satisfies strong spectral recursion. For the one-dimensional case, however, we do.

Theorem 4.18. *Let G be a one-dimensional simplicial complex. Then G satisfies strong spectral recursion if and only if G is a cograph.*

Proof. We know that if G satisfies strong spectral recursion, then all deletion-contraction subcomplexes of G satisfy spectral recursion. In particular, by Corollary 4.16, we know that all induced subcomplexes of G (being deletion-contraction subcomplexes) satisfy spectral recursion. Thus G must be P_4-free (for if G contained a P_4, then G would contain an induced subcomplex that does not satisfy spectral recursion, a contradiction), so G is a cograph.

Now let G be a cograph. We have to show that every deletion-contraction subcomplex of G satisfies spectral recursion. We claim that every deletion-contraction subcomplex of G is a cograph. To see this, let G' be a deletion-contraction subcomplex of G, so there is a deletion-contraction sequence $\{G_0, \ldots, G_n\}$ with $G_0 = G$ and $G_n = G'$. If $G_i = G_{i-1} \setminus v$ for all i, then G' is an induced subcomplex of G, so by Corollary 4.13, G' satisfies spectral recursion.

Assume then that there is a G_i with $G_i = G_{i-1}/v$. Since G_{i-1} is a graph (as it is a subcomplex of G), G_{i-1} contains no two-simplices, so the resulting graph after contracting v can contain no one-simplices. Thus G_i is the disjoint union of a collection of points. Since G' is a subcomplex of G_i, this means that G' is also the disjoint union of a collection of points. Since K_1 is a cograph by definition and disjoint unions of cographs are cographs, we have that G' is a cograph, so by Theorem 4.12, G' satisfies spectral recursion, completing the proof. □

4.4 Two-Dimensional Complexes

We do not have many results on two-dimensional complexes. We began our study of two-dimensional complexes by looking for a two-dimensional analogue of the path on four vertices P_4. In other words, we were hoping to find a two-dimensional complex K such that a two-dimensional complex satisfied spectral recursion if and only if it did not contain K as an induced subcomplex. We were unsuccessful in this endeavor. This section is organized around a series of examples, illustrating our attempts to discover a "minimal" two-dimensional complex which failed to satisfy spectral recursion.

We begin with a seemingly trivial result.

Proposition 4.19. *Let K be a d-dimensional complex. If the p-skeleton $K^{(p)}$ does not satisfy spectral recursion for some $p \leq d$, then K does not satisfy spectral recursion.*

Proof. This is just the contrapositive of Lemma 4.11. □

The reason we state Proposition 4.19 is because, along with Conjecture 4.14, this would imply that if a two-dimensional complex K contained P_4 as an induced subcomplex of its 1-skeleton, then we would immediately know that K does not satisfy spectral recursion. This fact generalizes to higher dimensions. Assuming Conjecture 4.14 is true, the d-dimensional complexes K that do not satisfy spectral recursion can be separated into two categories:

1. The p-skeleton $K^{(p)}$ contains an induced subcomplex that does not satisfy spectral recursion, for some $p < d$.

2. The p-skeleton $K^{(p)}$ satisfies spectral recursion for all $p < d$.

Case 1 is mostly uninteresting, as we learn nothing new about the complex, simply that it was built from a complex that did not satisfy spectral recursion. Thus we are mainly looking for complexes that fall in Case 2, where the reason the d-dimensional complex K fails to satisfy spectral recursion is truly "d-dimensional."

First, we prove a partial converse to Lemma 4.4

Proposition 4.20. *Let K and L be two disjoint complexes, let $v \in K^{(0)}$, and let $K \star L$ denote the join of K and L. If $K \star L$ satisfies spectral recursion with respect to v, then so does K.*

Proof. By Duval [6], we know that if K and L are disjoint simplicial complexes and $v \in K^{(0)}$, then

$$\begin{aligned} (K \star L) \setminus v &= (K \setminus v) \star L \\ (K \star L)/v &= (K/v) \star L \\ ((K \star L) \setminus v, (K \star L)/v) &= (K \setminus v, K/v) \star L \\ S_{K \star L}(t,q) &= S_K(t,q) S_L(t,q) \end{aligned}$$

Since $K \star L$ satisfies spectral recursion with respect to v, we have by equation 3,

$$\begin{aligned} S_{K \star L} &= qS_{(K \star L)\setminus v} + tqS_{(K\star L)/v} + (1-q)S_{((K\star L)\setminus v,(K\star L)/v)} \\ &= qS_{(K\setminus v)\star L} + tqS_{(K/v)\star L} + (1-q)S_{(K\setminus v, K/v)\star L} \\ &= qS_{K\setminus v}S_L + tqS_{K/v}S_L + (1-q)S_{(K\setminus v,K/v)}S_L \\ &= S_L(qS_{K\setminus v} + tqS_{K/v} + (1-q)S_{(K\setminus v,K/v)}) \end{aligned}$$

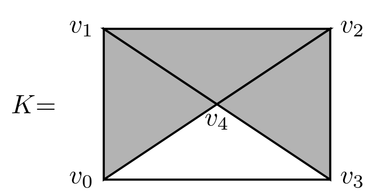
But we know $S_{K\star L} = S_K S_L$, meaning that $S_K = qS_{K\setminus v} + tqS_{K/v} + (1-q)S_{(K\setminus v,K/v)}$, so K satisfies spectral recursion with respect to v. □

Proposition 4.20, together with Lemma 4.4, implies that K satisfies spectral recursion with respect to v if and only if $K \star L$ does as well. In fact, we have a formula for calculating the eigenvalues of $K \star L$ in terms of the eigenvalues of K and L; the following comes from [6]:

$$\mathrm{Spec}_{d-1}((K,K') \star (L,L')) = \bigcup_{\substack{i+j=d \\ \lambda \in \mathrm{Spec}_{i-1}((K,K')) \\ \mu \in \mathrm{Spec}_{j-1}((L,L'))}} \lambda + \mu \tag{4}$$

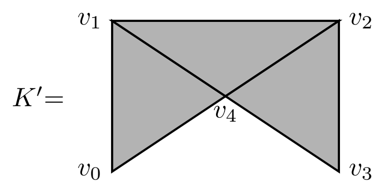
Thus if K satisfies spectral recursion in dimension $i-1$ and L does not satisfy spectral recursion in dimension $j-1$, then $K \star L$ will not satisfy spectral recursion in dimension $d-1$ where $d = i+j$.

Example 4.21. Let K be the following two-dimensional simplicial complex:



A brute-force calculation shows that K satisfies spectral recursion with respect to v_4, but does not satisfy spectral recursion with respect to any of the other vertices; see Appendix A for more details. It is clear that K contains no induced P_4 in its 1-skeleton. From the standpoint above, K would then fall into Case 2 above, those complexes such that $K^{(p)}$ satisfy spectral recursion for all $p < \dim(K)$. However, we can easily explain why K does not satisfy spectral recursion using what we have just described.

Consider K', the subcomplex of K created by removing edge $\{v_0, v_3\}$:



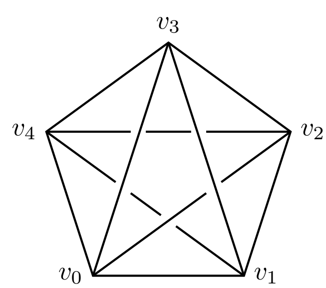
Observe that $K' = P_4 \star \{v_4\}$. Thus, by Proposition 4.20, K' satisfies spectral recursion with respect to v_4 and does not satisfy spectral recursion with respect to the other four vertices. In fact, since P_4 fails to satisfy spectral recursion in dimension 1, while v_4 satisfies spectral recursion in dimension 0, by equation 4, K' will fail to satisfy spectral recursion in dimension 2.

Now look back at our original complex K. The only difference between K and K' is that K has an additional 1-simplex. However, since $\{v_0, v_3\}$ is not contained in any 2-simplex, its presence will not affect the calculation of Δ_2. Thus, since K' fails to satisfy spectral recursion in dimension 2, so does K. While K technically fails to satisfy spectral recursion for a "new" reason (in the sense that $K^{(p)}$ satisfies spectral recursion for all $p < \dim(K)$) we gained no new information about the types of complexes that fail to satisfy spectral recursion.

Our goal now is to attempt to find "minimal" two-dimensional complexes that do not satisfy spectral recursion. A complex K that fails to satisfy spectral recursion is *minimal* if all proper induced subcomplexes of K satisfy spectral recursion. Notice that any complex K that contains P_4 as an induced subcomplex of $K^{(1)}$ is not minimal.

We start by looking at complexes with 4 vertices. Notice that any two-dimensional complex on 4 vertices will not contain P_4 as an induced subcomplex, since it must contain a 3-cycle in order to contain a 2-simplex. See Appendix B for all two-dimensional complexes on 4 vertices, along with justification as to why each one satisfies spectral recursion.

We now look at complexes with 5 vertices. We focus on those complexes K such that $K^{(1)} = K_5$, the complete graph on five vertices. The reason we do this is to guarantee that P_4 is not an induced subcomplex of K, so that any complex we find that fails spectral recursion will do so for a purely "two-dimensional" reason. Thus the 1-skeleton of all complexes will be as follows:



With these requirements, there are $\binom{5}{3} = 10$ different 2-simplices we have to choose from. If any complex has more than one 2-simplex, then every pair of 2-simplices must intersect in at least one vertex, as there are only five vertices. There are $\binom{10}{k}$ different complexes with k 2-simplices. From this we see that there is a one-to-one correspondence between the complexes with k 2-simplices and the complexes with $10-k$ 2-simplices, as $\binom{10}{k} = \binom{10}{10-k}$. It should be noted, however, many of these $\binom{10}{k}$ complexes will be combinatorially identical. For example, there is only one complex with one 2-simplex, and there are only two distinct complexes with two 2-simplices: the 2-simplices intersect in a vertex or they intersect in an edge. All three of these complexes satisfy spectral recursion. See Appendix C for all complexes on five vertices with one or two 2-simplices, along with justification as to why each one satisfies spectral recursion.

Things become more interesting when we look at complexes with three 2-simplices. It turns out that there are four distinct complexes of this form. We give an example of each of them by listing their 2-simplices:

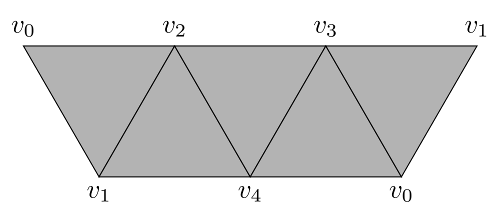
1. $\{v_0, v_1, v_3\}, \{v_0, v_1, v_4\}, \{v_0, v_3, v_4\}$
2. $\{v_0, v_1, v_2\}, \{v_0, v_3, v_4\}, \{v_1, v_2, v_3\}$
3. $\{v_0, v_1, v_3\}, \{v_0, v_2, v_3\}, \{v_0, v_3, v_4\}$
4. $\{v_0, v_3, v_4\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}$

We can check to see that the first three satisfy spectral recursion, but the last one does not; see Appendix C for the calculations. It turns out that the last complex is in fact the complex K from Example 4.21, except with edges $\{v_0, v_2\}$ and $\{v_1, v_3\}$ also present, and so fails to satisfy spectral recursion for the same reason the first example did.

Here is where our true analysis ends. At the time of writing, we were unable to calculate the combinatorially distinct complexes with four and five 2-simplices (once we know the combinatorially distinct complexes with zero through five 2-simplices, by symmetry we know the combinatorially distinct complexes with six through ten 2-simplices). At first glance, there are $\binom{10}{4} = 210$ possible complexes with four 2-simplices and $\binom{10}{5} = 252$ possible complexes with five 2-simplices. By looking at the symmetries of K_5 we were able to reduce those numbers to at most 36 combinatorially distinct complexes with four 2-simplices and at most 45 combinatorially distinct complexes with five 2-simplices. However, both of these numbers can probably be lowered significantly.

So far, the only two-dimensional complexes we have found that do not satisfy spectral recursion have failed to do so because of their 1-skeleton (it contains an induced P_4) or because of the way the 1-skeleton interacts with the 2-simplices (like Example 4.21). None of the complexes looked at so far have failed to satisfy spectral recursion for a truly "two-dimensional" reason. We end this discussion with one such example.

Example 4.22. Let K be the following two-dimensional complex:



The 1-skeleton of K is K_5; we have represented K with two copies of v_0 and v_1 so as to draw K in the plane. The complex K is in fact a triangulation of the Möbius strip. The complex K does not satisfy spectral recursion with respect to any of its vertices, as the following discussion shows. First notice that K is vertex transitive, i.e. given any two vertices v_i and v_j, there is an automorphism f of K with $f(v_i) = v_j$. Thus if we show that K does not satisfy spectral recursion with respect to one vertex v_i, we will know that K does not satisfy spectral recursion with respect to all vertices. We have the following spectrum polynomials for K, $K \setminus v_0$, K/v_0, and $(K \setminus v_0, K/v_0)$:

$$\begin{aligned}
S_K(t,q) &= q^5 + 5tq^5 + t^2 + 2t^2q^{\frac{1}{2}(5-\sqrt{5})} + 2t^2q^{\frac{1}{2}(5+\sqrt{5})} + 5t^2q^5 + 2t^3q^{\frac{1}{2}(5-\sqrt{5})} + 2t^3q^{\frac{1}{2}(5+\sqrt{5})} + t^3q^5 \\
S_{K\setminus v_0}(t,q) &= q^4 + 4tq^4 + t^2 + t^2q^2 + 4t^2q^4 + t^3q^2 + t^3q^4 \\
S_{K/v_0}(t,q) &= q^4 + tq^{2-\sqrt{2}} + tq^2 + tq^{2+\sqrt{2}} + tq^4 + t^2q^{2-\sqrt{2}} + t^2q^2 + t^2q^{2+\sqrt{2}} \\
S_{(K\setminus v_0, K/v_0)}(t,q) &= t^2 + t^2q + t^2q^3 + t^3q + t^3q^3
\end{aligned}$$

It is clear from these calculations that equation 3 does not hold, and so K does not satisfy spectral recursion. We cannot explain this failure with anything we have previously encountered. The complex K is not the join of any one-dimensional complexes, and it is not, like Example 4.21, the join of one-dimensional complexes with some number of edges added on. Thus K fails to satisfy spectral recursion not because of its 1-simplices, but because of its 2-simplices. It is the first and so far only such two-dimensional complex known.

In conclusion, the study of the spectral recursion formula is just beginning. While the class of simplicial complexes that satisfy spectral recursion is known to be closed under join and disjoint union, there is no general characterization of all complexes that satisfy spectral recursion. Our approach to this question was to instead attempt to characterize all complexes that do *not* satisfy spectral recursion; it was then hoped that, by using Conjecture 4.14, we could classify all complexes that do satisfy spectral recursion.

We were unsuccessful in this endeavor even in the lowest dimensions, one and two. We made significant strides in dimension one, directly calculating all one-dimensional complexes that satisfy strong spectral recursion. In dimension two, we were less fortunate. While we were only able to find one minimal two-dimensional complex that failed to satisfy spectral recursion for a purely two-dimensional reason (Example 4.22), we do not believe that this is the only such complex. It seems unlikely that our method of analysis concerning this problem will lead to any significant results. Perhaps those complexes that satisfy spectral recursion form a new class of simplicial complexes, independent of usual classes of complexes already studied.

Appendix

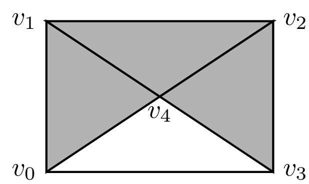
In this Appendix we provide calculations and explanations as to why certain simplicial complexes discussed in the body of the thesis do or do not satisfy the spectral recursion formula,

$$S_K(t,q) = qS_{K\setminus v}(t,q) + tqS_{K/v}(t,q) + (1-q)S_{(K\setminus v,K/v)}(t,q). \tag{5}$$

When possible we quote a theorem of some sort (i.e. a given complex is a matroid complex, is the join of two complexes that satisfy spectral recursion, etc.). If we cannot do this then we will provide the spectrum polynomials of the complex K and the associated complexes $K \setminus v$, K/v, and $(K \setminus v, K/v)$ for all combinatorially distinct vertices. After listing the spectrum polynomials, we leave it to the reader to verify that equation 5 is or is not satisfied.

A Example 4.21

$K=$

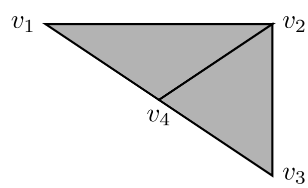


The complex K is the one from Example 4.21. We show computationally that K satisfies spectral recursion with respect to vertex v_4 but not with respect to any of the other vertices. The spectrum polynomial of K is

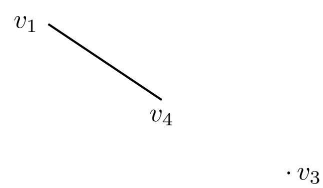
$$S_k(t,q) = q^5 + 2tq^3 + 3tq^5 + t^2 + t^2q^{3-\sqrt{2}} + 3t^2q^3 + t^2q^{3+\sqrt{2}} + 2t^2q^5 + t^3q^{3-\sqrt{2}} + t^3q^3 + t^3q^{3+\sqrt{2}}$$

Observe that v_0 and v_3 are combinatorially identical, as are v_1 and v_2. We have

$K \setminus v_0=$

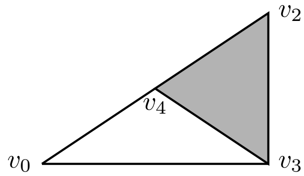


$K/v_0=$

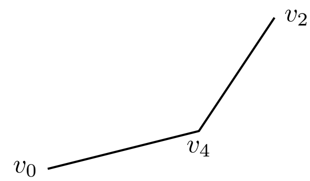


$$S_{K\setminus v_0}(t,q) = q^4 + tq^2 + 3tq^4 + 2t^2q^2 + 3t^2q^4 + t^3q^2 + t^3q^4$$
$$S_{K/v_0}(t,q) = q^3 + t + tq^2 + tq^3 + t^2q^2$$
$$S_{(K\setminus v_0,K/v_0)}(t,q) = tq^3 + t^2 + t^2q^{\frac{1}{2}(5-\sqrt{5})} + t^2q^3 + t^2q^{\frac{1}{2}(5+\sqrt{5})} + t^3q^{\frac{1}{2}(5-\sqrt{5})} + t^3q^{\frac{1}{2}(5+\sqrt{5})}$$

$K \setminus v_1 =$

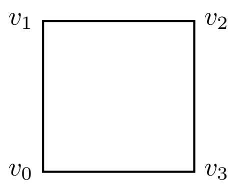


$K/v_1 =$

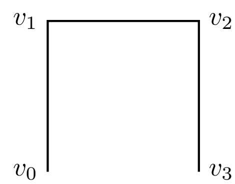


$$S_{K\setminus v_1}(t,q) = q^4 + tq^2 + 3tq^4 + t^2 + t^2q^2 + t^2q^3 + 2t^2q^4 + t^3q^3$$
$$S_{K/v_1}(t,q) = q^3 + tq + 2tq^3 + t^2q + t^2q^3$$
$$S_{(K\setminus v_1, K/v_1)}(t,q) = tq^3 + t^2 + t^2q^2 + t^2q^3 + t^3q^2$$

$K \setminus v_4 =$



$K/v_4 =$

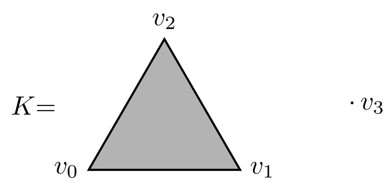


$$S_{K\setminus v_4}(t,q) = q^4 + 2tq^2 + 2tq^4 + t^2 + 2t^2q^2 + t^2q^4$$
$$S_{K/v_4}(t,q) = q^4 + tq^{2-\sqrt{2}} + tq^2 + tq^{2+\sqrt{2}} + tq^4 + t^2q^{2-\sqrt{2}} + t^2q^2 + t^2q^{2+\sqrt{2}}$$
$$S_{(K\setminus v_4, K/v_4)}(t,q) = t^2$$

B Two-Dimensional Complexes on Four Vertices

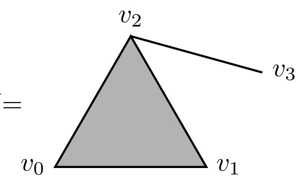
In this Appendix we look at all two-dimensional complexes on four vertices, and show that they all satisfy spectral recursion.

B.1 One 2-Simplex (I)



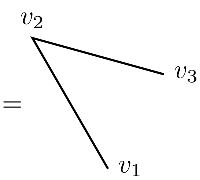
The complex K is the disjoint union of two matroid complexes (which satisfy spectral recursion by [6]), so by Lemma 4.4, K satisfies spectral recursion.

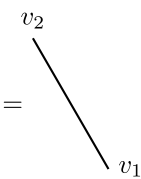
B.2 One 2-Simplex (II)

$K=$ 

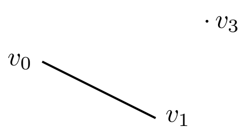
$$S_K(t,q) = q^4 + tq + tq^3 + 2tq^4 + t^2q + 2t^2q^3 + t^2q^4 + t^3q^3$$

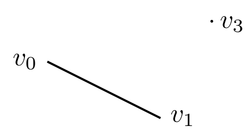
Vertices v_0 and v_1 are combinatorially identical, so we show through direct computation that K satisfies spectral recursion with respect to v_0, v_2 and v_3.

$K \setminus v_0=$ 

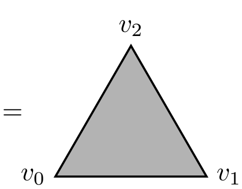
$K/v_0=$ 

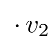
$$\begin{aligned} S_{K\setminus v_0}(t,q) &= q^3 + tq + 2tq^3 + t^2q + t^2q^3 \\ S_{K/v_0}(t,q) &= q^2 + 2tq^2 + t^2q^2 \\ S_{(K\setminus v_0, K/v_0)}(t,q) &= tq + t^2q \end{aligned}$$

$K \setminus v_2=$ 

$K/v_2=$ 

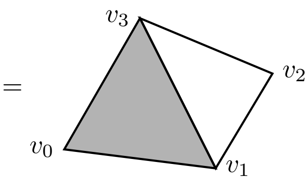
$$\begin{aligned} S_{K\setminus v_2}(t,q) &= q^3 + t + tq^2 + tq^3 + t^2q^2 \\ S_{K/v_2}(t,q) &= q^3 + t + tq^2 + tq^3 + t^2q^2 \\ S_{(K\setminus v_2, K/v_2)}(t,q) &= 0 \end{aligned}$$

$K \setminus v_3=$ 

$K/v_3=$ 

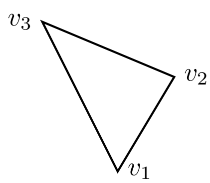
$$\begin{aligned} S_{K\setminus v_3}(t,q) &= q^3 + 3tq^3 + 3t^2q^3 + t^3q^3 \\ S_{K/v_3}(t,q) &= q + tq \\ S_{(K\setminus v_3, K/v_3)}(t,q) &= tq + tq^3 + t^2q + 2t^2q^3 + t^3q^3 \end{aligned}$$

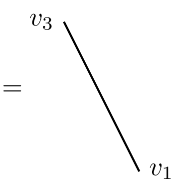
B.3 One 2-Simplex (III)

$K =$ 

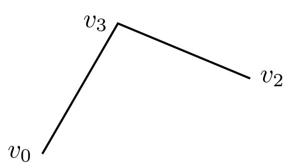
$$S_K(t,q) = q^4 + tq^2 + 3tq^4 + t^2 + t^2q^2 + t^2q^3 + 2t^2q^4 + t^3q^3$$

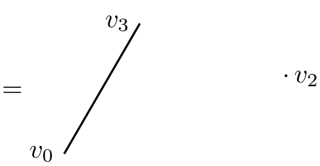
Vertices v_1 and v_3 are combinatorially identical, so we have to show that K satisfies spectral recursion with respect to vertices v_0, v_1, and v_2.

$K \setminus v_0 =$ 

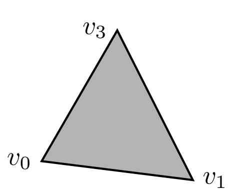
$K/v_0 =$ 

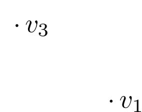
$$S_{K\setminus v_0}(t,q) = q^3 + 3tq^3 + t^2 + 2t^2q^3$$
$$S_{K/v_0}(t,q) = q^2 + 2tq^2 + t^2q^2$$
$$S_{(K\setminus v_0, K/v_0)}(t,q) = tq^2 + t^2 + t^2q^2$$

$K \setminus v_1 =$ 

$K/v_1 =$ 

$$S_{K\setminus v_1}(t,q) = q^3 + tq + 2tq^3 + t^2q + t^2q^3$$
$$S_{K/v_1}(t,q) = q^3 + t + tq^2 + tq^3 + t^2q^2$$
$$S_{(K\setminus v_1, K/v_1)}(t,q) = t^2$$

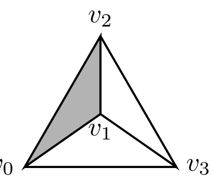
$K \setminus v_2 =$ 

$K \setminus v_2 =$ 

$$S_{K\setminus v_2}(t,q) = q^3 + 3tq^3 + 3t^2q^3 + t^3q^3$$
$$S_{K/v_2}(t,q) = q^2 + t + tq^2$$
$$S_{(K\setminus v_2, K/v_2)}(t,q) = tq^2 + t^2 + t^2q^2 + t^2q^3 + t^3q^3$$

B.4 One 2-Simplex (IV)

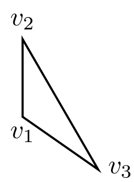
$K=$



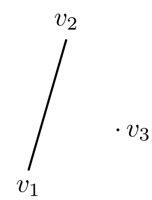
$$S_K(t,q) = q^4 + 4tq^4 + 2t^2 + t^2q^3 + 3t^2q^4 + t^3q^3$$

Vertices v_0, v_1, and v_2 are combinatorially identical, so we need to show that spectral recursion holds with respect to vertices v_0 and v_3.

$K \setminus v_0=$

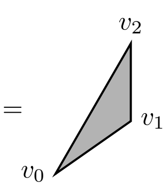


$K/v_0=$

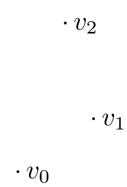


$$S_{K\setminus v_0}(t,q) = q^3 + 3tq^3 + t^2 + 2t^2q^3$$
$$S_{K/v_0}(t,q) = q^3 + t + tq^2 + tq^3 + t^2q^2$$
$$S_{(K\setminus v_0, K/v_0)}(t,q) = 2t^2$$

$K \setminus v_3=$

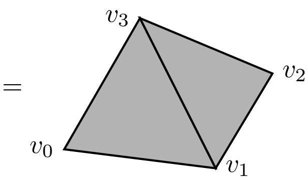


$K/v_3=$

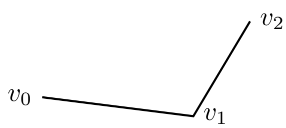


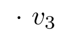
$$S_{K\setminus v_3}(t,q) = q^3 + 3tq^3 + 3t^2q^3 + t^3q^3$$
$$S_{K/v_3}(t,q) = q^3 + 2t + tq^3$$
$$S_{(K\setminus v_3, K/v_3)}(t,q) = 2t^2 + t^2q^3 + t^3q^3$$

B.5 Two 2-Simplices (I)

$K =$ 

Notice that K is the join of the two complexes L_1 and L_2 below:

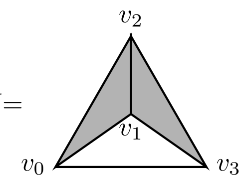
$L_1 =$ 

$L_2 =$ 

Both L_1 and L_2 satisfy spectral recursion as they are both P_4-free. Thus by Lemma 4.4, K satisfies spectral recursion.

B.6 Two 2-Simplices (II)

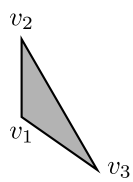
$K=$



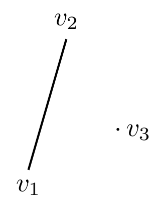
$$S_K(t,q) = q^4 + 4tq^4 + t^2 + t^2q^2 + 4t^2q^4 + t^3q^2 + t^3q^4$$

Notice that vertices v_0 and v_3 are combinatorially identical, as are vertices v_1 and v_2. Thus we need to show that K satisfies spectral recursion with respect to vertices v_0 and v_1.

$K \setminus v_0=$

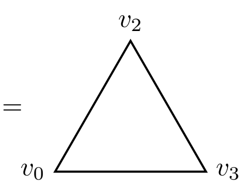


$K/v_0=$

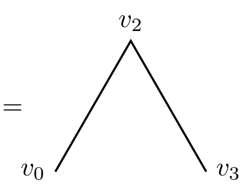


$$\begin{aligned} S_{K\setminus v_0}(t,q) &= q^3 + 3tq^3 + 3t^2q^3 + t^3q^3 \\ S_{K/v_0}(t,q) &= q^3 + t + tq^2 + tq^3 + t^2q^2 \\ S_{(K\setminus v_0, K/v_0)}(t,q) &= t^2 + t^2q^2 + t^3q^2 \end{aligned}$$

$K \setminus v_1=$

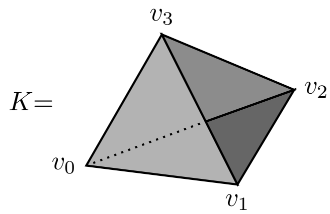


$K/v_1=$



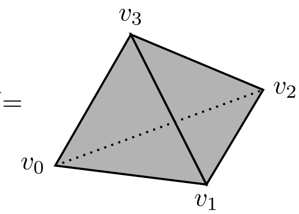
$$\begin{aligned} S_{K\setminus v_1}(t,q) &= q^3 + 3tq^3 + t^2 + 2t^2q^3 \\ S_{K/v_1}(t,q) &= q^3 + tq + 2tq^3 + t^2q + t^2q^3 \\ S_{(K\setminus v_1, K/v_1)}(t,q) &= t^2 \end{aligned}$$

B.7 Three 2-Simplices



This complex K has as its 1-skeleton K^4, the complete graph on four vertices, and contains 2-simplices $\{v_0, v_1, v_2\}$, $\{v_0, v_1, v_3\}$, and $\{v_0, v_2, v_3\}$ (so the only possible 2-simplex not in K is $\{v_1, v_2, v_3\}$). It is clear that K is a matroid complex, so by [6], K satisfies spectral recursion.

B.8 Four 2-Simplices

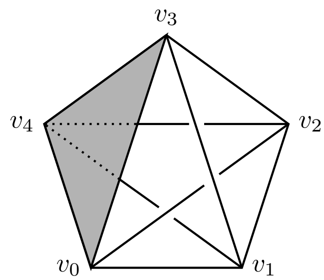
$K=$ 

The complex K is simply the boundary of the 3-simplex, i.e. it contains all possible 1-simplices and 2-simplices that can be formed from the four vertices shown. It is clear that K is a matroid complex, so by [6], K satisfies spectral recursion.

C Two-Dimensional Complexes on Five Vertices

C.1 One 2-Simplex

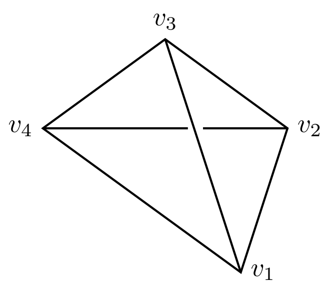
$K=$



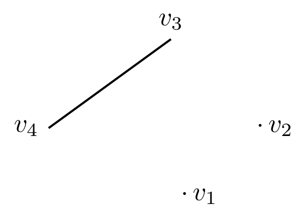
$$S_K(t,q) = q^5 + 5tq^5 + 5t^2 + t^2q^3 + 4t^2q^5 + t^3q^3$$

Vertices v_0, v_3, and v_4 are combinatorially identical, as are vertices v_1 and v_2. Thus we must show that K satisfies spectral recursion with respect to vertices v_0 and v_1.

$K \setminus v_0=$

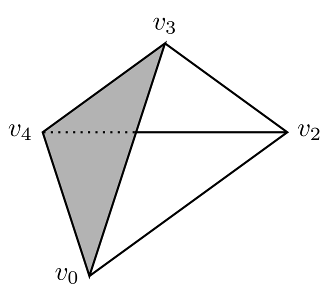


$K/v_0=$

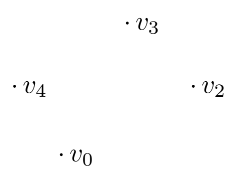


$$S_{K\setminus v_0}(t,q) = q^4 + 4tq^4 + 3t^2 + 3t^2q^4$$
$$S_{K/v_0}(t,q) = q^4 + 2t + tq^2 + tq^4 + t^2q^2$$
$$S_{(K\setminus v_0, K/v_0)}(t,q) = 5t^2$$

$K \setminus v_1=$



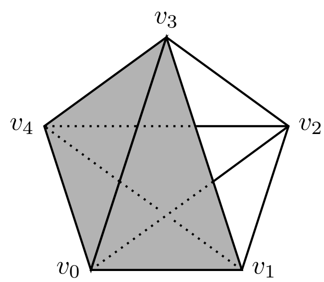
$K/v_1=$



$$S_{K\setminus v_1}(t,q) = q^4 + 4tq^4 + 2t^2 + t^2q^3 + 3t^2q^4 + t^3q^3$$
$$S_{K/v_1}(t,q) = q^4 + 3t + tq^4$$
$$S_{(K\setminus v_1, K/v_1)}(t,q) = 5t^2 + t^2q^3 + t^3q^3$$

C.2 Two 2-Simplices (I)

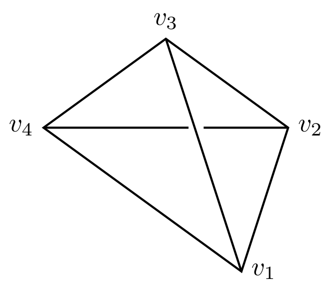
$K=$



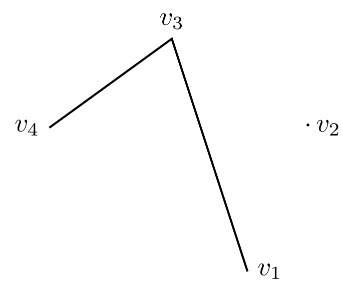
$$S_K(t,q) = q^5 + 5tq^5 + 4t^2 + t^2q^2 + t^2q^4 + 4t^2q^5 + t^3q^2 + t^3q^4$$

Complex K contains two 2-simplices $\{v_0, v_1, v_3\}$ and $\{v_0, v_3, v_4\}$. Vertices v_0 and v_3 are combinatorially identical, as are vertices v_1 and v_4. Thus we need to show that K satisfies spectral recursion with respect to vertices v_0, v_1, and v_2.

$K \setminus v_0 =$

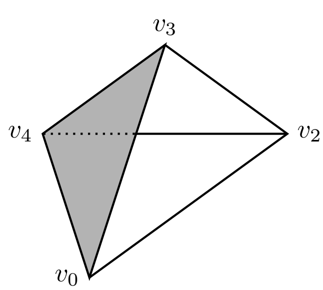


$K/v_0 =$

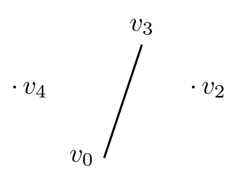


$$\begin{aligned} S_{K\setminus v_0}(t,q) &= q^4 + 4tq^4 + 3t^2 + 3t^2q^4 \\ S_{K/v_0}(t,q) &= q^4 + t + tq + tq^3 + tq^4 + t^2q + t^2q^3 \\ S_{(K\setminus v_0, K/v_0)}(t,q) &= 4t^2 \end{aligned}$$

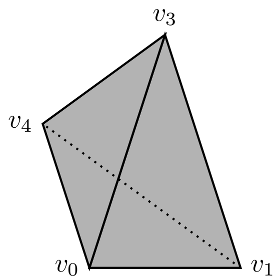
$K \setminus v_1 =$

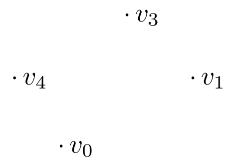


$K/v_1 =$



$$\begin{aligned} S_{K\setminus v_1}(t,q) &= q^4 + 4tq^4 + 2t^2 + t^2q^3 + 3t^2q^4 + t^3q^3 \\ S_{K/v_1}(t,q) &= q^4 + 2t + tq^2 + tq^4 + t^2q^2 \\ S_{(K\setminus v_1, K/v_1)}(t,q) &= 4t^2 + t^2q^2 + t^3q^2 \end{aligned}$$

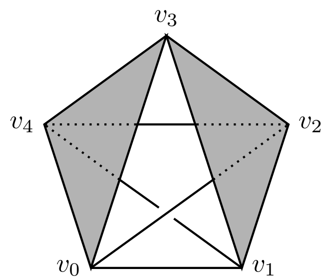
$K \setminus v_2=$ 

$K/v_2=$ 

$$\begin{aligned} S_{K\setminus v_2}(t,q) &= q^4 + 4tq^4 + t^2 + t^2q^2 + 4t^2q^4 + t^3q^2 + t^3q^4 \\ S_{K/v_2}(t,q) &= q^4 + 3t + tq^4 \\ S_{(K\setminus v_2, K/v_2)}(t,q) &= 4t^2 + t^2q^2 + t^2q^3 + t^3q^2 + t^3q^4 \end{aligned}$$

C.3 Two 2-Simplices (II)

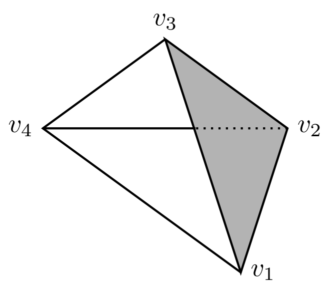
$K=$



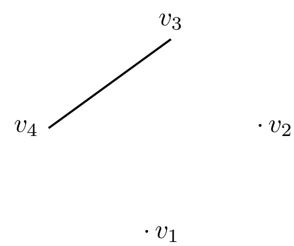
$$S_K(t,q) = q^5 + 5tq^5 + 4t^2 + 2t^2q^3 + 5t^2q^5 + 2t^3q^3$$

Vertices v_0, v_1, v_2, and v_4 are combinatorially identical. Thus we must show that K satisfies spectral recursion with respect to v_0 and v_3.

$K \setminus v_0=$

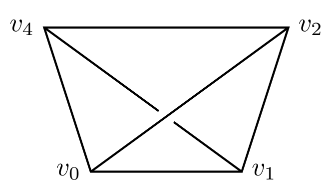


$K/v_0=$

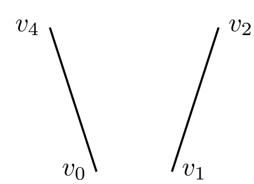


$$S_{K\setminus v_0}(t,q) = q^4 + 4tq^4 + 2t^2 + t^2q^3 + 3t^2q^4 + t^3q^3$$
$$S_{K/v_0}(t,q) = q^4 + 2t + tq^2 + tq^4 + t^2q^2$$
$$S_{(K\setminus v_0, K/v_0)}(t,q) = 4t^2 + t^2q^3 + t^3q^3$$

$K \setminus v_3=$



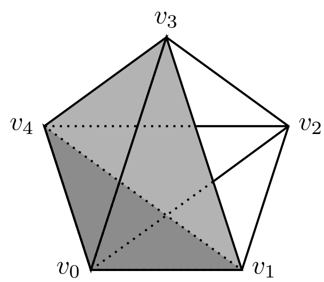
$K/v_3=$



$$S_{K\setminus v_3}(t,q) = q^4 + 4tq^4 + 3t^2 + 3t^2q^4$$
$$S_{K/v_3}(t,q) = q^4 + t + 2tq^2 + tq^4 + 2t^2q^2$$
$$S_{(K\setminus v_3, K/v_3)}(t,q) = 4t^2$$

C.4 Three 2-Simplices (I)

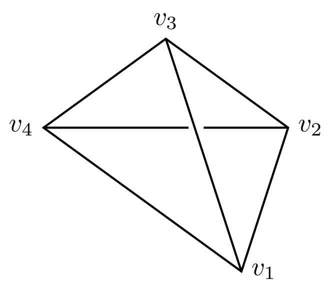
$K=$



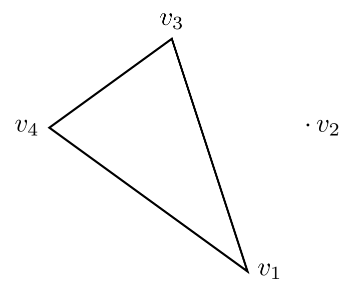
$$S_K(t,q) = q^5 + 5tq^5 + 3t^2 + t^2q + 2t^2q^4 + 4t^2q^5 + t^3q + 2t^3q^4$$

The complex K contains three 2-simplices, $\{v_0, v_1, v_3\}$, $\{v_0, v_1, v_4\}$, and $\{v_0, v_3, v_4\}$. Vertices v_1, v_3, and v_4 are combinatorially identical, so we need to show that K satisfies spectral recursion with respect to v_0, v_1, and v_2.

$K \setminus v_0=$

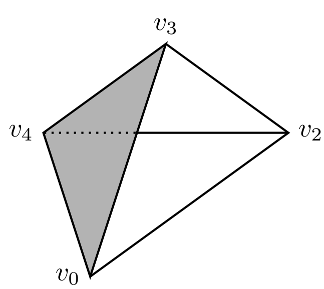


$K/v_0=$

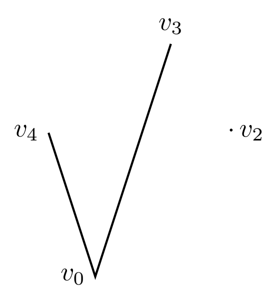


$$S_{K\setminus v_0}(t,q) = q^4 + 4tq^4 + 3t^2 + 3t^2q^4$$
$$S_{K/v_0}(t,q) = q^4 + t + 2tq^3 + tq^4 + t^2 + 2t^2q^3$$
$$S_{(K\setminus v_0, K/v_0)}(t,q) = 3t^2$$

$K \setminus v_1=$

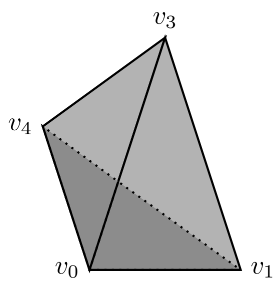


$K/v_1=$

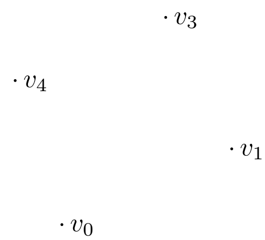


$$S_{K\setminus v_1}(t,q) = q^4 + 4tq^4 + 2t^2 + t^2q^3 + 3t^2q^4 + t^3q^3$$
$$S_{K/v_1}(t,q) = q^4 + t + tq + tq^3 + tq^4 + t^2q + t^2q^3$$
$$S_{(K\setminus v_1, K/v_1)}(t,q) = 3t^2 + t^2q + t^3q$$

$K \setminus v_2 =$

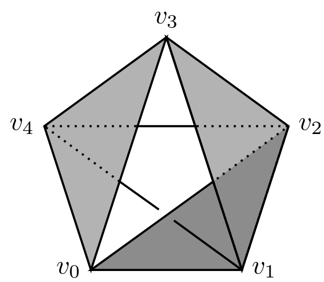


$K/v_2 =$



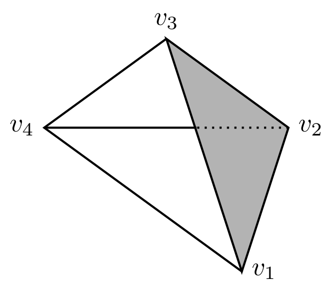
$$
\begin{aligned}
S_{K\setminus v_2}(t,q) &= q^4 + 4tq^4 + t^2q + 5t^2q^4 + t^3q + 2t^3q^4 \\
S_{K/v_2}(t,q) &= q^4 + 3t + tq^4 \\
S_{(K\setminus v_2, K/v_2)}(t,q) &= 3t^2 + t^2q + 2t^2q^4 + t^3q + 2t^3q^4
\end{aligned}
$$

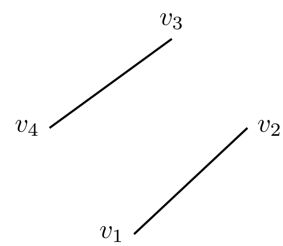
C.5 Three 2-Simplices (II)

$K=$ 

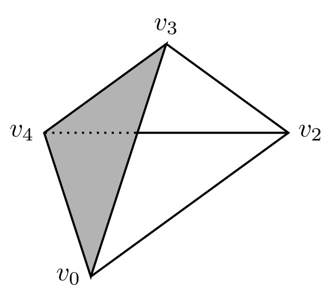
$$S_K(t,q) = q^5 + 5tq^5 + 3t^2 + t^2q^2 + t^2q^3 + t^2q^4 + 4t^2q^5 + t^3q^2 + t^3q^3 + t^3q^4$$

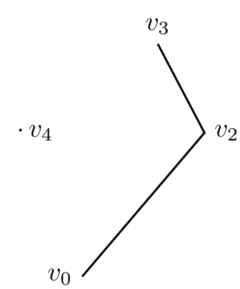
The complex K contains the 2-simplices $\{v_0, v_1, v_2\}$, $\{v_0, v_3, v_4\}$, and $\{v_1, v_2, v_3\}$. Vertices v_0 and v_3 are combinatorially identical, as are vertices v_1 and v_2. Thus we must show that K satisfies spectral recursion with respect to v_0, v_1, and v_4.

$K \setminus v_0=$ 

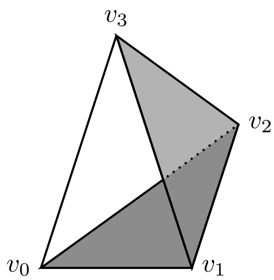
$K/v_0=$ 

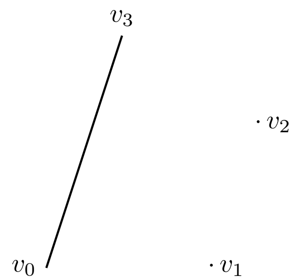
$$S_{K\setminus v_0}(t,q) = q^4 + 4tq^4 + 2t^2 + t^2q^3 + 3t^2q^4 + t^3q^3$$
$$S_{K/v_0}(t,q) = q^4 + t + 2tq^2 + tq^4 + 2t^2q^2$$
$$S_{(K\setminus v_0, K/v_0)}(t,q) = 3t^2 + t^2q^2 + t^3q^2$$

$K \setminus v_1=$ 

$K/v_1=$ 

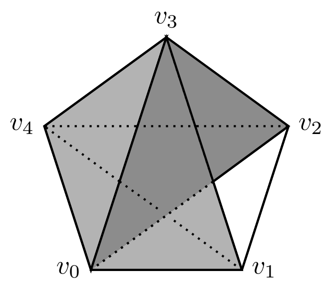
$$S_{K\setminus v_1}(t,q) = q^4 + 4tq^4 + 2t^2 + t^2q^3 + 3t^2q^4 + t^3q^3$$
$$S_{K/v_1}(t,q) = q^4 + t + tq + tq^3 + tq^4 + t^2q + t^2q^3$$
$$S_{(K\setminus v_1, K/v_1)}(t,q) = 3t^2 + t^2q^3 + t^3q^3$$

$K \setminus v_4 =$ 

$K/v_4 =$ 

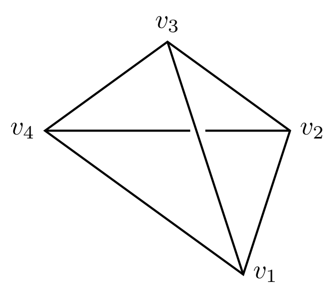
$$S_{K \setminus v_4}(t,q) = q^4 + 4tq^4 + t^2 + t^2q^2 + 4t^2q^4 + t^3q^2 + t^3q^4$$
$$S_{K/v_4}(t,q) = q^4 + 2t + tq^2 + tq^4 + t^2q^2$$
$$S_{(K \setminus v_4, K/v_4)}(t,q) = 3t^2 + t^2q^2 + t^2q^4 + t^3q^2 + t^3q^4$$

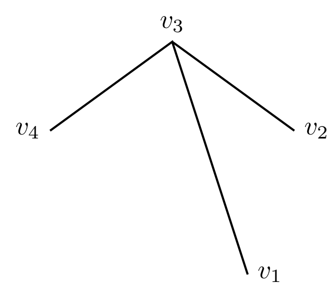
C.6 Three 2-Simplices (III)

$K=$ 

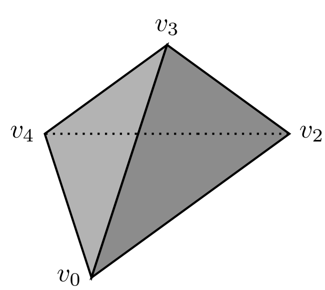
$$S_K(t,q) = q^5 + 5tq^5 + 3t^2 + 2t^2q^2 + 5t^2q^5 + 2t^3q^2 + t^3q^5$$

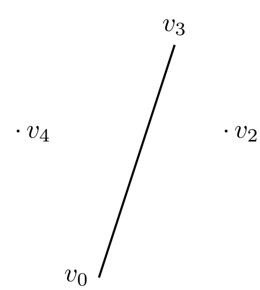
The complex K contains the 2-simplices $\{v_0, v_1, v_3\}$, $\{v_0, v_2, v_3\}$, and $\{v_0, v_3, v_4\}$. Vertices v_0 and v_3 are combinatorially identical, as are vertices v_1, v_2, and v_4. Thus we must show that K satisfies spectral recursion with respect to vertices v_0 and v_1.

$K \setminus v_0=$ 

$K/v_0=$ 

$$\begin{aligned} S_{K\setminus v_0}(t,q) &= q^4 + 4tq^4 + 3t^2 + 3t^2q^4 \\ S_{K/v_0}(t,q) &= q^4 + 2tq + 2tq^4 + 2t^2q + t^2q^4 \\ S_{(K\setminus v_0, K/v_0)}(t,q) &= 3t^2 \end{aligned}$$

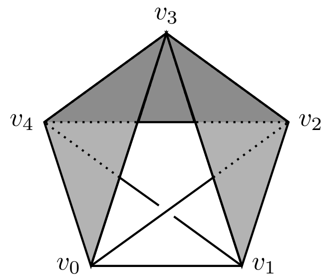
$K \setminus v_1=$ 

$K/v_1=$ 

$$\begin{aligned} S_{K\setminus v_1}(t,q) &= q^4 + 4tq^4 + t^2 + t^2q^2 + 4t^2q^4 + t^3q^2 + t^3q^4 \\ S_{K/v_1}(t,q) &= q^4 + 2t + tq^2 + tq^4 + t^2q^2 \\ S_{(K\setminus v_1, K/v_1)}(t,q) &= 3t^2 + 2t^2q^2 + 2t^3q^2 \end{aligned}$$

C.7 Three 2-Simplices (IV)

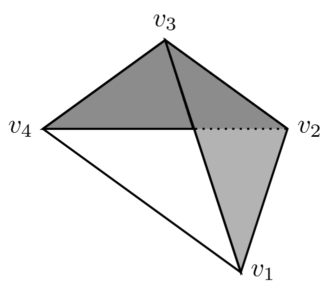
$K=$



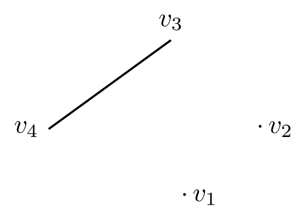
$$S_K(t,q) = q^5 + 5tq^5 + 3t^2 + t^2q^{3-\sqrt{2}} + t^2q^3 + t^2q^{3+\sqrt{2}} + 4t^2q^5 + t^3q^{3-\sqrt{2}} + t^3q^3 + t^3q^{3+\sqrt{2}}$$

The complex K has 2-simplices $\{v_0, v_3, v_4\}$, $\{v_1, v_2, v_3\}$, and $\{v_2, v_3, v_4\}$. Vertices v_0 and v_1 are combinatorially identical, as are vertices v_2 and v_4. This complex is similar to the one in Example 4.21. We show that K does not satisfy spectral recursion with respect to v_0 and v_2, yet satisfies spectral recursion with respect to v_3.

$K \setminus v_0=$

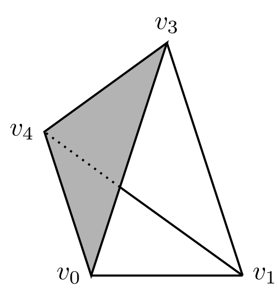


$K/v_0=$

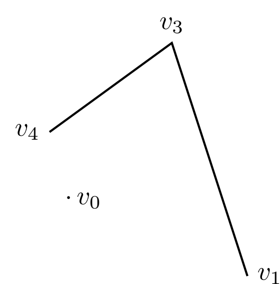


$$S_{K\setminus v_0}(t,q) = q^4 + 4tq^4 + t^2 + t^2q^2 + 4t^2q^4 + t^3q^2 + t^3q^4$$
$$S_{K/v_0}(t,q) = q^4 + 2t + tq^4 + tq^4 + t^2q^2$$
$$S_{(K\setminus v_0, K/v_0)}(t,q) = 3t^2 + t^2q^{\frac{1}{2}(5-\sqrt{5})} + t^2q^{\frac{1}{2}(5+\sqrt{5})} + t^3q^{\frac{1}{2}(5-\sqrt{5})} + t^3q^{\frac{1}{2}(5+\sqrt{5})}$$

$K \setminus v_2=$

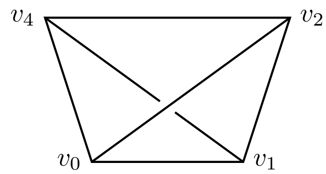


$K/v_2=$

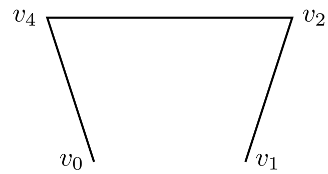


$$S_{K\setminus v_2}(t,q) = q^4 + 4tq^4 + 2t^2 + t^2q^3 + 3t^2q^4 + t^3q^3$$
$$S_{K/v_2}(t,q) = q^4 + t + tq + tq^3 + tq^4 + t^2q + t^2q^3$$
$$S_{(K\setminus v_2, K/v_2)}(t,q) = 3t^2 + t^2q^2 + t^3q^2$$

$K \setminus v_3 =$



$K/v_3 =$



$$S_{K\setminus v_3}(t,q) = q^4 + 4tq^4 + 3t^2 + 3t^2q^4$$
$$S_{K/v_4}(t,q) = q^4 + tq^{2-\sqrt{2}} + tq^2 + tq^{2+\sqrt{2}} + tq^4 + t^2q^{2-\sqrt{2}} + t^2q^2 + t^2q^{2+\sqrt{2}}$$
$$S_{(K\setminus v_4, K/v_4)}(t,q) = 3t^2$$

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